Quantum Algorithms, Part I

Note Title 10/13/2010

Simple Algorithms

- Deutch

$$f: \{0,1\} \longrightarrow \{0,1\}$$

question: Constant or Balanced?

Classical: 2 - queries

quantum: 1 - query.

- Deutch-Josza

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

question: Constant or Balanced?

Classical: $O(n)$ - queries

quantum: 1 - query.

- Bernstein – Vazirani

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

promise: $f(x) = x \cdot a$

question: find a.

Classical: n - queries

quantum: 1 - query.

Simon

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

promise: $f(x) = f(x \oplus a)$.

question: find a.

Classical: $O(2^n)$ queries

quantum: $O(n)$ queries.

- Factoring (shor)

$$f : \{0,1\}^n \longrightarrow \{0,1\}^n$$

promise: $f(x) = f(x+r)$

question: find r

Classical: $T = A\, e^{[(\ln n)^{1/3} (\ln(\ln n))^{2/3}]}$

quantum: $T = B\, (\ln n)^3$

Discrete Logarithms (shor-Kitaev)

$$f : \{0,1\}^n \longrightarrow \{0,1\}^n$$

promise: $a^s = b$

question: find $s = \log_a b$

Classical: exponential

quantum: polynomial

- Grover

$$f(x) = \begin{cases} 1 & x = w \\ 0 & x \neq w \end{cases}$$

question : Find where w is in the database.

classical : $O(N)$ queries.

quantum : $O(\sqrt{N})$ queries.

Fourier Transform

over $\mathbb{Z}_N$: $x \in \{0, 1, 2, \dots N-1\}$.

$$x_k \longrightarrow y_k := \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{\frac{2\pi i}{N} kl} x_l$$

QFT: U: $U|x_k\rangle := \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{\frac{2\pi i}{N} kl} |x_l\rangle$

- Phase Estimation

$\mathcal{U}$: unitary operator

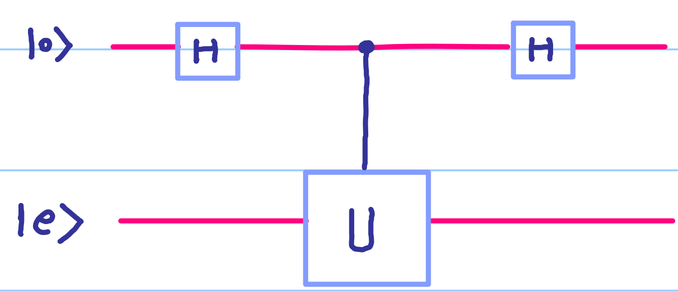
$|e\rangle$: eigenvector of $\mathcal{U}$.

$$\mathcal{U}|e\rangle = e^{i\phi}|e\rangle$$

question: find ϕ

The simplest example: $e^{i\phi} = \pm 1$.

$$\mathcal{U}|e\rangle = \pm|e\rangle.$$

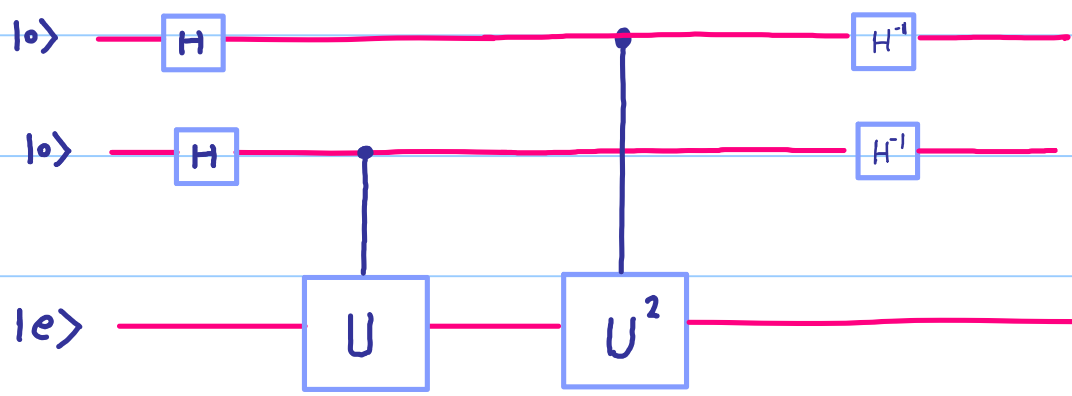


$$(|0\rangle + |1\rangle)|e\rangle \longrightarrow |0\rangle|e\rangle + |1\rangle\,\mathcal{U}|e\rangle =$$

$$(|0\rangle + \epsilon|1\rangle)|e\rangle \longrightarrow \begin{cases} |+\rangle & \epsilon = 1 \\ |-\rangle & \epsilon = -1 \end{cases}$$

The next simple example:

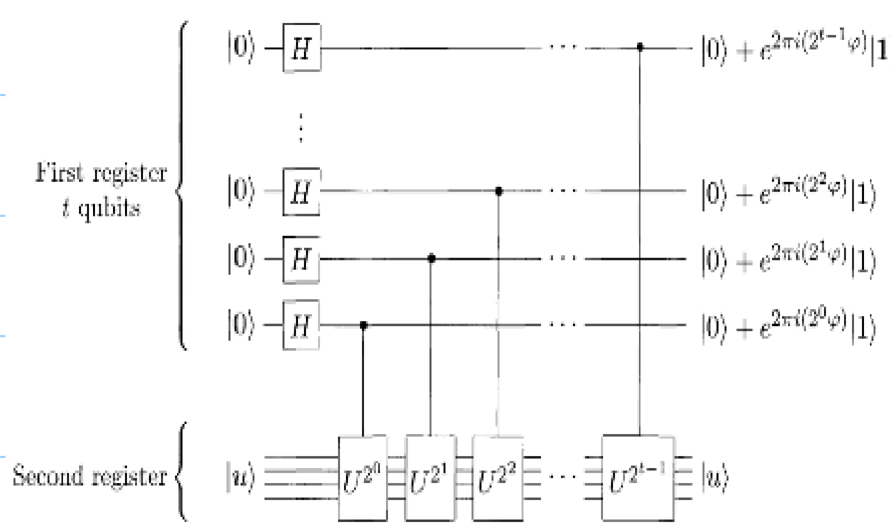
$$\mathcal{U}|e\rangle = e^{\frac{2i\pi}{4}k}|e\rangle \qquad e^{i\phi} = \left\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3}{2}\pi}\right\}$$



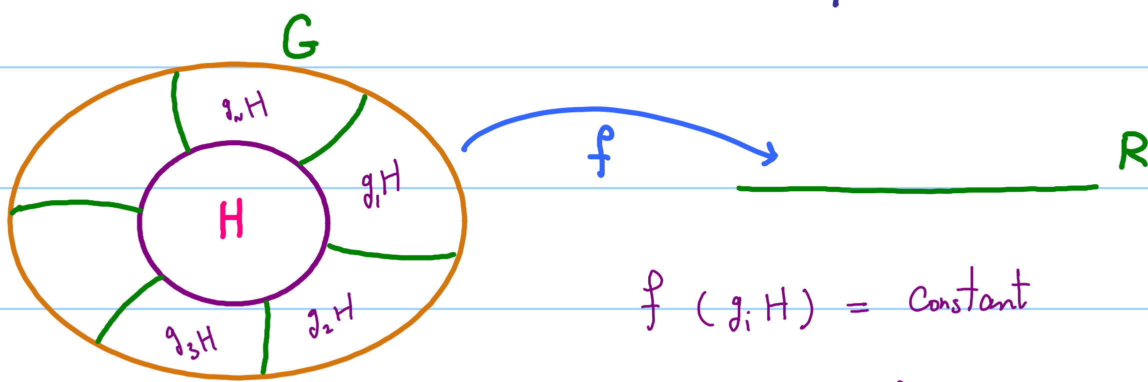
$$(|00\rangle+|01\rangle+|10\rangle+|11\rangle)|e\rangle = (|0\rangle+|1\rangle+|2\rangle+|3\rangle)|e\rangle$$

$$\to \left(|0\rangle + e^{\frac{2\pi i}{4}k}|1\rangle + e^{\frac{2\pi i}{4}2k}|2\rangle + e^{\frac{2\pi i}{4}3k}|3\rangle\right)|e\rangle$$

$$= \left(\sum_{y=0}^{3} e^{\frac{2\pi i}{4}ky}|y\rangle\right)|e\rangle \xrightarrow[\text{Fourier}]{\text{Inverse}} |k\rangle|e\rangle.$$



Hidden Subgroup problem



$f(g_i H) = \text{Constant}$

$f(g_1 H) \neq f(g_2 H)$

Problem: find a set of generators of H.

THE HIDDEN SUBGROUP PROBLEM - REVIEW AND OPEN PROBLEMS

CHRIS LOMONT, CYBERNET

ABSTRACT. An overview of quantum computing and in particular the Hidden Subgroup Problem are presented from a mathematical viewpoint. Detailed proofs are supplied for many important results from the literature, and notation is unified, making it easier to absorb the background necessary to begin research on the Hidden Subgroup Problem. Proofs are provided which give very concrete algorithms and bounds for the finite abelian case with little outside references, and future directions are provided for the nonabelian case. This summary is current as of October 2004.

Quantum Algorithms, Part II; Quantum Adiabatic Computation

Note Title 10/18/2010

Quantum Computation by Adiabatic Evolution

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SAT problem

- Variables: $x_1, x_2, x_3, \ldots, x_n$
- Clause: $C_1 = x_1 \vee \bar{x}_2 \vee x_3$

  $C_2 = \bar{x}_3 \vee x_7 \vee x_8$
- Formula: $F = C_1 \wedge C_2 \wedge C_3 \ldots \wedge C_m$
- $H(0)$ $|\psi(0)\rangle$
- $H(T)$ $|\psi(T)\rangle$

$$H(0) \longrightarrow H(T)$$

$$|\psi(0)\rangle \longrightarrow |\psi(T)\rangle$$

$$H(t) = H_{C_1}(t) + H_{C_2}(t) + \dots + H_{C_M}(t)$$

$\boxed{\text{Successfull only if} \quad T \sim poly(n)}$

- the adiabatic Theorem:

$$i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

$$H(t) = \tilde{H}\left(\frac{t}{T}\right) = \tilde{H}(s)$$

$$t : 0 \longrightarrow T \qquad s : 0 \longrightarrow 1$$

$$\tilde{H}(s)\,|l;s\rangle = E_l(s)\,|l;s\rangle$$

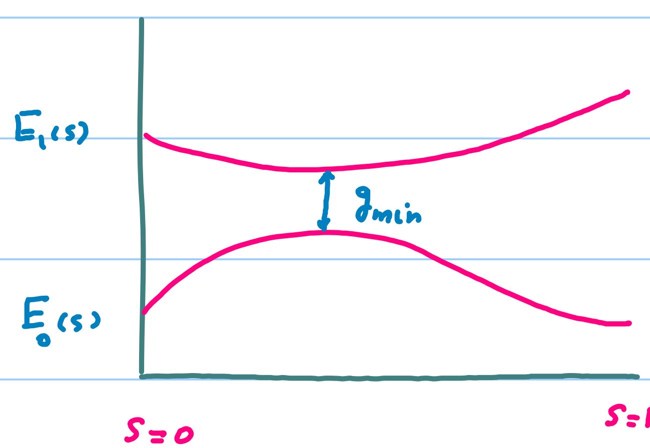
$$E_0(s) \leq E_1(s) \leq \cdots \leq E_{N-1}(s)$$

- Adiabatic theorem: $\quad |\psi(0)\rangle = |l=0, s=0\rangle$

if $\quad E_0(s) < E_1(s) \quad$ then

$$\lim_{T\to\infty} |\langle l=0, s=1|\psi(T)\rangle| = 1$$

More precise version:



$$g_{min} = \min_{0\leq s\leq 1}\left(E_1(s) - E_0(s)\right).$$

$$\mathcal{E} := \max_{0 \leq s \leq 1} \left| \langle l=1, s | \frac{dH}{ds} | l=0, s \rangle \right|$$

$$F := \left| \langle l=0, s=1 | \psi(s=1) \rangle \right|$$

$$\boxed{\text{if} \quad \frac{\mathcal{E}}{g_{min}^2} \leq \epsilon \quad \longrightarrow \quad F \geq \sqrt{1-\epsilon^2}}$$

- 3-SAT

- The Final or Problem Hamiltonian

$$h_c^p(z_1, z_2, z_3) = \begin{cases} 0 & \text{if } z_1, z_2, z_3 \text{ satisfies } C \\ 1 & \text{otherwise} \end{cases}$$

Example: $C = z_1 \vee z_2 \vee \bar{z}_3$

$$h_C^P = (1-z_1)(1-z_2)\, z_3 \qquad H^P = \sum_C h_C^P$$

- Notation $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\frac{1}{2}(1-\sigma_z)\,|0\rangle = 0 \qquad \frac{1}{2}(1-\sigma_z)\,|1\rangle = |1\rangle$$

$$\frac{1}{2}(1-\sigma_z)\,|z\rangle = z\,|z\rangle$$

$$\frac{1}{2}(1+\sigma_z)\,|z\rangle = (1-z)\,|z\rangle$$

if $C = z_1 \vee z_2 \vee \bar{z}_3$

$$\hat{h}_C^P = \frac{1}{2}(1+\hat{\sigma}_{z_1})\,\frac{1}{2}(1+\hat{\sigma}_{z_2})\,\frac{1}{2}(1-\hat{\sigma}_{z_3})$$

or

$$\hat{h}_C^P = \frac{1}{2}(1+\hat{z}_1)\,\frac{1}{2}(1+\hat{z}_2)\,\frac{1}{2}(1-\hat{z}_3).$$

$$\hat{H}^P = \sum_c \hat{h}_c^P .$$

Initial Hamiltonian $H^B = \sum_c h_c^B$

$h_C^B = ?$ if $z_1, z_2, \dots z_k \in C$

- $h_c^B = \sum_{i=1}^{k} \frac{1}{2}(1 - \hat{X}_i)$
- No Interaction

$|x=0\rangle \equiv |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $|x=1\rangle \equiv |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

The ground state of H^B $|\psi(0)\rangle = |x_1=0\rangle|x_2=0\rangle \dots |x_N=0\rangle$.

$= |\swarrow \swarrow \swarrow \swarrow \swarrow \dots \swarrow\rangle$

Note: $|\psi(0)\rangle = \frac{1}{2^{n/2}} \sum_{z_1 \dots z_N} |z_1\rangle|z_2\rangle \dots |z_N\rangle$

Adiabatic Evolution: $H(t) = \left(1-\frac{t}{T}\right) H_B + \frac{t}{T} H_P$

- $\tilde{H}(s) = (1-s) H_B + s H_P$

- $H_c(t) = \left(1-\frac{t}{T}\right) H_c^B + \frac{t}{T} H_c^P$

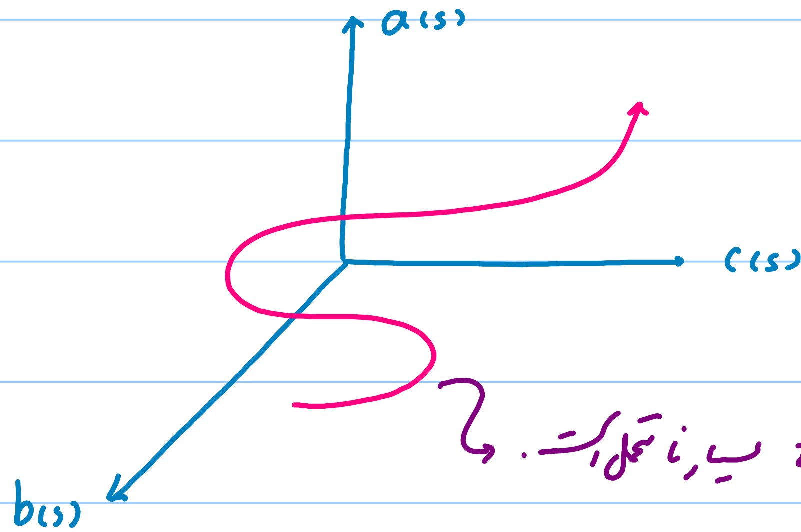
- $\tilde{H}_c(s) = (1-s) H_c^B + s H_c^P$

- $\tilde{H}(s) = \sum_c \tilde{H}_c(s)$

- The important point $\longrightarrow$ $\boxed{g_{min} \neq 0 \quad \text{مهمه}}$

$$\tilde{H}(s) = \begin{bmatrix} a(s) & b(s)+i\,c(s) \\ b(s)-i\,c(s) & d(s) \end{bmatrix}$$

$E_0(s) = E_1(s)$ only if $a(s) = d(s)$, $b(s) = c(s) = 0$

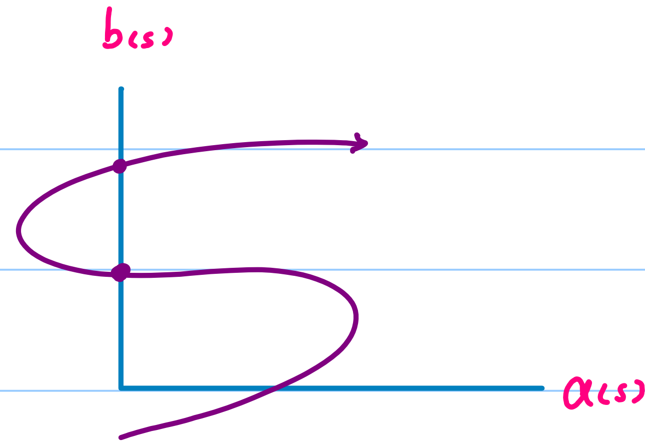


قطع کردن محور z بسیار نامحتمل است.

But if $\tilde{H}$ has a symmetry, $[\tilde{H}(s), X] = 0$

$\rightarrow \quad X\tilde{H}(s)X = \tilde{H}(s) \quad \rightarrow \quad a(s) = d(s), \; c(s) = 0$

$$\rightarrow \quad \tilde{H}(s) = \begin{bmatrix} a(s) & b(s) \\ b(s) & a(s) \end{bmatrix}$$



حالا قطع کردن محور z محتمل است.

کافی نیست که $g_{min} \neq 0$ باشد، g_{min} زیاد خیلی باشد که

$$T \sim O(n^p) \quad \text{NOT} \quad T \sim O(a^n)$$

T فقط به g_{min} بستگی دارد نه به ε.

$$\tilde{H}(s) = (1-s)H^B + sH^P$$

- $$\frac{d}{ds}\tilde{H}(s) = H^P - H^B$$

$$\mathcal{E} = \max_{0 \leq s \leq 1} |\langle l=1, s | H^P - H^B | l=0, s \rangle| \leq \lambda_{max}(H^P - H^B)$$

$$H^P = \sum_c h_c^P$$

$$\hat{h}_c^P = \frac{1}{2}(1 + \hat{z}_1)\frac{1}{2}(1 + \hat{z}_2)\frac{1}{2}(1 - \hat{z}_3).$$

$$\text{Spectrum}(\hat{h}_c^P) = \begin{cases} 0 \\ 1 \end{cases} \longrightarrow \text{Spectru}\,(H^P) \in \{0, 1, 2, \dots M\}$$

$$h_c^B = \sum_{i=1}^{K} \frac{1}{2}(1 - \hat{x}_i)$$

$$\text{Spectrum}(\hat{h}_c^B) \in \{-K_c, \dots, K_c\}$$

$$\text{Spectrum}(\hat{H}^B) \in \{-K_c M, \dots K_c M\}$$

$\lambda_{max}(H^P - H^B) \sim O(n^p)$. $\leftarrow$ $M \sim O(n^q)$?

$\rightarrow$ $T_{max} \geq \dfrac{\mathcal{E}}{g_{min}^2}$ $\rightarrow$ only g_{min} is important.

of bits. Thus $\mathcal{E}$ grows at most like a polynomial in n and the distinction between polynomial and exponential running time depends entirely on $g_{\min}$. ?

We make no claims about the size of $g_{\min}$ for any problems other than the examples given in Section 4. We will give three examples where $g_{\min}$ is of order $1/n^p$ so the evolution time T is polynomial in n. Each of these problems has a regular structure that made calculating $g_{\min}$ possible. However, the regularity of these problems also makes them classically computationally simple. The question of whether there are computationally difficult problems that could be solved by quantum adiabatic evolution we must leave to future investigation.

- Examples:

1-Qubit Example:

$H^B = \frac{1}{2}(1 - X) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $\qquad C = Z$

$H^P = \frac{1}{2}(1 + Z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\tilde{H}(s) = \begin{bmatrix} \frac{1+s}{2} & \frac{-1+s}{2} \\ \frac{-1+s}{2} & \frac{1-s}{2} \end{bmatrix}$$

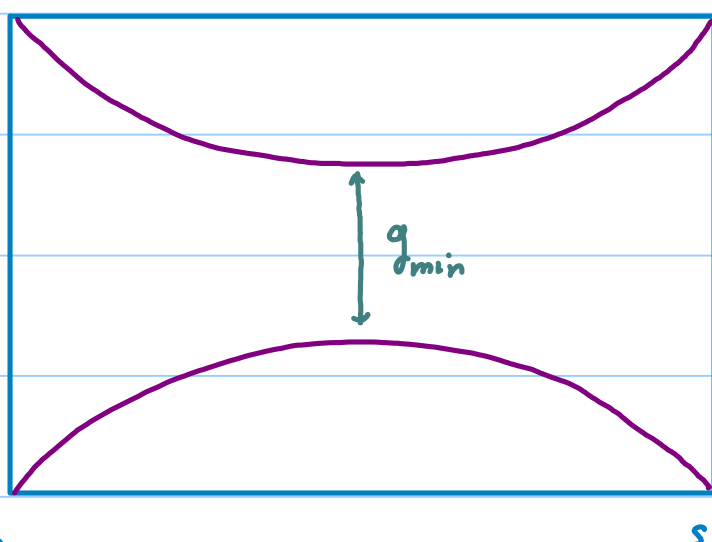
$$(1-s-\lambda)(1+s-\lambda) - (1-s)^2 = 0$$

$$\lambda^2 - 2\lambda + 1 - s^2 - (1-s)^2 = 0$$

$$\lambda^2 - 2\lambda + 2s(1-s) = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-2s(1-s)}}{2}$$

$$\lambda = \frac{1 \pm \sqrt{(1-s)^2 + s^2}}{2}$$



$$\Delta E = \sqrt{(1-s)^2 + s^2}$$

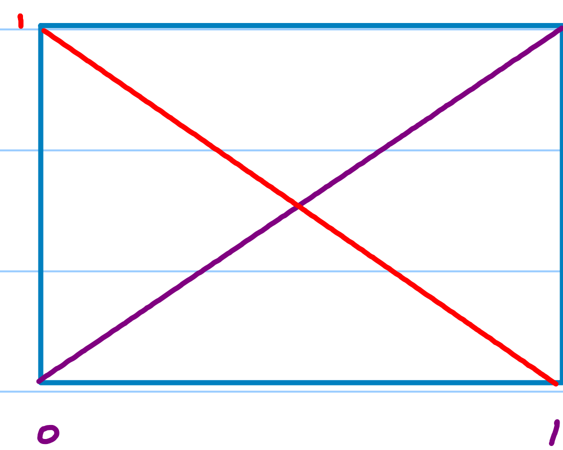
$$g_{min} = \frac{1}{\sqrt{2}}$$

- An important question:

why not start from $H^B = \frac{1}{2}(1-Z)$?

$$\rightarrow \quad \tilde{H}(s) = (1-s)\,H^B + s\,H^P = (1-s)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + s\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & 1-s \end{bmatrix}$$



- The levels cross due to symmetry. $[\tilde{H}(s), Z] = 0$

Adiabatic Evolution : if $|\psi(0)\rangle = |0\rangle$

$|\psi(T)\rangle = |0\rangle$ Not $|1\rangle$!

- A small perturbation

$$H(s) = \begin{bmatrix} s & \epsilon(1-s) \\ \epsilon(1-s) & 1-s \end{bmatrix}$$

$$(s-\lambda)(1-s-\lambda) - \epsilon^2(1-s)^2 = 0$$

$$\lambda^2 - \lambda + s(1-s) - \epsilon^2(1-s)^2 = 0$$

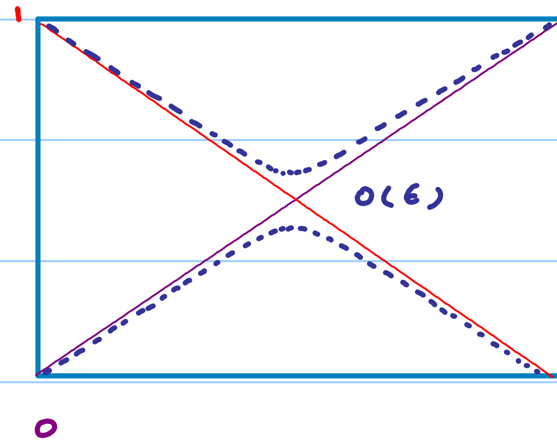
$$\lambda = \frac{1}{2}\left\{1 \pm \sqrt{1-4s(1-s)+\epsilon^2(1-s)^2}\right\}$$

$$\lambda = \frac{1}{2}\left\{1 \pm \sqrt{(1-2s)^2+\epsilon^2(1-s)^2}\right\}$$

$$\rightarrow \quad \Delta E = \sqrt{(1-2s)^2+\epsilon^2(1-s)^2}$$

$$g_{min} = O(\epsilon)$$

Level Repulsion



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two qubits: Example 1) two bit disagree

$$C = x_1 \oplus x_2 = \quad C = x_1 + x_2 - 2x_1 x_2 .$$

$$H^{B} = \tfrac{1}{2}(1 - X_1) + \tfrac{1}{2}(1 - X_2)$$

x_1	x_2	$x_1 \oplus x_2$
0	0	0
0	1	1
1	0	1
1	1	0

$$H^{P} = I - |01\rangle\langle 01| - |10\rangle\langle 10|$$

$$C = x_1 \oplus x_2 = (\bar{x}_1 \wedge x_2) \vee (x_1 \wedge \bar{x}_2) \qquad \text{or}$$

$$C = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2) = C_1 \wedge C_2$$

- $h_{C_1} = \frac{1}{2}(1+\hat{Z}_1)\,\frac{1}{2}(1+\hat{Z}_2)$

  $h_{C_2} = \frac{1}{2}(1-\hat{Z}_1)\,\frac{1}{2}(1-\hat{Z}_2)$

- $H_C = h_{C_1} + h_{C_2} = \frac{1}{4}(1+\hat{Z}_1)(1+\hat{Z}_2) + \frac{1}{4}(1-\hat{Z}_1)(1-\hat{Z}_2)$

  $= \frac{1}{2}(1+\hat{Z}_1\hat{Z}_2)$

$$H_C = \frac{1}{2}\left(I + \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \right) = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}$$

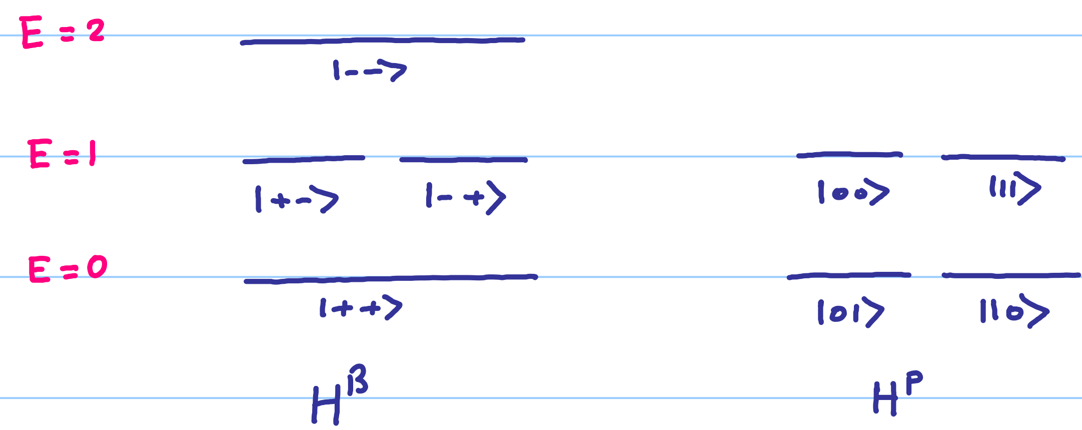
$H_C = I - |01\rangle\langle 01| - |10\rangle\langle 10|.$

$$\tilde{H}(s) = (1-s)\,H^B + s\,H^P$$

$$= \frac{1}{2}(1-s)\left\{ 1 - X_1 + 1 - X_2 \right\} + \frac{s}{2}\left\{ 1 + Z_1 Z_2 \right\}$$

$[\tilde{H}, P] = 0$ $\quad$ P = permutation operator

$P|\alpha\beta\rangle = |\beta\alpha\rangle$



$|l=0, s\rangle$ = Symmetric $\quad |++\rangle \longrightarrow \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$s=0 \qquad\qquad s=1$

$|l=1, s\rangle$ = Antisymmetric,

$$\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$|l=2, s\rangle$ = symmetric

$$\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- Eigenstates of $\tilde{H}(s)$ have definite parity.

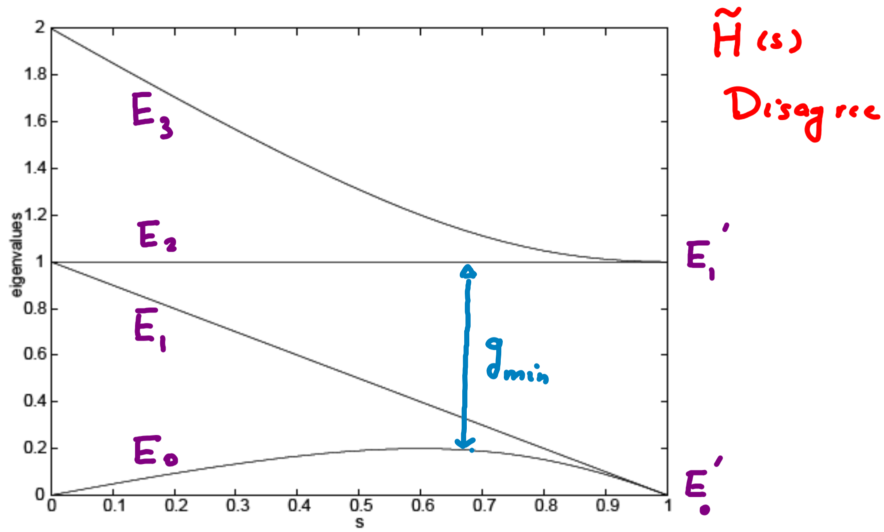


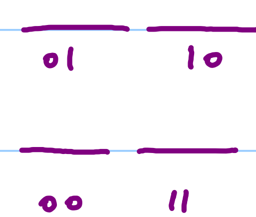
Figure 4: *The four eigenvalues of $\tilde{H}(s)$ associated with "2-bit disagree". The same levels are associated with "2-bit agree".*

No Transition is possible from $|l=0,s\rangle \to |l=1,s\rangle$.

Example 2) Two bit agree.

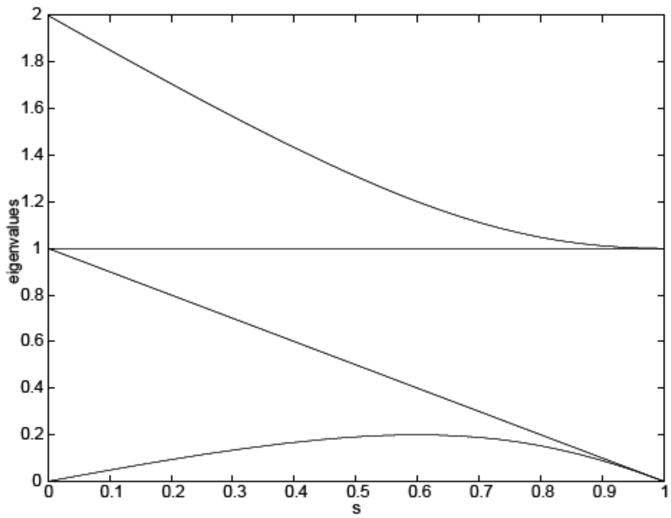
$C = x_1 \oplus \bar{x}_2$ C is satisfied by 00, 11.

$H^P = \frac{1}{2}(1 - Z_1 Z_2)$



$$(1 \otimes X_2)\, H^{P}_{\text{Disagree}}\, (1 \otimes X_2) = H^{P}_{\text{agree}}$$

$$(1 \otimes X_2)\, \tilde{H}_{\text{Dis}}(s)\, (1 \otimes X_2) = \tilde{H}_{\text{agree}}(s)$$



$\tilde{H}_{\text{agree}}(s)$

Figure 4: *The four eigenvalues of $\widetilde{H}(s)$ associated with "2-bit disagree". The same levels are associated with "2-bit agree".*

Example 3) 2-bit imply. C is satisfied by 00, 01, 11.

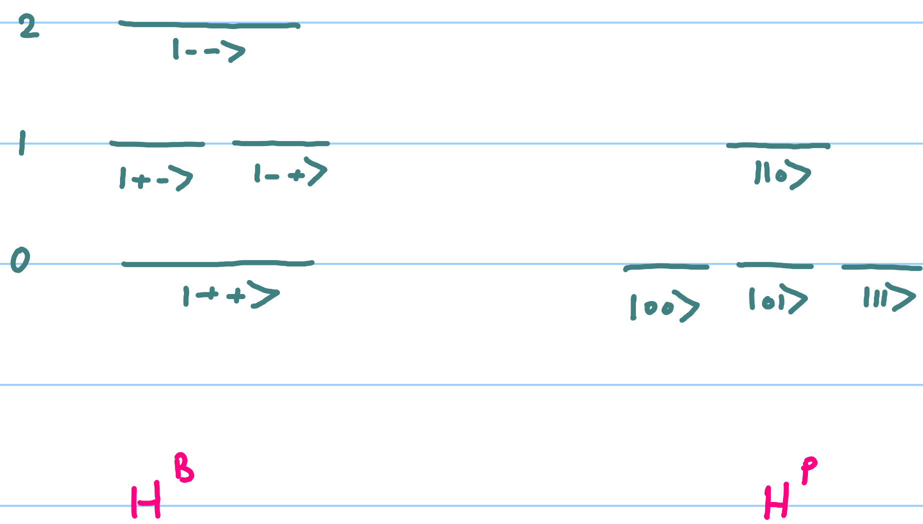
x_1	x_2	C
0	0	1
0	1	1
1	0	0
1	1	1

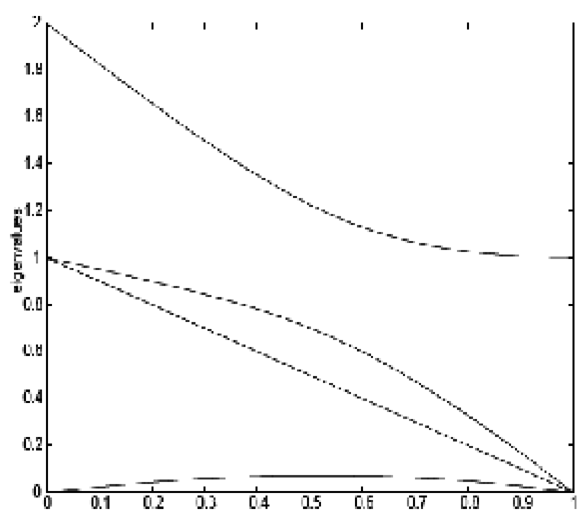
$$C = \bar{x}_1 \vee x_2$$

$$h^{P}_{C} = \frac{1}{2}(1 + Z_1)\frac{1}{2}(1 - Z_2)$$

$$H^P = \frac{1}{4}\left(1 + Z_1 - Z_2 - Z_1 Z_2 \right)$$

$$H^B = \frac{1}{2}\left(1 - X_1 \right) + \frac{1}{2}\left(1 - X_2 \right)$$





$\leftarrow H_{imply}$

Figure 5: The four eigenvalues of $\tilde{H}(s)$ associated with the 2-bit imply clause.

3-qubits $\qquad C = (\text{Imply})_{12} \wedge (\text{Disagree})_{13} \wedge (\text{Agree})_{23}$

$$C = (\bar{x}_1 \vee x_2) \wedge (x_1 \oplus x_3) \wedge (x_2 \oplus \bar{x}_3)$$

x_1	x_2	x_3	$(\text{IMP})_{12}$	$(\text{Dis})_{13}$	$(\text{Ag})_{23}$	C
0	0	0	1	0	1	0
0	0	1	1	1	0	0
0	1	0	1	0	0	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	0	0	0
1	1	0	1	1	0	0
1	1	1	1	0	1	0

$$H^P = (H^{\text{IMP}})_{12} + (H^{\text{Dis}})_{13} + (H^{\text{Agree}})_{23}$$

$$H^P = \frac{1}{4}\left(1 + Z_1 - Z_2 - Z_1 Z_2\right) + \frac{1}{2}\left(1 + Z_1 Z_3\right) + \frac{1}{2}\left(1 - Z_2 Z_3\right).$$

$$H^B = \frac{1}{2}(h_1^B + h_2^B) + \frac{1}{2}(h_1^B + h_3^B) + \frac{1}{2}(h_2^B + h_3^B)$$

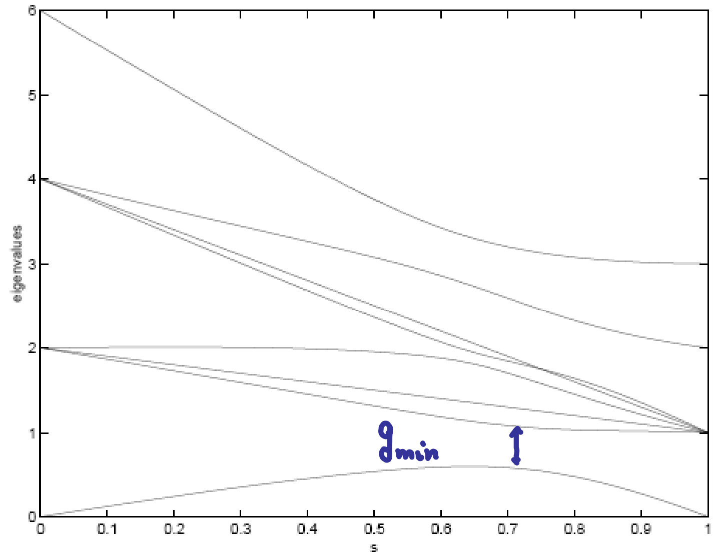


Figure 6: *The eight levels of $\widetilde{H}(s)$ for the 3-bit problem with H_P and H_B given by (3.5).*

n-qubit problems

Example 1) 2-Sat on a Ring: Agree (A) & Disagree (D)

- $F = A \wedge D \wedge D \wedge D \wedge A \wedge A \wedge D$
- there are two satisfying assignments:

1 1 0 1 0 0 0 1 or 0 0 1 0 1 1 1 0

- the Hamiltonian $\quad F = A \wedge D \wedge D \wedge D \wedge A \wedge A \wedge D$

$$H = \frac{1}{2}(1 - Z_1 Z_2) + \frac{1}{2}(1 + Z_2 Z_3) + \frac{1}{2}(1 + Z_3 Z_4) + \frac{1}{2}(1 + Z_4 Z_5)$$
$$+ \frac{1}{2}(1 - Z_5 Z_6) + \frac{1}{2}(1 - Z_6 Z_7) + \frac{1}{2}(1 + Z_7 Z_1).$$

- تمام همیلتونی‌های از این نوع، طیف یکسان دارند.

- $$|w\rangle = |1\,1\,0\,1\,0\,0\,0\,1\rangle$$

$$\mathcal{U} = I \otimes I \otimes X \otimes I \otimes X \otimes X \otimes X \otimes I$$

under $\mathcal{U}$: $\quad (Z_3, Z_5, Z_6, Z_7) \longrightarrow (-Z_3, -Z_5, -Z_6, -Z_7)$

$$\mathcal{U} H \mathcal{U}^{\dagger} = \sum_{k=1}^{n} \frac{1}{2}(1 - Z_k Z_{k+1}) = \sum_k h^{\text{agree}}_{k,k+1} = H^{P}_{ag}$$

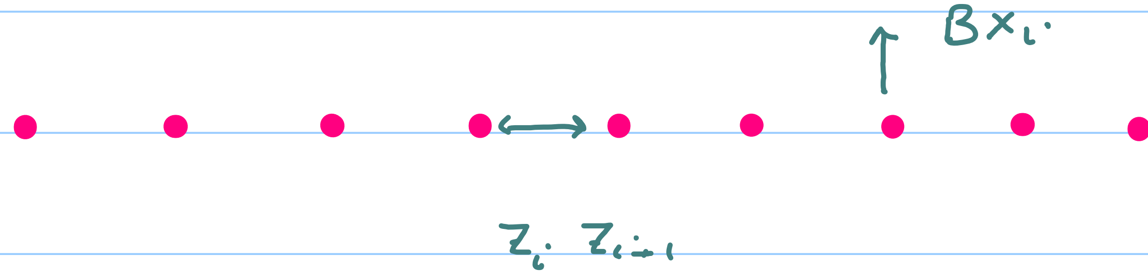
So all the spectrums are like $\sum_k h_{k,k+1}^{\text{agree}}$

and we need only consider

$$\boxed{H = (1-s)\sum_{k=1}^{N}(1 - X_k) + s\sum_{k=1}^{N}\frac{1}{2}(1 - Z_k Z_{k+1}).}$$

Ising Model in Transverse Field. (ITF)

$$H = \text{cte} - (1-s)\sum_k X_k - \frac{s}{2}\sum_k Z_k Z_{k+1}$$



- Free-Fermions. Jordan-Wigner Transformations.

- The XY Model

$$H = \sum_k X_k X_{k+1} + Y_k Y_{k+1} + B \sum_k Z_k .$$

$$\sigma_k^+ = \frac{1}{2}(X_k + i Y_k) , \qquad \sigma_k^- = \frac{1}{2}(X_k - i Y_k)$$

$$\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ = \frac{1}{2}(X_k X_{k+1} + Y_k Y_{k+1}).$$

$$\sigma_k^+ \sigma_k^- = \frac{1}{2}(1 + Z_k).$$

$$H \equiv \sum_k \sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ + B \sum_k \sigma_k^+ \sigma_k^-$$

Jordan-Wigner Transformation:

$$b_k = Z_1 Z_2 \cdots Z_{k-1} \sigma_k^+$$

$$b_k^\dagger = Z_1 Z_2 \cdots Z_{k-1} \sigma_k^-$$

$$b_k^2 = 0, \quad b_k^{\dagger 2} = 0 \qquad \{b_k, b_l\} = 0 \quad \{b_k^\dagger, b_l^\dagger\} = 0$$

$$b_k b_l = (Z_1 \cdots Z_{k-1} \sigma_k^+)(Z_1 \cdots Z_k \cdots Z_{l-1} \sigma_l^+) = -b_l b_k .$$

$$\{b_k, b_l^\dagger\} = \delta_{kl} \mathbb{1}$$

$$b_k b_{k+1}^\dagger = (Z_1 Z_2 \cdots Z_{k-1}) \sigma_k^+ (Z_1 Z_2 \cdots Z_k) \sigma_{k+1}^-$$

$$= \sigma_k^+ z_k \sigma_{k+1}^- = -\sigma_k^+ \sigma_{k+1}^-$$

$$b_k^\dagger b_{k+1} = (z_1 z_2 \dots z_{k-1}) \sigma_k^- (z_1 z_2 \dots z_k) \sigma_{k+1}^+$$

$$= \sigma_k^- z_k \sigma_{k+1}^+ = \sigma_k^- \sigma_{k+1}^+$$

$$b_k^\dagger b_k = (z_1 z_2 \dots z_{k-1}) \sigma_k^- (z_1 \dots z_{k-1}) \sigma_k^+$$

$$= \sigma_k^- \sigma_k^+ = \frac{1}{2}(1 - z_k).$$

$$\rightarrow \quad H = \sum_k x_k x_{k+1} + y_k y_{k+1} + B \sum_k z_k$$

$$H \equiv \sum_k \sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ + B \sum_k \sigma_k^+ \sigma_k^-$$

$$H = \sum_k b_k^\dagger b_{k+1} - b_k b_{k+1}^\dagger - B \sum_k b_k^\dagger b_k$$

$$H = (b_1^\dagger \; b_2^\dagger \; \dots \; b_N^\dagger) \begin{bmatrix} -B & 1 & & & & \\ 1 & -B & 1 & & & \\ & 1 & -B & 1 & & \\ & & 1 & \ddots & & \\ & & & & -B & 1 \\ & & & & 1 & -B \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$

$$H = \sum_m w_m c_m^\dagger c_m$$

$$c_k = \sum_k S_{kl} b_l .$$

$\downarrow$ unitary.

$w_m :=$ eigenvalues of $\mathcal{U}$.

c_k, $c_k^\dagger$ are also fermions, since S is unitary.

$$u_{k-1} + u_{k+1} - B u_k = w u_k \qquad u_k = A e^{ik\alpha}$$

$$e^{-i\alpha} + e^{i\alpha} - B = w \rightarrow w = B + 2\cos\alpha$$

$$e^{iN\alpha} = 1 \longrightarrow \alpha = \frac{2\pi}{N} m \rightarrow w_m = B + 2\cos\frac{2\pi}{N} m .$$

Back to our original problem.

$$\tilde{H}(s) = (1-s)\sum_{i=1}^{N}(1-X_i) + \frac{1}{2}s\sum_{i=1}^{N}(1-Z_i Z_{i+1}).$$

$$b_j = X_1 X_2 \ldots\ldots X_{j-1}\,\sigma_j^+ \qquad \sigma_j^+ = \frac{1}{2}(Z_j - iY_j)$$

$$b_j^\dagger = X_1 X_2 \ldots\ldots X_{j-1}\,\sigma_j^- \qquad \sigma_j^- = \frac{1}{2}(Z_j + iY_j)$$

$$\tilde{H}(s) = \sum_{j=1}^{n}\left\{2(1-s)b_j^\dagger b_j + \frac{s}{2}\left(1-(b_j^\dagger - b_j)(b_{j+1}^\dagger + b_{j+1})\right)\right\}.$$

$$\beta_p = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} e^{i\pi pj/n} b_j \quad \text{for } p = \pm 1, \pm 3, \ldots, \pm(n-1)$$

to

$$b_j = \frac{1}{\sqrt{n}}\sum_{p=\pm 1,\pm 3,\ldots,\pm(n-1)} e^{-i\pi pj/n}\beta_p$$

$$\tilde{H}(s) = \sum_{p=1,3,\ldots,(n-1)} A_p(s)$$

$$A_p(s) = 2(1-s)\left[\beta_p^\dagger\beta_p + \beta_{-p}^\dagger\beta_{-p}\right] + s\left\{1 - \cos\frac{\pi p}{n}\left[\beta_p^\dagger\beta_p - \beta_{-p}\beta_{-p}^\dagger\right] + i\sin\frac{\pi p}{n}\left[\beta_{-p}^\dagger\beta_p^\dagger - \beta_p\beta_{-p}\right]\right\}.$$

$$A_p(s) = \begin{bmatrix} s + s\cos \pi p/n & is(\sin \pi p/n) \\ -is(\sin \pi p/n) & 4 - 3s - s\cos \pi p/n \end{bmatrix} .$$

igenvalues of $A_p(s)$ are

$$E_p^{\pm}(s) = 2 - s \pm \left\{(2-3s)^2 + 4s(1-s)(1-\cos \pi p/n)\right\}^{\frac{1}{2}} .$$

Ground State Energy. $E_0 = \sum_p E_p^-(s)$.

The next level: $E_1 = E_1^+(s) + \sum_{p>1} E_p^-(s)$

$$g_{min} = \left(E_1^+(s) - E_1^-(s)\right)_{s=\frac{2}{3}} \simeq \frac{4\pi}{3n}.$$

$T \geq c n^3$ For any 2-SAT of Agree-Disagree.

The Grover Algorithm

$$H^P|z\rangle = \begin{cases} |z\rangle & \text{if } z \neq \omega \\ 0 & \text{if } z = \omega \end{cases}$$

$$H^{p} = (1 - |w\rangle\langle w|)$$

$$\tilde{H} = (1-s)\sum_{k=1}^{N}(1 - X_k) + s\,(1 - |w\rangle\langle w|)$$

without loss of generality $|w\rangle \to |000\cdots 0\rangle$

$$\tilde{H} = (1-s)\sum_{k=1}^{N}(1 - X_k) + s - s(|0\rangle\langle 0|)^{\otimes N}$$

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$$g_{min} = 2 \cdot 2^{-n/2}$$

$$\rightarrow \quad T \sim 2^{n} \sim N$$

No Advantage.

Quantum Search by Local Adiabatic Evolution

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The adiabatic theorem has been recently used to design quantum algorithms of a new kind, where the quantum computer evolves slowly enough so that it remains near its instantaneous ground state which tends to the solution [1]. We apply this time-dependent Hamiltonian approach to the Grover's problem, i. e., searching a marked item in an unstructured database. We find that, by adjusting the evolution rate of the Hamiltonian so as to keep the evolution adiabatic on each infinitesimal time interval, the total running time is of order $\sqrt{N}$, where N is the number of items in the database. We thus recover the advantage of Grover's standard algorithm as compared to a classical search, scaling as N. This is in contrast with the constant-rate adiabatic approach developed in [1], where the requirement of adiabaticity is expressed only globally, resulting in a time of order N.

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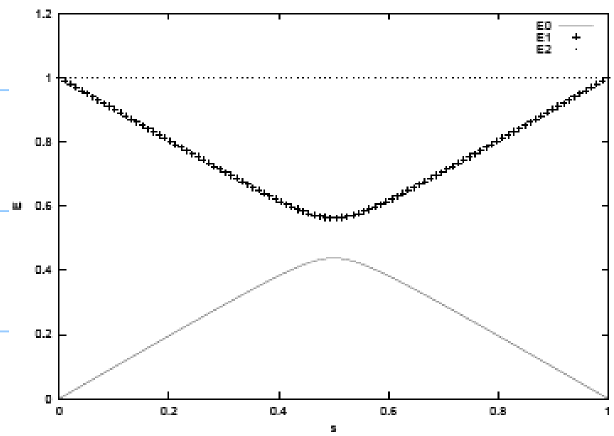

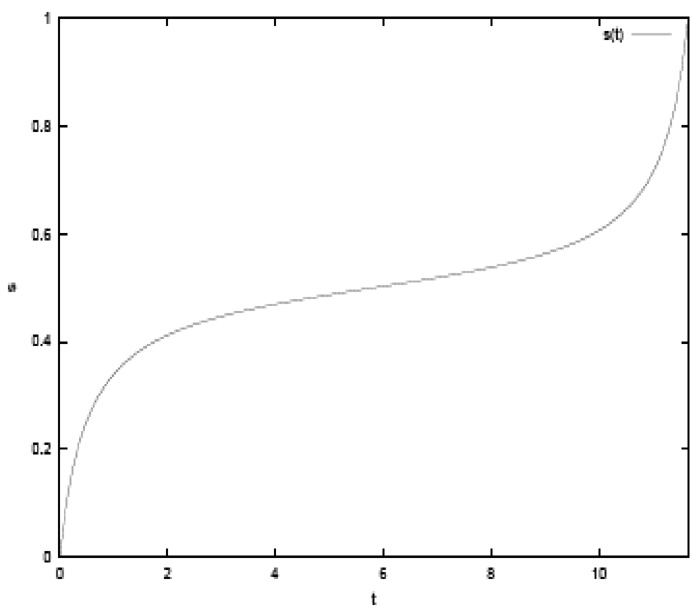

A Quantum Adiabatic Evolution Algorithm Applied to Random Instances of an NP-Complete Problem

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