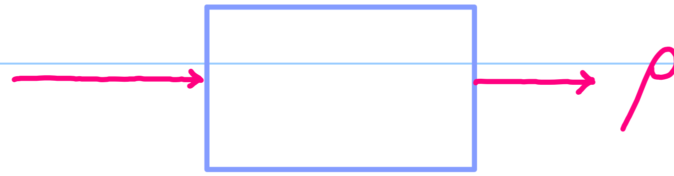
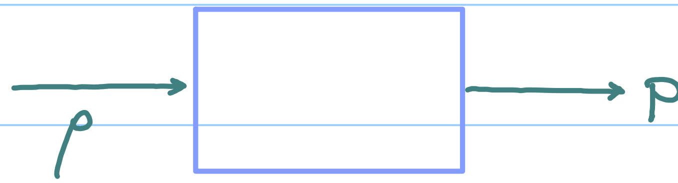
Quantum Channels and Maps; part

Note Title 10/22/2010

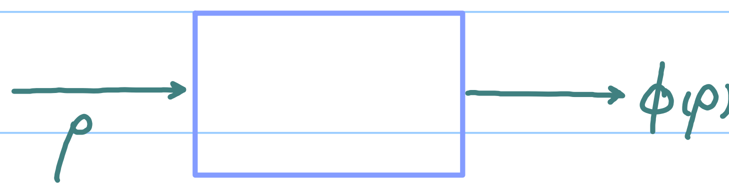
Preparator:



Measurement



Channel

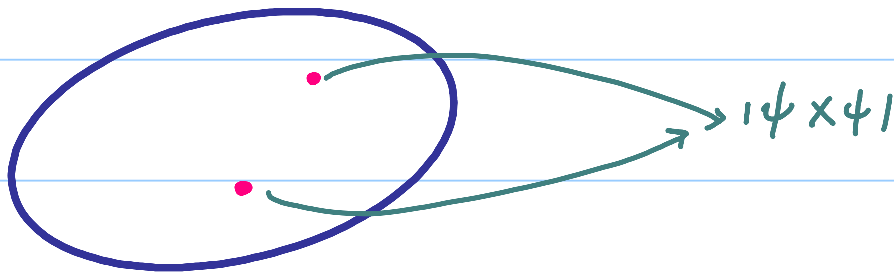


Theorem 7. *(Stinespring's Theorem.)* Let $\mathcal{E} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ be a quantum channel. There exists an environment Hilbert space $\mathcal{H}_E$, unitary operator $U : \mathcal{H} \otimes \mathcal{H}_E \to \mathcal{H} \otimes \mathcal{H}_E$ and an initial state of the environment $\xi \in \mathcal{S}(\mathcal{H}_E)$ such that

$$\mathcal{E}(\varrho) = \mathrm{tr}_E[U(\varrho \otimes \xi)U^*] \tag{5.12}$$

for all $\varrho \in \mathcal{S}(\mathcal{H})$. The triple $\langle \mathcal{H}_E, U, \xi \rangle$ is called a *Stinespring dilation* of the channel $\mathcal{E}$.

Contraction to pure states:



$$C_\psi(\rho) = |\psi\rangle\langle\psi|$$

$$E_k := |\psi\rangle\langle k| \qquad E_k^\dagger = |k\rangle\langle\psi| \qquad E_k \rho E_k^\dagger = |\psi\rangle\langle k|\rho|k\rangle\langle\psi|$$

$$\sum_k E_k \rho E_k^\dagger = |\psi\rangle\langle\psi|$$

Convex Combination of Channels

$$(1-\lambda)\,\mathcal{E} + \lambda\,\mathcal{F}$$

Contraction to mixtures:

$$\phi := \sum_{\psi} \lambda_{\psi} C_{\psi}$$

$$\phi(\rho) = \sum_{\psi} \lambda_{\psi} |\psi\rangle\langle\psi|$$

Contraction into the total mixture.

$$\Phi_0 : B(H_1) \longrightarrow B(H_1) \qquad \phi_0(\rho) = \frac{I}{d}$$

$$\tilde{A}_{jk} = |j\rangle\langle k| \qquad \tilde{A}^{\dagger}_{jk} = |k\rangle\langle j|$$

$$\sum_{jk} \tilde{A}_{jk} \tilde{A}^{\dagger}_{jk} = d\, I \qquad \sum_{jk} \tilde{A}^{\dagger}_{jk} \tilde{A}_{jk} = d\, I$$

$$A_{jk} = \frac{1}{\sqrt{d}} |j\rangle\langle k|$$

$$\sum_{jk} A_{jk} \rho A_{jk}^{\dagger} = \frac{1}{d} \sum_{jk} |j\rangle\langle k|\rho|k\rangle\langle j| = \frac{1}{d} \mathrm{tr}(\rho)\, I.$$

Random Unitary Channels

$$\mathcal{E}(\rho) = \sum_{i} p_i\, U_i \rho U_i^{\dagger} \quad , \quad \sum_i p_i = 1$$

$$\mathcal{E}(\rho) = \int dU\, p(U)\, U \rho U^{\dagger} \qquad \int dU\, p(U) = 1.$$

Relation with Classical Stochastic Maps

$$\Phi(\rho) := \sum_k A_k \rho A_k^\dagger \rightarrow$$

$$[\phi(\rho)]_{\mu\nu} = \sum_k (A_k)_{\mu\alpha} \rho_{\alpha\beta} (A_k^*)_{\nu\beta} = \sum_k (A_k)_{\mu\alpha} (A_k^*)_{\nu\beta} \rho_{\alpha\beta}$$

$$[\phi(\rho)]_{\mu\nu} = \sum_k (A_k \otimes A_k^*)_{\mu\nu,\alpha\beta} \rho_{\alpha\beta}$$

$$\Phi \Longleftrightarrow \mathcal{A} = \sum_k A_k \otimes A_k^*$$

$$\hat{\phi} \Longleftrightarrow \hat{\mathcal{A}} = \sum_k A_k^\dagger \otimes (A_k^\dagger)^*$$

λ is an eigenvalue of $\mathcal{A}$ $\longleftrightarrow$ λ^* is an eigenvalue of $\hat{\mathcal{A}}$

if ϕ is Trace-preserving $\Rightarrow$ $\hat{\phi}$ is unital $\Rightarrow$ $\hat{\phi}(I) = I$

$\rightarrow$ $\hat{\phi}$ has eigenvalue 1 $\Rightarrow$ ϕ also has an eigenvalue 1

$\rightarrow$ $\exists\, P \mid \phi(P) = P$.

$$|P_t\rangle = \begin{pmatrix} P_t(1) \\ P_t(2) \\ \vdots \\ P_t(N) \end{pmatrix} \qquad P_{t+1}(n) = \sum_{n'} w(n' \to n)\, P_t(n')$$

$$|P_{t+1}\rangle = Q\,|P_t\rangle \qquad Q = \begin{bmatrix} w(1\to 1) & w(2\to 1) & w(N\to 1) \\ w(1\to 2) & w(2\to 2) & w(N\to 2) \\ & & \\ w(1\to N) & w(2\to N) & w(N\to N) \end{bmatrix}$$

$$\langle S| = (1, 1, \dots, 1) \qquad \langle S|\,Q = \langle S| \longrightarrow$$

Q has eigenvalue 1. $\longrightarrow$ $\exists\ |P\rangle \ \ |\ \ Q\,|P\rangle = |P\rangle$

Bi-stochastic Map.
$$\begin{cases} \sum_{n'} w(n \to n') = 1 \\ \sum_{n} w(n \to n') = 1 \end{cases}$$

Distances Between Channels

Minimal Distance $$D_{inf}(\mathcal{E}_1, \mathcal{E}_2) = \frac{1}{2} \inf_{\rho} \| \mathcal{E}_1(\rho) - \mathcal{E}_2(\rho) \|_{tr}$$

Maximal Distance $$D_{max}(\mathcal{E}_1, \mathcal{E}_2) = \frac{1}{2} \sup_{\rho} \| \mathcal{E}_1(\rho) - \mathcal{E}_2(\rho) \|_{tr}$$

Average Distance $$D_{av}(\mathcal{E}_1, \mathcal{E}_2) = \int_{B(H)} d\rho \, \| \mathcal{E}_1(\rho) - \mathcal{E}_2(\rho) \|_{tr}$$

Φ_0 := Contraction to total mixture

σ_U := unitary channel

مثلا:

$$\Phi_0(\rho) = \frac{1}{d} I \ , \quad \sigma_U(\rho) = U\rho U^{\dagger}$$

$$D_{min}(\sigma_U, \Phi_0) = 0$$

$$D_{max}(\sigma_U, \Phi_0) = \frac{1}{2}\left(1 - \frac{1}{d}\right)$$

Quantification of Noise :

$$\Delta_{sup}(\mathcal{E}) := \frac{1}{2} \sup_{\rho} \| \mathcal{E}(\rho) - \rho \|$$

C_ψ := Contraction to pure state ψ

$$\Delta_{sup}(C_\psi) = \frac{1}{2} \sup_{\rho} \| |\psi\rangle\langle\psi| - \rho \|$$

$$\rho = |\psi^\perp\rangle\langle\psi^\perp| \rightarrow \Delta_{sup}(C_\psi) = 1.$$

Contraction to maximally mixed state

$$\Delta_{sup}(\phi_0) = \frac{1}{2} \sup_{\rho} \| \frac{1}{d} I - \rho \| = 1 - \frac{1}{d}$$

Output purity

Noise

$|\psi\rangle \rightarrow \boxed{\text{Channel}} \rightarrow \rho$

p-Norm: $\|A\|_p := \left(\mathrm{Tr}[A^p]\right)^{1/p}$ $\quad 1 \le p$

$$\|A\|_1 = (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$$

$$\|A\|_2 = \sqrt{(\lambda_1^2 + \lambda_2^2 + \cdots \lambda_n^2)}$$

if ρ is pure $\longrightarrow$ $\|\rho\|_p = 1$.

if ρ_1 is more pure than ρ_2 $\longrightarrow$ $\|\rho_1\|_p \geq \|\rho_2\|_p$

example. $\rho_1 = 0.8\,|0\rangle\langle 0| + 0.2\,|1\rangle\langle 1|$

$\rho_2 = 0.7\,|0\rangle\langle 0| + 0.3\,|1\rangle\langle 1|$.

$$\|\rho_1\|_2 = \frac{1}{100}(64+4) = 0.68$$

$$\|\rho_2\|_2 = \frac{1}{100}(49+9) = 0.58$$

نکته ۱:

$$\lim_{p\to 1^+} \frac{1-x^p}{p-1} = -x\ln x.$$

اثبات: با قاعدهٔ هوپیتال رفع ابهام کنیم نتیجه بدست می‌آید.

نکته ۲:

$$\lim_{p\to 1^+} \frac{1-(x_1^p+x_2^p+\dots x_N^p)}{p-1} = -\sum_i x_i \ln x_i$$

$$\frac{d}{dp}\left(-\sum_{i=1}^{N} x_i^p\right)\Big|_{p=1} = H(x).$$

$$\Rightarrow \quad \frac{d}{dp} -(\|\rho\|_p)^p \Big|_{p=1} = S(\rho)$$

$$\Rightarrow \quad -S[\phi] = \frac{d}{dp}[\gamma_p(\phi)]^p\Big|_{p=1}$$

Additivity problem

$$\boxed{\quad\phi_1\quad}$$

$$\boxed{\quad\phi_2\quad}$$

$$C(\phi_1 \otimes \phi_2) \overset{?}{=} C(\phi_1) + C(\phi_2)$$

Multiplicativity of output p-Norm.

if $\nu_p(\phi_1 \otimes \phi_2) = \nu_p(\phi_1)\, \nu_p(\phi_2) \rightarrow$

$$-S[\phi_1 \otimes \phi_2] = \frac{d}{dp}\left[\nu_p(\phi_1)\, \nu_p(\phi_2)\right]^p \Big|_{p=1} =$$

$$= S(\phi_1)\,[\nu_p(\phi_2)]^p + [\nu_p(\phi_1)]^p\, S(\phi_2)\, \Big|_{p=1}$$

But $\lim\limits_{p\to 1} \nu_p(\phi) = \lim\limits_{p\to 1} \sup\limits_{\rho} \nu_p(\phi(\rho)) = \sup\limits_{\rho} \nu_1(\phi(\rho))$

$$= \sup_{\rho} \mathrm{Tr}(\phi(\rho)) = \sup_{\rho} \mathrm{Tr}(\rho) = 1.$$

So: $\forall \quad \nu_p(\phi_1 \otimes \phi_2) = \nu_p(\phi_1)\, \nu_p(\phi_2)$

$$S(\phi_1 \otimes \phi_2) = S(\phi_1) + S(\phi_2)$$