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# On the Uniquely List Colorable Graphs

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**Abstract**

Let $G$ be a graph with $\nu$ vertices, and let $S_1, S_2, \ldots, S_\nu$ be a list of colors on its vertices, each of size $k$. If there exists a unique proper coloring for $G$ from this list of colors, then $G$ is called *uniquely $k$–list colorable graph*. We characterize all uniquely 2–list colorable graphs, and discuss uniquely $k$–list colorable graphs by introducing some open problems. We also show the connection of this concept with the defining sets in graphs and critical sets in latin squares.

**AMS subject classification: 05**
**Keywords:** *graph coloring, list coloring, uniquely list colorable, Hall's theorem, defining set.*

## 1 Introduction and Preliminaries

We consider simple graphs which are finite, undirected, with no loops or multiple edges. For the necessary definitions and notations we refer the reader to standard texts, such as [2] and [3]. Following the latter book we usually use $\nu$ for $|V(G)|$. In this section we mention some of the definitions and results which are referred to throughout the paper.

Let $G$ be a graph with vertex set $\{1, 2, \ldots, \nu\}$, and $S_1, S_2, \ldots, S_\nu$ a list of colors on its vertices. If there exists a proper coloring $c$ for $G$ such that $c(v) \in S_v$ for all $v \in V(G)$; then $G$ is called to have a *list coloring*. By a proper coloring we mean: for adjacent vertices $u$ and $v$ we have $c(u) \neq c(v)$. The idea of list coloring of graphs is due independently to V. G. Vizing [9] and to P. Erdös, A. L. Rubin, and H. Taylor [4]. For a recent survey on list coloring we refer the interested reader to N. Alon [1]. It is interesting to note that if the assumed graph is the complete graph $K_\nu$, then a list coloring of $G$ is nothing but a system of distinct representatives–SDR– for the sets $S_1, S_2, \ldots, S_\nu$. The following theorem of Marshal Hall, which is a corollary of the celebrated Marriage Theorem of P. Hall, is of a great interest to us.

**Theorem** [5]**.** *If $\nu$ sets $S_1, S_2, \ldots, S_\nu$ have an SDR and the smallest of these sets contains $t$ objects, then if $t \geq \nu$, at least there are $t(t-1)\cdots(t-\nu+1)$ different SDRs, and if $t < \nu$, there are at least $t!$ different SDRs.*

**Corollary.** *If $\nu$ sets $S_1, S_2, \ldots, S_\nu$ have an SDR and the smallest of these sets is at least of size 2, then there are at least two different SDRs. Or, equivalently, if there exists a list coloring for the*

*complete graph $K_\nu$ with the list of colors $S_1, S_2, \ldots, S_\nu$, each of size at least 2, then there are two different list colorings of $K_\nu$.*

The following natural question is a motivation of this note.

**Question 1.** *For which graphs does the result of the above corollary hold?*

We say that $G$, a graph with $\nu$ vertices, *has the property $M(k)$*, M for Marshal Hall, if for any list of colors $S_1, S_2, \ldots, S_\nu$, with $|S_i| \geq k$; having a coloring for $G$ implies that there exists also a different coloring for $G$. Note that in this definition, without loss of generality, we can assume $|S_i| = k$; for $i = 1, 2, \ldots, \nu$. Conversely, a graph $G$ with $\nu$ vertices, is called a *uniquely $k$–list colorable graph*, if there exists $S_1, S_2, \ldots, S_\nu$, a list of colors on its vertices, each of size $k$, such that there is a unique coloring for $G$ from this list of colors. So $G$ is uniquely $k$–list colorable if and only if it does not have the property $M(k)$. It can be seen from the definition that the graphs having the property $M(2)$ are solutions of Question 1.
From the above corollary we see the following,

**Example 1.** *Complete graphs $K_\nu$ have the property $M(2)$.*

**Example 2.** *The graph $K_4 - e$ is a uniquely 2–list colorable graph. Thus it does not have the property $M(2)$.*
To see this we give a list of colors, each of size two, as in Figure 1.

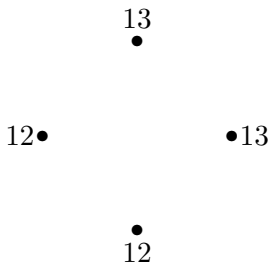


Figure 1: $K_4 - e$

In Section 2 we characterize all uniquely 2–list colorable graphs. In Section 3 some applications of uniquely 2–list colorable graphs is discussed. In Section 4 we introduce some generalizations and open problems.

## 2 Uniquely 2–List Colorable Graphs

In this section we state some results concerning graphs which are uniquely 2–list colorable. These include some lemmas and theorems which will deduce the proof of the main theorem, given at the end of this section.
The following proposition is clear from the definition of the property $M(2)$.

**Proposition 1.** *A graph has the property $M(2)$ if and only if at least one of its components has the property $M(2)$.*

In the following theorem we introduce some families of graphs which have the property $M(2)$. For brevity, we omit the proof of this and some other theorems. We refer the interested reader to a forthcoming paper.

**Theorem 1.** *Cycles $C_\nu$, complete graphs $K_\nu$, and complete bipartite graphs $K_{m,n}$ have the property $M(2)$.* □

Next theorems are useful in our discussion.

**Theorem 2.** *If an induced subgraph $H$ of a connected graph $G$ is uniquely 2–list colorable, then $G$ is uniquely 2–list colorable also.*

**Proof.** We prove by induction on $\nu(G) - \nu(H)$. If $\nu(G) - \nu(H) = 0$, then $G = H$, and there is nothing to prove. Now, let $\nu(G) - \nu(H) = k$, where $k > 0$. Since $G$ is connected, there exists at least a vertex $v \in V(G) - V(H)$, which is adjacent to a vertex $u$ in $H$. We claim that $H_1 =< V(H) \cup \{v\} >$ is also uniquely 2–list colorable. The graph $H$ being uniquely 2–list colorable, there exist a list of colors $S_1, S_2, \ldots, S_{\nu(H)}$ with $|S_i| = 2$, such that $H$ has a unique coloring $c_1, c_2, \ldots, c_{\nu(H)}$ $(c_i \in S_i)$ with this list. Now we assign the list $\{c(u), x\}$, where $x$ is a new color, for the vertex $v$ in the induced subgraph $H_1$, and keep the list of other vertices of $H$ unchanged. It is easy to see that $H_1$ with the assigned list is uniquely list colorable. But for $H_1$ we have $\nu(G) - \nu(H_1) = k - 1$, so the result follows by induction. □

**Theorem 3.** *Let $G$ be a graph which is the union of two graphs $G_1$ and $G_2$ which are joined in exactly one vertex, and both have the property $M(2)$. Then $G$ has the property $M(2)$ also.*

**Proof.** Let $G_1$ and $G_2$ be joined in a vertex $x$. Assume that a list of colors for the vertices of $G$ is given, each of size 2, and there is a coloring $c$ for $G$ with this list. Suppose that the list given to the vertex $x$ is $\{a, b\}$ and $c(x) = a$. Now with this list of colors, $G_1$ also has a coloring different from $c$ say $c_1$. If $c_1(x) = a$, then by changing the colors of vertices of $G_1$ in $G$ to $c_1$ coloring, we obtain a different coloring for $G$. So assume that $c_1(x) = b$. Similarly in $c_2$, the second coloring of $G_2$, if $c_2(x) = a$ we are done. Thus let $c_2(x) = b$. Then by coloring $G_1$ with $c_1$ and $G_2$ with $c_2$, which are consistent in $x$, we obtain a different coloring for $G$. □

**Corollary.** *A connected graph $G$ has the property $M(2)$ if and only if each of its blocks has the property $M(2)$.*

**Proof.** It follows from Theorem 2, Theorem 3, and by induction. □

So we only need to study 2–connected graphs having the property $M(2)$. The following lemma will be useful.

**Lemma 1.** *Suppose that a graph $G$ has a triangle $abc$ such that, there exists a path $av_1v_2 \ldots v_kb$ from the vertex $a$ to the vertex $b$ and disjoint from $c$, in which $v_k$ is not adjacent to $c$. Then $G$ is uniquely 2–list colorable.*

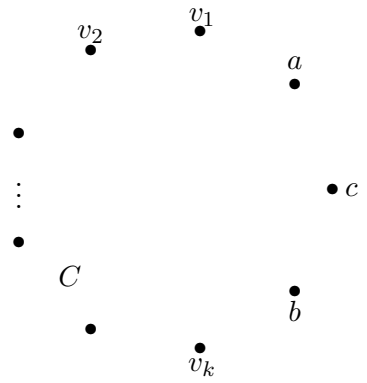


Figure 2: A subgraph of $G$

**Proof.** A subgraph of $G$ is shown in Figure 2. There may be more edges in the induced subgraph on these vertices but $c$ is not adjacent to $v_k$. We give the following list of colors to the vertices: $\{1,2\}$ for $b$ and $c$, $\{2,3\}$ for $a$, $\{3,4\}$ for $v_1$, $\{4,5\}$ for $v_2$, ..., $\{k+1,k+2\}$ for $v_{k-1}$, and $\{k+2,1\}$ for $v_k$. This list of colors shows that an induced subgraph of $G$ is a uniquely 2–list colorable graph, therefore by Theorem 2, $G$ is uniquely 2–list colorable. □

**Lemma 2.** *If a 2–connected graph $G$ which contains a triangle has the property $M(2)$, then $G$ is a complete graph.*

**Proof.** We prove by contradiction. Assume that $G$ is not a complete graph and let $H$ be a (maximal) clique in $G$. So by assumption $V(G) - V(H) \neq \emptyset$, and there exists $v \in V(G) - V(H)$ which is adjacent to a vertex $b$ in $H$. Since $H$ is a largest complete subgraph of $G$, there exists a vertex in $H$, which is not adjacent to $v$. We consider two cases:

(i) $v$ is adjacent to another vertex $a$ in $H$. Let $c$ be a vertex in $H$ which is not adjacent to $v$. Then the subgraph induced on the vertices $\{a,b,c,v\}$ is $K_4 - e$, and as in Example 2, is uniquely 2–list colorable. Thus by Lemma 2, $G$ is uniquely 2–list colorable.

(ii) $v$ is adjacent only to $b$ in $H$. Then since $G$ is 2–connected, there exists a path from $v$ to a vertex $a$ (different from $b$) in $H$. Let $c$ be a third vertex in $H$ different from $a$ and $b$. Since $G$ contains a triangle such a vertex exists. By our assumption in this case, $c$ is not adjacent to $v$. Now applying Lemma 1 follows that $G$ is a uniquely 2–list colorable graph and this is a contradiction. □

**Theorem 4.** *If a 2–connected graph $G$ which contains an odd cycle has the property $M(2)$, then $G$ is either a complete graph or an odd cycle.*

**Proof.** The proof of this theorem is a lengthy and involved argument similar to the previous two lemmas. For brevity, we do not mention it here. □

Now we turn our attention to the bipartite graphs.

**Lemma 3.** *If a bipartite 2–connected graph $G$ which contains a square has the property $M(2)$, then $G$ is a complete bipartite graph.*

**Proof.** The proof is similar to the proof of Lemma 1 and Lemma 2, but a little more complicated and lengthy. So, we do not mention it here. □

**Theorem 5.** *If a bipartite 2–connected graph $G$ has the property $M(2)$, then $G$ is either an even cycle or a complete bipartite graph.*

**Proof.** Again, the proof of this theorem is similar to the Theorem 4 and we do not mention it here. □

Now we are ready to prove the main theorem.

**Theorem** (MAIN)**.** *A connected graph has the property $M(2)$ if and only if every block of which is either a cycle, a complete graph, or a complete bipartite graph.*

**Proof.** The result follows by Proposition 1 and Theorems 1, 3, 4, and 5. □

## 3 Applications

Graphs having the property $M(2)$ have applications in the discussion of defining sets in graph colorings [6], and in the critical sets in latin squares ([6] and [7]). In a given graph $G$, a set of vertices $S$ with an assignment of colors is said to be a *defining set of vertex coloring*, if there exists a unique extension of the colors of $S$ to a $\chi(G)$–coloring of the vertices of $G$. We denote by $d_v(G)$ the minimum size of a defining set for a vertex coloring of $G$. Example 1 is applied to prove a theorem on the defining set of graphs in [6]. Now we generalize that theorem using the concept of $M(2)$ property.

**Theorem 6.** *For every graph $M$ having the property $M(2)$, and every graph $G$ such that $\chi(G) \leq \chi(M)$, we have:*

$$d_v(G \times M) \geq |V(G)|(\chi(M) - 1) - 2|E(G)|$$

**Proof.** First, note that we have $\chi(G \times M) = \chi(M) = k$. Suppose that $V(G) = \{1, \ldots, m\}$ and $V(M) = \{1, ..., n\}$. We note $G \times M$ has $mn$ vertices and contains $m$ horizontal copies of $M$, say $M^{(1)}, M^{(2)}, \ldots, M^{(m)}$ and $n$ vertical copies of $G$, say $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$. A horizontal copy $M^{(i)}$ and a vertical copy $G^{(j)}$ have only one vertex $(i, j)$ in common. Now let $S$ be a defining set for $G \times M$ and let $d_1, d_2, \ldots, d_m$ be the degree sequence of $G$. We will show that for each $i$, at least $k - d_i - 1$ vertices of $M^{(i)}$ belong to $S$. In contrary, suppose that less than $k - d_i - 1$ vertices of $M^{(i)}$ belong to $S$. In other words, the color of at least $n - k + d_i + 2$ vertices of $M^{(i)}$, say $t_1, t_2, \ldots$ are not given. Assume that we have colored all of the vertices of $G \times M$, except $t_j$'s. Let $T_j$ be the set of colors that can be used as the color of $t_j$. We note that the at most $(k - d_i - 2) + d_i$ neighbors of $t_j$ are colored, so we have $|T_j| \geq k - (k - d_i - 2 + d_i) = 2$. Let $T$ be the subgraph of $G \times M$ induced by $t_j$'s. Since $S$ is a defining set, there is a list coloring with this list of colors for $T$. But by Theorem 2, $M$ having the property $M(2)$ implies that $T$ has the property $M(2)$ also. Therefore there is another list coloring for $T$ which contradicts the fact that $S$ is a defining set. □

The technique used in the proof of the above theorem, can be used in many problems about defining sets. In fact, the main theorem of the previous section, can be used to generalize theorems which use the M. Hall's theorem in their proof.

## 4 Generalizations and Open Questions

The motivation for defining the property $M(2)$, was the corollary of M. Hall's theorem. One can also ask the following natural question:

**Question 2.** *For which graphs does the result of M. Hall's theorem hold?*

The following theorem answers this question:

**Theorem 7.** *Let $G$ be a graph with property $M(2)$. Assume that some lists of colors are assigned to the vertices of $G$ and there is a list coloring for $G$. Then there is a vertex $v$ in $G$, such that, for every color $c$ in the list of $v$, there is a coloring for $G$ in which $v$ is colored with $c$.*

**Proof.** Suppose in contrary that for every vertex $v$, there is a color $c_v$ in its list, such that $v$ cannot be colored with $c_v$ in any list coloring. Now assign a list of two colors to $v$, one of the colors is the color of $v$ in the list coloring, and the other is $c_v$. By the definition of $c_v$'s, $G$ must be colored uniquely with this list. This contradicts the fact that $G$ has the property $M(2)$. □

**Corollary.** *Let $G$ be a graph with property $M(2)$. If some lists of colors are assigned to the vertices of $G$, and the smallest of these lists has $t$ elements, then the existence of a list coloring for $G$, implies that the number of different list colorings for $G$ with these lists of colors is at least $t!$ if $t < v$, and at least $t(t-1)\cdots(t-v+1)$ if $t \geq v$.*

**Proof.** The result follows immediately by Theorem 7, Theorem 2, and by induction. □

Similar questions such as Question 1 can be asked again about the property $M(k)$.

**Problem 1.** *For each $k$, characterize all uniquely $k$–list colorable graphs.*

This question seems to be very difficult. It is easy to see that if a graph has the property $M(k)$, then it has the property $M(k+1)$. Equivalently if $G$ is a uniquely $(k+1)$–list colorable graph, then $G$ is also a uniquely $k$–list colorable graph. Example 2, given earlier, is a special case of the following when $k = 2$.

**Example 3.** *Let $G$ be a complete $(2k-1)$–partite graph with the parts $X_1$, $X_2$, ..., $X_{2k-1}$; such that $|X_1| = |X_2| = \ldots = |X_{k-1}| = 2$ and $|X_k| = |X_{k+1}| = \ldots = |X_{2k-1}| = 1$. We assign a list of colors to the vertices of $G$ as follows. Each list, consists of $k$ colors and we have:*
$\{1,2,3,\ldots,k\},\ \{1,k+1,k+2,\ldots,2k-1\}$ *for the vertices in $X_1$,*
$\{1,2,3,\ldots,k\},\ \{1,2,k+1,\ldots,2k-2\}$ *for the vertices in $X_2$,*
$\{1,2,3,\ldots,k\},\ \{1,2,3,k+1,\ldots,2k-3\}$ *for the vertices in $X_3$, ...,*
$\{1,2,3,\ldots,k\},\ \{1,2,3,\ldots,k-1,k+1\}$ *for the vertices in $X_{k-1}$, and*
$\{1,2,3,\ldots,k\},\ \{1,2,3,\ldots,k-1,k+1\},\ldots,\{1,2,3,\ldots,k-1,2k-1\}$ *for the vertices in $X_k$, $X_{k+1}$, ... , $X_{2k-1}$ respectively. $G$ is a uniquely $k$–list colorable graph.*

**Proof.** Since $G$ is complete $(2k-1)$–partite graph, at least $(2k-1)$ different colors are needed to color its vertices. But the union of all the lists given above, has $(2k-1)$ colors in total. So vertices in each part $X_i$, $i = 1,\ldots,k-1$, must have the same color. This forces the color 1 for the vertices of $X_1$, 2 for the vertices of $X_2$, ... , and $k-1$ for the vertices of $X_{k-1}$. Thus, colors of the remaining vertices are determined uniquely. □

In the following theorem, many other graphs with the property $M(k)$ are introduced.

**Theorem 8.** *Let $H$ be an induced subgraph of a graph $G$, such that for each $v \in V(H)$, there are at most $l$ edges connecting $v$ to $G - H$. If $H$ has the property $M(k)$, then $G$ has the property $M(k + l)$.*

**Proof.** Suppose that we have assigned lists of size $k + l$ to the vertices of $G$, such that there is a list coloring with this list. Assume that we know the color of vertices of $G$, except for the vertices in $H$. Now, there are at least $k$ possibilities for the color of each vertex in $H$ and this, together with the hypothesis that $H$ has the property $M(k)$, implies that there is a different coloring for $H$ which induces a different coloring for $G$. □

**Example 4.** *Let $G$ be the graph of Example 3. Let $H$ be a subgraph of $G$ that contains exactly one vertex from each part of $G$. Note that $H$ is a complete graph $K_{2k-1}$. Since there are $k - 1$ vertices in $G - H$ and $H$ has the property $M(2)$, the Theorem 8 shows that $G$ has the property $M(k + 1)$.*

**Corollary.** *Every graph $G$ has the property $M(\delta+2)$ where $\delta$ is the minimum degree of the vertices of $G$.*

**Proof.** If in Theorem 8 we let $H = \{v\}$; where $v$ is a vertex of degree $\delta$ in $G$, then we will find that $G$ has the property $M(\delta + 2)$. □

The above corollary, leads us to the following definition: The *M-number* of a graph $G$, denoted by $m(G)$, is defined as the least integer $k$ such that $G$ has the property $M(k)$. Note that by this notation, the above corollary can be expressed as the following:

For every graph $G$, we have $m(G) \leq \delta + 2$.

We can construct a graph $G$ with $M$-number 3, by adding a new vertex and connecting it to exactly one of the vertices of a uniquely 2–list colorable graph. Then by Theorem 2 and corollary of Theorem 8, we have $m(G) = 3 = \delta + 2$. This shows that for many graphs, equality in the result of the above statement holds. So we can propose the following problem:

**Problem 2.** *Characterize all graphs with $m(G) = \delta + 2$.*

The following theorem, is a generalization of Theorem 3.

**Theorem 9.** *Let $G$ be a graph which is the union of two graphs $G_1$ and $G_2$ joined together in exactly one vertex. Then $m(G) \leq m(G_1) + m(G_2) - 2$.*

**Proof.** Let $m(G_1) = k$, $m(G_2) = l$, and $v$ be the intersection of $G_1$ and $G_2$. Suppose that lists of size $k + l - 2$ are assigned to the vertices of $G$, such that there is a unique list coloring $c$ which induces the colorings $c_1$ and $c_2$ on $G_1$ and $G_2$, respectively. Assume that the color of $v$ in this unique coloring is 1. Let $A_1$ ($A_2$, respectively) be the set of colors $x \neq 1$, such that there does *not* exist any list coloring for $G_1$ ($G_2$, respectively) in which $v$ is colored with $x$. If $|A_1| \geq k - 1$, we can assign the list $A_1 \cup \{1\}$ to $v$. $G_1$ having the property $M(k)$ implies that there is a coloring, say $c_1'$, for $G_1$ other than $c_1$ with this list of colors. The definition of $A_1$ implies that the color of $v$ in $c_1'$ is 1. Therefore, the colorings $c_1'$ and $c_2$ together give a new coloring for $G$, contradicting the

uniqueness of $c$. So, we have $|A_1| \leq k-2$. Similarly $|A_2| \leq l-2$. Therefore $|A_1 \cup A_2| \leq k+l-4$, which implies that there is a color $x \neq 1$ which is neither in $A_1$, nor in $A_2$. This means that there is a coloring $c_1'$ for $G_1$ and a coloring $c_2'$ for $G_2$, in which $v$ is colored with $x$. Hence, $c_1'$ and $c_2'$ together give a coloring for $G$ other than $c$. This completes the proof. □

In a paper by Thomassen [8], it is shown that every planar graph is 5-choosable. He has proved that even if the list of a vertex of the graph contains only one element, there is still a list coloring for graph. This means that if we assign lists of size greater than or equal to 5 to the vertices of a planar graph, then there is a vertex which can accept any color in its list. Therefore, for any planar graph $G$, we have $m(G) \leq 5$. It remains an open question to find a planar graph such that $m(G) = 5$.

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