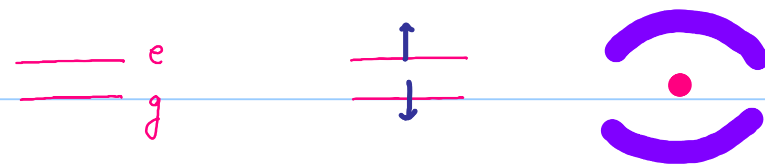
Continuous Variables_ I

Note Title 9/25/2010

qubit	qutrit	qudit	CV
$d=2$	$d=3$	d	∞

qubit:



$\underset{H}{\uparrow}\ ,\ \underset{V}{\rightarrow}$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma_1 \qquad \sigma_2 \qquad \sigma_3 \qquad \sigma_0$

$$\sigma_{\mu\nu} = \sum_j (-1)^{j\nu} |j+\mu\rangle\langle j|$$

$\sigma_{00} = I$, $\sigma_{01} = Z$ $\quad \sigma_{10} = X$ $\quad \sigma_{11} = Y$

$$\mathcal{U} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$Z : \ |0\rangle, |1\rangle \qquad\qquad X : \ |+\rangle, |-\rangle$$

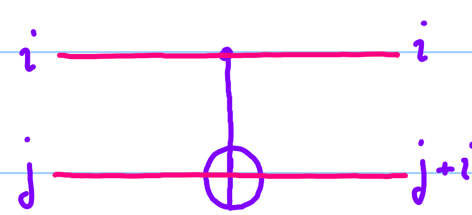
$$|+\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right)$$

$$|-\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right)$$

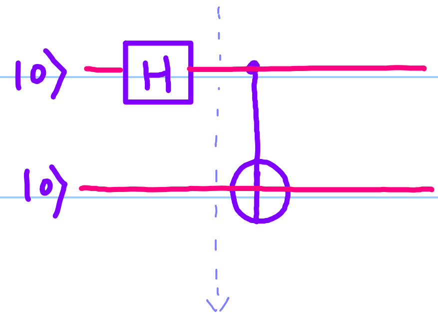
$$H = |+\rangle\langle 0| + |-\rangle\langle 1| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad H_{ij} = \frac{1}{\sqrt{2}} (-1)^{ij}$$

$$H Z H^{\dagger} = X \longrightarrow H^{\dagger} X H = Z$$

$$\text{CNOT} \, |i, j\rangle = |i, \, i+j\rangle$$

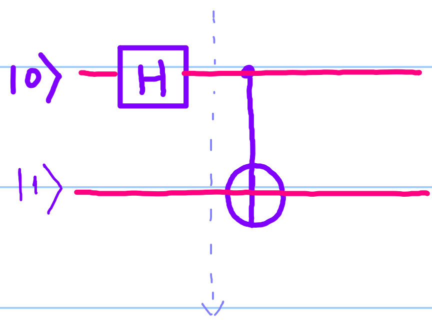


$$|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle \pm |11\rangle \right) \qquad |\phi_{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle \pm |10\rangle \right)$$



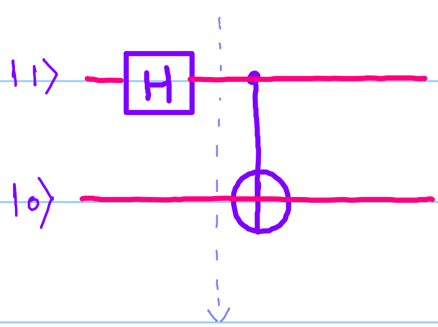
$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi_+\rangle$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$



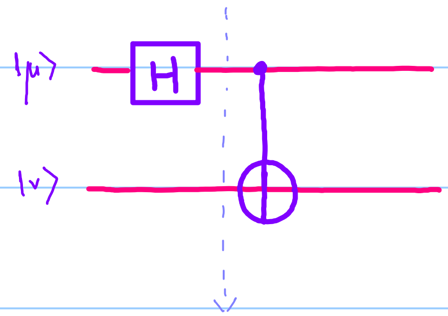
$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$



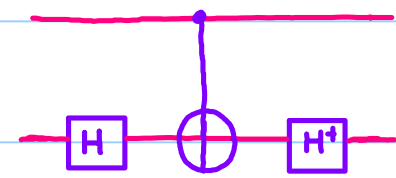
$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$$



$$|\psi_{\mu\nu}\rangle = \frac{1}{\sqrt{2}} \sum_{j} (-1)^{j\nu} |j+\mu, j\rangle$$

$$CZ = (I \otimes H^{\dagger})\, CX\, (I \otimes H)$$



Many other things are simple or Rather Simple for qubits :

• Seperability of bi-partite systems

1) For pure states : $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

if $ad - bc = 0 \longleftrightarrow \psi = \phi \otimes \chi$

2) for mixed states

Given a density Matrix $\rho = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ \rho_{41} & \cdot & \cdot & \rho_{44} \end{bmatrix}$

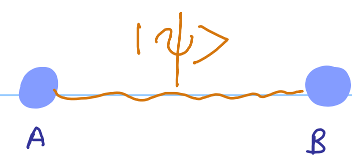
if $\rho^{T_B} \geq 0 \longleftrightarrow \rho$ is seperable.

Peres PPT criteria. Peres PRL (1998)

Positive partial Transpose. (PPT).

• Entanglement Measure for bi-partite systems.

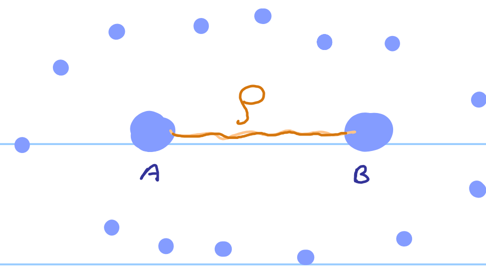
1) for pure states



$$\mathcal{E} = S(\rho_A) = -tr(\rho_A \ln \rho_A)$$

$$\rho_A = tr_B(|\psi\rangle\langle\psi|).$$

2) for Mixed states



$$\mathcal{E} = \mathcal{E}(\rho_D)$$

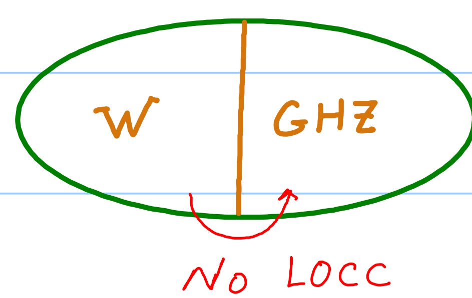
$$\rho_D = \frac{1}{2}\begin{pmatrix} 1+\sqrt{1-C^2} & \\ & 1-\sqrt{1-C^2} \end{pmatrix}$$

$$C = \max\left(0,\ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \right)$$

$$\lambda_i\text{'s} = +\sqrt{\text{eigenvalues of}}\quad \rho\tilde{\rho}\,,$$

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\,\rho^*\,(\sigma_y \otimes \sigma_y)$$

- Three-partite Entanglement



$$|W\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$$

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

Choi Matrix: $\quad Z_\phi := \sum_{ij} \phi(E_{ij}) \otimes E_{ij} \quad \in B \otimes A$

1) if ϕ trace preserving $\equiv$ $tr(\phi(\rho)) = tr(\rho) \longrightarrow$

$$tr(\phi(E_{ij})) = tr(E_{ij}) = \delta_{ij}$$

$$\rightarrow \quad Tr_1(Z_\phi) = I$$

if $Tr(Z_\phi) = I \longrightarrow Tr\, \phi(E_{ij}) = \delta_{ij}$

$$\rightarrow \quad \mathrm{Tr}(\phi(\rho)) = \mathrm{Tr}\Big(\phi\big(\sum_{ij} \rho_{ij} E_{ij}\big)\Big) =$$

$$= \mathrm{Tr}\Big(\sum_{ij} \rho_{ij}\, \phi(E_{ij})\Big) = \sum_{ij} \rho_{ij}\, \mathrm{Tr}(\phi(E_{ij}))$$

$$= \sum_{ij} \rho_{ij}\, \delta_{ij} = \mathrm{Tr}(\rho). \longrightarrow \phi \text{ is TP.}$$

So $\quad \phi$ is TP $\longleftrightarrow \mathrm{Tr}_1 \mathcal{Z}(\phi) = \mathbb{1}$

2) $\quad \phi$ is unital $\equiv \phi(I) = I$

$$\rightarrow \quad \mathrm{Tr}_2(\mathcal{Z}_\phi) = \sum_{ij} \phi(E_{ij})\, \delta_{ij}$$

$$= \phi(I) = I.$$

Conversely:

if $Tr_2(Z_\phi) = I \rightarrow Tr_2\left(\sum_{ij} \phi(E_{ij}) \otimes E_{ij}\right) = \mathbb{1}$

$\rightarrow \sum_{ij} \phi(E_{ij}) \otimes \delta_{ij} = \mathbb{1} \rightarrow \phi\left(\sum_i E_{ii}\right) = \mathbb{1} \rightarrow$

$\phi(\mathbb{1}) = \mathbb{1} \rightarrow$ ϕ is unital

So: unital $\phi \longleftrightarrow Tr_2(Z_\phi) = I$.

- Hermiticity preserving $\equiv$ if $\rho = \rho^\dagger \rightarrow \phi(\rho) = [\phi(\rho)]^\dagger$

you can show that for such a ϕ:

$$[\phi(E_{ij})]^\dagger = \phi(E_{ji})$$

Then $(Z_\phi)^\dagger = \sum_{ij} [\phi(E_{ij})]^\dagger \otimes E_{ij}^\dagger =$

$$= \sum_{ij} \phi(E_{ji}) \otimes E_{ji} = Z_\phi .$$

So HP $\Phi \longleftrightarrow$ Hermitian Z_ϕ

Finally

Φ is Completely positive $\longleftrightarrow$ Z_ϕ is positive.

- Positive Maps on Qubits:

32. arXiv:quant-ph/0101003 [pdf, ps, other]

An Analysis of Completely-Positive Trace-Preserving Maps on 2x2 Matrices

Mary Beth Ruskai, Stanislaw Szarek, Elisabeth Werner

Comments: A significantly expanded version of quant-ph/0005004. 34 pages, 3 figures Final version accepted for publication in Lin. Alg. Appl

Journal-ref: Lin. Alg. Appl. 347, 159--187 (2002)

Subjects: **Quantum Physics (quant-ph)**; Mathematical Physics (math-ph); Operator Algebras (math.OA); Rings and Algebras (math.RA)

Bloch sphere $\qquad \rho = w_0 I + \vec{w}.\vec{\sigma}$

Φ : A completely positive Map.

$$\Phi(\rho) = w_0 I + (w_0 \vec{t} + T\vec{w}).\vec{\sigma}$$

$$\begin{pmatrix} w_0 \\ \vec{w} \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ \vec{t} & T \end{pmatrix}}_{\mathcal{T}} \begin{pmatrix} w_0 \\ \vec{w} \end{pmatrix}$$

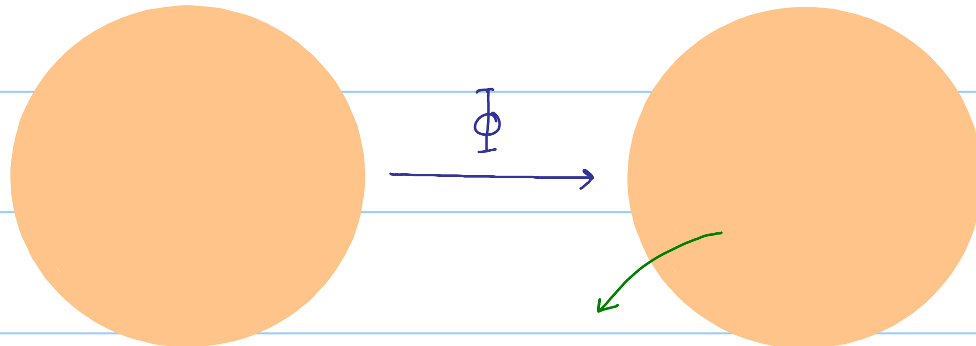
King & Ruskai : $\qquad$ Change of Basis

[13] C. King and M.B. Ruskai, "Minimal entropy of states emerging from noisy quantum channels" *IEEE Trans. Info. Theory* **47**, 192–209 (2001).

$$\mathcal{T} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{bmatrix} \qquad (\lambda_1 \pm \lambda_2) \leq (1 \pm \lambda_3)$$

Complete Positivity.

Bloch sphere



$$\left(\frac{x_1 - t_1}{\lambda_1}\right)^2 + \left(\frac{x_2 - t_2}{\lambda_2}\right)^2 + \left(\frac{x_3 - t_3}{\lambda_3}\right)^2 = 1.$$

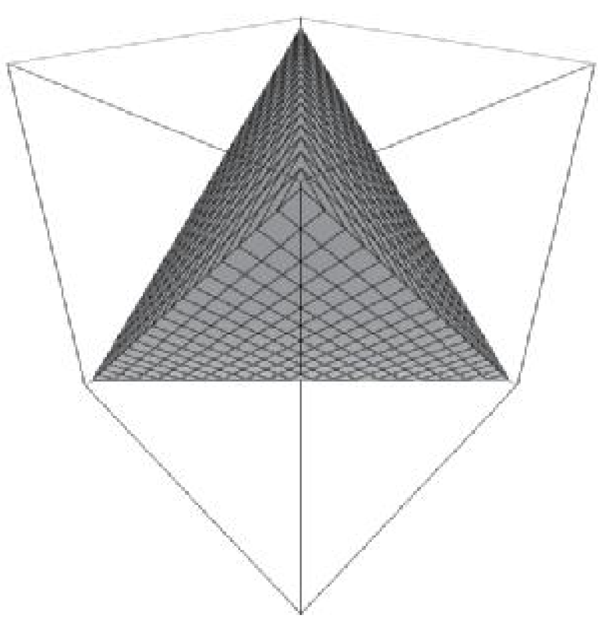
$$\Phi(\rho) = U\,\Phi_{t,\lambda}(V\rho V^\dagger)\,V^\dagger$$

$$(\lambda_1 \pm \lambda_2) \leq 1 \pm \lambda_3 \longrightarrow \lambda_1 + \lambda_2 + \lambda_3 \leq 1$$

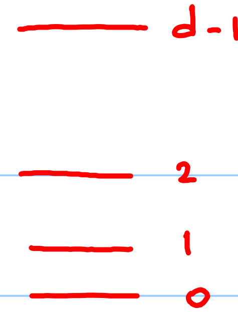
$$\lambda_1 - \lambda_2 + \lambda_3 \leq 1$$

$$\lambda_1 + \lambda_2 - \lambda_3 \leq 1$$

$$\lambda_1 - \lambda_2 - \lambda_3 \leq 1$$



Qudits



qudit: $|0\rangle, |1\rangle, \dots |d-1\rangle$

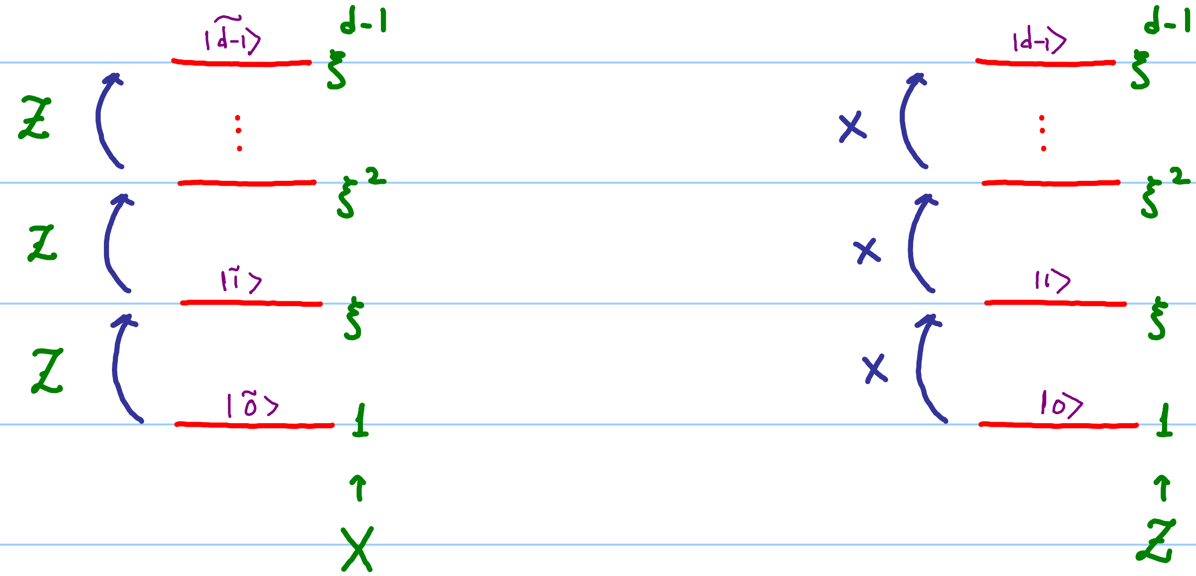
$$(-1)^2 = 1 \longrightarrow \xi^d = 1 \qquad \xi = e^{2\pi i/d}$$

$$X = \sum_{i=0}^{d-1} |i+1\rangle\langle i| \qquad Z = \sum_{i=0}^{d-1} \xi^i |i\rangle\langle i|$$

$$XZ = \xi ZX \qquad X^d = I \qquad Z^d = I.$$

eigenvalues of $X, Z = \{1, \xi, \xi^2, \dots \xi^{d-1}\}$

$$Z|i\rangle = \xi^i |i\rangle \qquad X|\tilde{i}\rangle = \xi^i |\tilde{i}\rangle$$



$$|\tilde{i}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \xi^{ij} |j\rangle$$

$$H_{ij} = \frac{1}{\sqrt{d}} \xi^{ij}$$

Many things for qudits are NOT simple:

- No Bloch sphere.
- Seperability

- pure state:

$$|\psi\rangle = \psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{02}|02\rangle +$$
$$\psi_{10}|10\rangle + \psi_{11}|11\rangle + \psi_{12}|12\rangle +$$
$$\psi_{20}|20\rangle + \psi_{21}|21\rangle + \psi_{22}|22\rangle$$

$$|\phi_1\rangle \otimes |\phi_2\rangle = (a_1|0\rangle + b_1|1\rangle + c_1|2\rangle) \otimes (a_2|0\rangle + b_2|1\rangle + c_2|2\rangle)$$

6 parameters instead of 9. $\longrightarrow$ 3 equations.

$$\psi_{00}\psi_{11}\psi_{22} = \psi_{01}\psi_{12}\psi_{20} = \psi_{02}\psi_{21}\psi_{10}$$
$$= \psi_{01}\psi_{10}\psi_{22} = \psi_{00}\psi_{12}\psi_{21} = \psi_{02}\psi_{11}\psi_{20}.$$

Another criteria: $\rho_A = tr_B(|\psi\rangle\langle\psi|)$.

if $|\psi\rangle$ is separable $\longrightarrow$ $\rho_A = |\phi_1\rangle\langle\phi_1| \longrightarrow \rho_A^2 = \rho_A$.

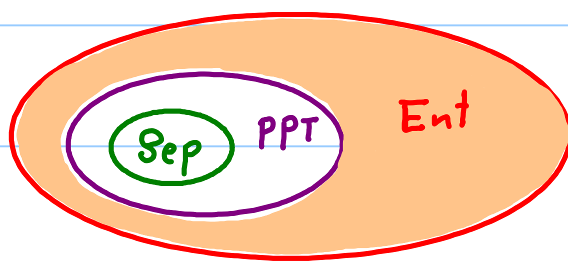
- **Mixed Case :** Seperable $\longrightarrow$ PPT.

True in every dimension.

NPT $\longrightarrow$ Entangled.

But what if a state is PPT ?

$\hookrightarrow$ Bound Entangled.



- **Entanglement**
  - pure $\quad |\psi\rangle_{AB} \longrightarrow \mathcal{E} = -\mathrm{tr}(\rho_A \ln \rho_A)$
  - Mixed $\quad \rho_{AB} \longrightarrow$?

$$\rho_{AB}^{T_B} = \text{Partial Transpose} \rightarrow \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_d \end{bmatrix}$$

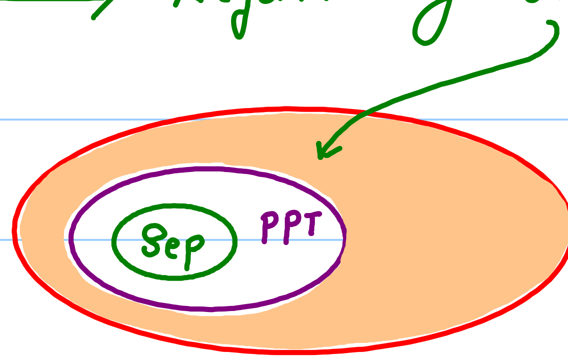
$$\mathcal{N}(\rho) = \frac{\| \rho^{T_B} \|_1 - 1}{2}$$

where $\| X \|_1 = \sum_{i=1}^{d} | \lambda_i |$

$$\mathcal{N}(\rho) = \frac{1}{2} \left(\sum_{i=1} (\lambda_i^+ - \lambda_i^-) - \sum_{i=1} (\lambda_i^+ + \lambda_i^-) \right)$$

$$= - \sum_{i=1} \lambda_i^- > 0$$

$\mathcal{N}(\rho) > 0 \longrightarrow$ Negative eigenvalue



- Positive maps on qudits ?

Analysis of complete positivity conditions for quantum qutrit channels

Agata Chęcińska*

Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Warszawa 00–681, Poland and ICFO-Institut de Ciencies Fotoniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain

Krzysztof Wódkiewicz

Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Warszawa 00–681, Poland and Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131-1156, USA

(Dated: December 15, 2008)

Generalized Bloch sphere:

$$\rightarrow \quad \rho = \frac{1}{d}\left(I + \vec{w}\cdot\vec{T} \right)$$

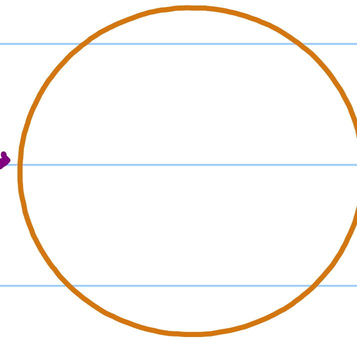
T basis of Hermitian Traceless Matrices for SU(d).

Normalization: $\mathrm{Tr}(T_i T_j) = d(d-1)\,\delta_{ij}$

Then $Tr(\rho^2) \le 1 \rightarrow \frac{1}{d^2}\left(d + d(d-1)\, w\cdot w\right) < 1$

$\rightarrow \vec{w}\cdot\vec{w} \le 1.$

7-dim. sphere



But Not every $w \longrightarrow$ a positive ρ. !!

For qutrits: $\rho = \frac{1}{3}\left(1 + \sqrt{3}\, n\cdot\lambda\right)$

$\lambda_1, \dots, \lambda_8$ (Gellman Matrices)

$Tr(\lambda_i \lambda_j) = 2\delta_{ij}$

$\lambda_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad \lambda_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \quad \dots$

$\lambda_i \lambda_j = d_{ijk}\, \lambda_k.$

$$\rho = \frac{1}{3}\left(1 + \sqrt{3}\ n\cdot\lambda \right)$$

Pure states: $\vec{n}\cdot\vec{n} = 1 \qquad \vec{n} * \vec{n} = \vec{n}$

where $(A * B)_i = d_{ijk} A_j \cdot B_k$.

$$\Phi : \vec{n} \mapsto \vec{n}' = \Lambda\vec{n} + \vec{t}, \tag{15}$$

where $\Lambda = diag\{\Lambda_1, ..., \Lambda_8\}$ consists of 8 damping coefficients and $\vec{t}$ is an eight dimensional translation. The image of the set of pure states under this transformation is

$$\sum_{i=1}^{8} \left(\frac{n'_i - t_i}{\Lambda_i}\right)^2 = 1, \tag{16}$$

together with the condition for $*$-product $\vec{n} * \vec{n} = \vec{n}$

$$\frac{n'_i - t_i}{\Lambda_i} = d_{ijk} \frac{n'_j - t_j}{\Lambda_j} \frac{n'_k - t_k}{\Lambda_k}. \tag{17}$$

$$\begin{aligned}
1 - \Lambda_8 + \tfrac{3}{2}(\Lambda_4 - \Lambda_5) &\geq 0,\\
1 - \Lambda_8 - \tfrac{3}{2}(\Lambda_4 - \Lambda_5) &\geq 0,\\
1 - \Lambda_8 + \tfrac{3}{2}(\Lambda_6 - \Lambda_7) &\geq 0,\\
1 - \Lambda_8 - \tfrac{3}{2}(\Lambda_6 - \Lambda_7) &\geq 0,\\
1 - \Lambda_8 + \tfrac{3}{2}(\Lambda_1 - \Lambda_2) + \tfrac{3}{2}(\Lambda_8 - \Lambda_3) &\geq 0,\\
1 - \Lambda_8 - \tfrac{3}{2}(\Lambda_1 - \Lambda_2) + \tfrac{3}{2}(\Lambda_8 - \Lambda_3) &\geq 0.
\end{aligned} \tag{21}$$