QMII_Lec1_Addition of Angular Momentum

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QM II - Winter 2020

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- ☐ Rotations & Representations
- ☐ Composite system
- ☐ Clebsch-Gordan coefficients (CGC)
  - Examples

Rotations SO(2) & SO(3)

Group: A set with an operation such that

$\forall a, b \in G$

- $a.b \in G$
- $(a.b).c = a.(b.c)$
- $\exists e \in G: \quad e.a = a.e = a$
- $\exists a^{-1} \in G: \quad a.(a^{-1}) = (a^{-1}).a = e$

One example is the group of rotations, SO(3)

Rotations are a subset of $\mathcal{L}(\mathbb{R}^3)$.

These are the transformations that preserve both the length of vectors & the angle between them.

If a basis is set for $\mathbb{R}^3$, then Rotations can be shown with matrices. and form a group under matrix multipication. It's a group under matrix multipication. It's easy to check that

$\forall\ R \in SO(3)$

$*\quad R\,R^T = R^T . R = \mathbb{1}$ $\longrightarrow$ To preserve the angle.

$*\quad \det(R) = 1$ $\longrightarrow$ To preserve the length

Ⓐ Check that this forms a group.

We use $R_n(\theta)$ to denote rotation around $\vec{n}$ for θ.

This is a continous group.

Representation

$D : G \longrightarrow \mathcal{L}(V)$ $\qquad \forall g_1, g_2 \in G$

$$D(g_1)\,D(g_2) = D(g_1 g_2)$$

Examples: $G = \{a, a^2, e = a^3\}$

$*\quad D(g) = 1$ $\qquad *\quad D(a) = e^{\frac{2i\pi}{3}}$

$*\quad D(a) = R_z\left(\frac{2\pi}{3}\right)$ $\qquad *\quad D(a) = \widetilde{D}\left(R_z\left(\frac{2\pi}{3}\right)\right)$ For any $\widetilde{D}$

We are interested in the action of rotation on quantum states

$$D(R_n(\theta)) : |\psi_1\rangle \longrightarrow |\psi_2\rangle$$

So, D should be a unitary representation.

$$\underset{\substack{\downarrow \\ D_n(\theta)}}{D}(R_n(\theta)) = e^{i\theta A} \quad \text{for } A \text{ some Hermitian operator.}$$

It is possible to show that there should be 3 operators $\{A_1, A_2, A_3\}$ to generate all rotations &

$$[A_i, A_j] = i\varepsilon_{ijk} A_k$$

(This comes from $D(g_1)D(g_2) = D(g_1 g_2)$).

This resembles the angular momentum operators. $(A_i = \hbar J_i)$
In fact we can show that

$$D_n(\theta) = e^{i\theta \frac{\vec{n}\cdot\vec{J}}{\hbar}} .$$

Examples

* 2-D Representation

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D(R_n(\theta)) = e^{i\frac{\theta}{\hbar}\vec{n}\cdot\vec{J}}$$

We have already seen that these act like rotations

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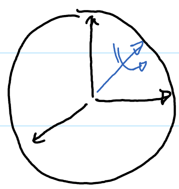
e.g



$$e^{i\frac{\theta}{2}\sigma_z}\ \sigma_x\ e^{-i\frac{\theta}{2}\sigma_z} = \cos(\theta)\ \sigma_x + i\sin(\theta)\ \sigma_y$$

* 3-D Representation

(A) Find the unitary representation of rotation for $j=1$.

(A) Would $J_x = \hbar\left(\begin{array}{c|c} 0 & \\ \hline & \frac{\sigma_x}{2}\end{array}\right)$, $J_y = \hbar\left(\begin{array}{c|c} 0 & \\ \hline & \frac{\sigma_y}{2}\end{array}\right)$

$J_z = \hbar\left(\begin{array}{c|c} 0 & \\ \hline & \frac{\sigma_z}{2}\end{array}\right)$ work? What's the difference?

(A) Would $\left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2\end{array}\right)$ be a representation

if D_1 & D_2 are two valid representation of

SO(3)?

Composite system

Knowing the j, we can easly find the proper representation that is not a composition of other representations. e.g for spin $\{\sigma_x, \sigma_y, \sigma_z\}$ generates all the rotations for the spin ($j = 1/2$).

Now, what if we have multiple spins?

$$D_n^{(1)}(\theta) = e^{i\frac{\theta}{\hbar}\vec{n}.\vec{J}^{(1)}}$$

$$D_n^{(2)}(\theta) = e^{i\frac{\theta}{\hbar}\vec{n}.\vec{J}^{(2)}}$$

Composite system

$$D_n^{(1)}(\theta) \otimes D_n^{(2)}(\theta) = e^{i\frac{\theta}{\hbar}\vec{n}.(\vec{J}^{(1)}+\vec{J}^{(2)})}$$

So we introduce $J^{tot} = \vec{J}^{(1)} + \vec{J}^{(2)}$

$$J_x^{tot} = \vec{J}_x^{(1)} + \vec{J}_x^{(2)}$$

$$J_y^{tot} = J_y^{(1)} + J_y^{(2)}$$

$$J_z^{tot} = J_z^{(1)} + J_z^{(2)}$$

$$J_\pm^{tot} = J_\pm^{(1)} + J_\pm^{(2)}$$

This gives a representation for the composite system

$$D_n(\theta) = e^{i\frac{\theta}{\hbar}\vec{n}\,\vec{J}^{tot}}.$$

What's the dimension of this representation? $\underline{\underline{4}}$

Is it one irreducible rep with $j = \frac{3}{2}$? or combination of a couple of representations?

$$\left[\begin{array}{c|c} D^{1/2} & \\ \hline & D^{1/2} \end{array}\right] \text{ or } \left[\begin{array}{c|c} D^{0} & \\ \hline & D^{1} \end{array}\right] \text{ or } \left[\begin{array}{c} D^{3/2} \end{array}\right]$$

To answer this question, we should find J^{tot}.

What's the total angular momentum of composite system?

Example: $\left(j=\frac{1}{2} \otimes j=\frac{1}{2}\right)$

$$|\psi\rangle = |1/2, m_1\rangle \otimes |1/2, m_2\rangle$$

$$J^2 = \vec{J}\cdot\vec{J} = (\vec{J}_1+\vec{J}_2)\cdot(\vec{J}_1+\vec{J}_2)$$

$$= \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + 2\vec{J}_1\cdot\vec{J}_2 \qquad : \vec{J}_1 = \vec{J}_1 \otimes \mathbb{1}$$

$$\vec{J}_1\cdot\vec{J}_2 = \underbrace{J_{1,x}J_{2,x} + J_{1,y}J_{2y}}_{\frac{J_{1,+}J_{2,-} + J_{1,-}J_{2,+}}{2}} + J_{1z}J_{2z}$$

$$\Rightarrow \vec{J}^2 = \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + 2J_{1,z}J_{2,z} + \left(J_{1+}J_{2-} + J_{1-}J_{2+}\right)$$

Now, what's $\vec{J}^2$ for the states in this basis?

$$\left\{ \underbrace{|1/2, \pm 1/2\rangle}_{|\pm 1/2\rangle} \otimes \underbrace{|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle}_{|\pm 1/2\rangle} \right\}$$

New notation $\longrightarrow$

$\Rightarrow$ ✓ $$\vec{J}^2 |\tfrac{1}{2}\rangle\otimes|1/2\rangle = \hbar^2 \underbrace{\left(\tfrac{3}{4}+\tfrac{3}{4}+2\left(\tfrac{1}{4}\right)+0\right)}_{2\hbar^2} |1/2\rangle\otimes|1/2\rangle$$

✓ $$\vec{J}^2 |{-\tfrac{1}{2}}\rangle \otimes |{-\tfrac{1}{2}}\rangle = \hbar^2 \underbrace{\left(\tfrac{3}{4} + \tfrac{3}{4} + 2\left(\tfrac{1}{4}\right) + 0\right)}_{2\hbar^2} |{-\tfrac{1}{2}}\rangle \otimes |{-\tfrac{1}{2}}\rangle$$

$$J^2 |\tfrac{1}{2}\rangle \otimes |{\tfrac{-1}{2}}\rangle = \hbar^2 \left(2\left(\tfrac{3}{4}\right) - 2\left(\tfrac{1}{4}\right)\right) |\tfrac{1}{2}, \tfrac{-1}{2}\rangle$$

$$+\hbar^2 |\tfrac{-1}{2}, \tfrac{1}{2}\rangle \longrightarrow \text{Not an eigen-state}$$

$$J^2 |{-\tfrac{1}{2}}\rangle \otimes |\tfrac{+1}{2}\rangle = \hbar^2 \left(2\left(\tfrac{3}{4}\right) - 2\left(\tfrac{1}{4}\right)\right) |\tfrac{-1}{2}\rangle |\tfrac{+1}{2}\rangle$$

$$+\hbar^2 |\tfrac{+1}{2}, \tfrac{-1}{2}\rangle \longrightarrow \text{Not an eigen-state}$$

What are the eigen basis of $\vec{J}^2$?

$$\left\{ |\tfrac{1}{2}\rangle|\tfrac{1}{2}\rangle,\ |\tfrac{-1}{2}\rangle|\tfrac{-1}{2}\rangle,\ \alpha|\tfrac{1}{2}\rangle|\tfrac{-1}{2}\rangle + \beta|\tfrac{-1}{2}\rangle|\tfrac{1}{2}\rangle,\ \alpha' \ \text{''} \ +\beta' \ \text{''} \right\}$$

So, we need to find $\{\alpha, \beta, \alpha', \beta'\}$.

It's easy to see that $\alpha = \beta = \frac{1}{\sqrt{2}}$

$\alpha' = -\beta' = \frac{1}{\sqrt{2}}$.

(A) Try to write the last two states as

$|\psi_1\rangle \otimes |\psi_2\rangle$

(A) Out of $\{\vec{J}^2, \vec{J}_1^2, \vec{J}_2^2, J_{1,z}, J_{2,z}, J_z\}$ which ones do we know for all the eigen basis elements? What are the corresponding values? Make a table

Ⓐ Can you determine $J_{1,z}$ & $J_{2,z}$ for all the states in the set?

Ⓐ What happens to the state $\frac{|\frac{1}{2}\rangle|\frac{-1}{2}\rangle - |\frac{-1}{2}\rangle|\frac{1}{2}\rangle}{\sqrt{2}}$ if rotated? What does that tell us about the total j of this state?

Is it the same for $\frac{|\frac{1}{2}\rangle|\frac{-1}{2}\rangle + |\frac{-1}{2}\rangle|\frac{1}{2}\rangle}{\sqrt{2}}$?

Doing the assignment above we should find that

$$\vec{J}^2 |\pm\tfrac{1}{2}\rangle|\pm\tfrac{1}{2}\rangle = 2\hbar^2 |\pm\tfrac{1}{2}\rangle|\pm\tfrac{1}{2}\rangle \rightarrow j = 1$$

$$\vec{J}^2 \left(\frac{|\frac{1}{2}\rangle|\frac{-1}{2}\rangle + |\frac{-1}{2}\rangle|\frac{1}{2}\rangle}{\sqrt{2}} \right) = 2\hbar^2 \left(\frac{|\frac{1}{2}\rangle|\frac{-1}{2}\rangle + |\frac{-1}{2}\rangle|\frac{1}{2}\rangle}{\sqrt{2}} \right)$$

$$\vec{J}^2 \left(\frac{|\frac{1}{2}\rangle|\frac{-1}{2}\rangle - |\frac{-1}{2}\rangle|\frac{1}{2}\rangle}{\sqrt{2}} \right) = 0$$

So $\vec{J}^2 = \hbar^2 \begin{bmatrix} \boxed{\begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix}} & \\ & \boxed{0} \end{bmatrix}$, Does this make sense?

This indicates that we could end up with different values of angular momentum j^{tot}.

Addition of angular momentum

Assume that we are given two particles in states $|j_1, m_1\rangle$ & $|j_2, m_2\rangle$. We want to know the properties of the combined system in terms of the $\underbrace{\{\vec{J}^2, J_z, \vec{J}_1^2, \vec{J}_2^2\}}$.

What's the advantage? Why not use the original set $\{J_1^2, J_2^2, J_{1,z}, J_{2,z}\}$?

* It is easier to understand rotations in this basis
* Often, we do not have access to the sub-system individually and it is easier to use global properties of the system e.g. j^{tot}, j_z^{tot} to describe the states.

Notation

Starting fr. $|j_1, m_1\rangle \otimes |j_2, m_2\rangle \longrightarrow |m_1, m_2\rangle$

Starting from $|j_1, m_1\rangle \otimes |j_2, m_2\rangle \longrightarrow |m_1, m_2\rangle$

Going to $|j, m, j_1, j_2\rangle \longrightarrow |j, m\rangle$

Our goal is to find $|j, m\rangle$. We know that

$$|jm\rangle = \sum_{m_1, m_2} C(j, m; m_1, m_2)\, |m_1, m_2\rangle$$

$$C(j, m; m_1, m_2) = \langle m_1, m_2 | j, m\rangle$$

These are known as Clebsch-Gordan coefficients (CGC)

We want to find these CGC. It's helpful to make some remarks first.

* $m_1 + m_2 = m$

$$\underbrace{\langle m_1, m_2 | J_z | j, m\rangle}_{m\hbar} = \underbrace{\langle m_1, m_2 | J_{1,z} + J_{2,z} | jm\rangle}_{(m_1+m_2)\hbar}$$

$$\Rightarrow \boxed{m_1 + m_2 = m}$$

* $|j_{max} = j_1 + j_2,\ m_{max} = m_1 + m_2\rangle = |m_1 = j_1, m_2 = j_2\rangle$

$$\Rightarrow C(j_1+j_2, j_1+j_2; j_1, j_2) = 1$$

$* \; |j_1-j_2| \leq j \leq j_1+j_2$ Take $m_1 \leq j_1$ & $m_2 \leq j_2 \rightarrow$ Max values

$$m = m_1 + m_2 \leq j_1 + j_2 \quad . \quad m_{max} = j = j_1 + j_2$$

What about the min ?

$$d = (2j_1+1)(2j_2+1) = \sum_{j=j_{min}}^{j_1+j_2} (2j+1)$$

$$\Rightarrow j_{min} = |j_1 - j_2|$$

$* \quad j : |j_1-j_2| \, , \, |j_1-j_2|+1 , \ldots \; j_1+j_2$

Notation $\quad j_1 \otimes j_2 = |j_1-j_2| \oplus (|j_1-j_2|+1) \oplus \cdots \oplus (j_1+j_2)$

Clebsch - Gordan coefficients (CGC)

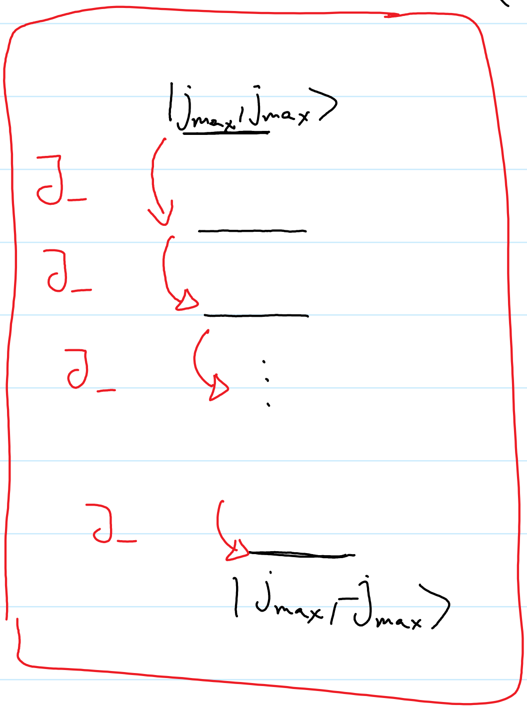
$$|j_{max} = j_1 + j_2 , m = j_{max}\rangle = \underbrace{|j_1, m_1 = j_1\rangle \otimes |j_2, m_2 = j_2\rangle}_{|j_1, j_2\rangle \; \rightarrow \text{ Our notation.}}$$

$$|j_{max}, m = j_{max} - 1\rangle = J_-^{(tot)} |j_{max}, j_{max}\rangle$$

$$= \left(J_-^{(1)} + J_-^{(2)} \right) | j_1, j_2 \rangle$$

Similarly for $|j_{max}, m = j_{max} - \ell \rangle =$

$$\left(J_-^{tot} \right)^\ell | j_1, j_2 \rangle$$



But this is only one block. how to go to

$$j^{tot} = j_{max} - 1 \quad ?$$

Note $[J_\pm, J^{tot}] = 0 \to$ So

$J_\pm$ can't help.

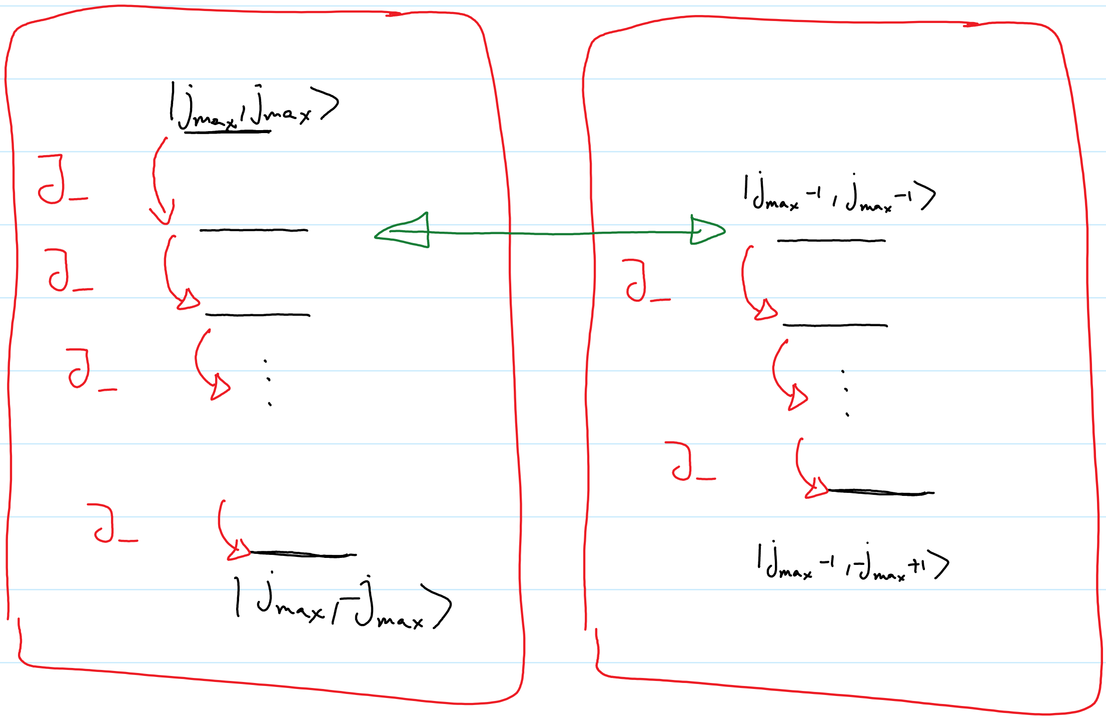
Here we use the fact that the subspace of $m = j_{max} - 1$ has $\dim = 2$ since

$$| j^{tot} = j_{max} - 1, m = j_{max} - 1 \rangle = \alpha | j_1, j_2 - 1 \rangle + \beta | j_1 - 1, j_2 \rangle$$

Also we already have $| j_{max}, j_{max} - 1 \rangle$ and know that

$$\langle j_{max} - 1, j_{max} - 1 | j_{max}, j_{max} - 1 \rangle = 0.$$ This determines

the $| j_{max} - 1, j_{max} - 1 \rangle$



Similarly, we can go to $j_{max}-2$ block and so on. Note that for the top state in $j^{tot} = j_{max}-2$, we need to use the orthogonality to two states

$$|j_{max}, j_{max}-2\rangle \;\&\; |j_{max}-1, j_{max}-2\rangle.$$

Addition of angular momentum

Example:

Spin 1 $\otimes$ Spin $\frac{1}{2}$

We want to start with $|m_1, m_2\rangle$ basis and find $|j, m\rangle$ interms of $|m_1, m_2\rangle$.

First, let's find the possible values of j:

$$1 \otimes 1/2 \to j_{max} = 3/2 \quad , \quad j_{min} = 1/2 .$$

$$j \in \left\{ \frac{3}{2}, \frac{1}{2} \right\} .$$

Next, we construct the easiest one:

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = |\underset{m_1}{1}, \underset{m_2}{1/2}\rangle \qquad J_- |jm\rangle = \sqrt{j(j+1) - m(m-1)}$$

J_- ↓

$$\sqrt{\frac{3\times5}{4} - \frac{3\times1}{4}} \left|\frac{3}{2}, \frac{1}{2}\right\rangle = (J_{1,-} + J_{2,-}) \left|1, \frac{1}{2}\right\rangle$$

$$= \sqrt{2} \left|0, \frac{1}{2}\right\rangle + \sqrt{\frac{3}{4} - \frac{-1}{4}} \left|1, \frac{-1}{2}\right\rangle$$

$$\Rightarrow \left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} \left|0, \frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}} \left|1, \frac{-1}{2}\right\rangle$$

J_- ↓

$$\underbrace{\sqrt{\frac{3\times5}{4} + \frac{1}{4}}}_{2} \left|\frac{3}{2}, \frac{-1}{2}\right\rangle = \sqrt{\frac{2}{3}} \left(\sqrt{2} \left|-1, \frac{1}{2}\right\rangle + \left|0, \frac{-1}{2}\right\rangle \right) + \frac{1}{\sqrt{3}} \left(\sqrt{2} \left|0, \frac{-1}{2}\right\rangle + 0 \right)$$

$$\Rightarrow \left|\frac{3}{2}, \frac{-1}{2}\right\rangle = \frac{1}{\sqrt{3}} \left|-1, \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}} \left|0, \frac{-1}{2}\right\rangle$$

J_- ↓

$$\left|\frac{3}{2}, \frac{-3}{2}\right\rangle = |-1, 0\rangle$$

These makes the state with $j = 3/2$.

For $j = 1/2$ sector, we have

$$\overset{j\;\;\;m}{\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle} = \alpha \left|1, \tfrac{-1}{2}\right\rangle + \beta \left|0, \tfrac{1}{2}\right\rangle$$

and we know that

$$\left\langle \tfrac{3}{2}, \tfrac{1}{2} \middle| \tfrac{1}{2}, \tfrac{1}{2}\right\rangle = 0 \qquad \left(\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}} \left|1, \tfrac{-1}{2}\right\rangle + \sqrt{\tfrac{2}{3}} \left|0, \tfrac{1}{2}\right\rangle \right)$$

So

$$\frac{\alpha}{\sqrt{3}} + \frac{\sqrt{2}\beta}{\sqrt{3}} = 0 \Rightarrow \boxed{\alpha = -\sqrt{2}\beta} + \boxed{|\alpha|^2 + |\beta|^2 = 1}$$

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \beta \left(-\sqrt{2} \left|1, \tfrac{-1}{2}\right\rangle + \left|0, 1/2\right\rangle \right) \Rightarrow \beta = \frac{1}{\sqrt{3}}$$

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = -\sqrt{\tfrac{2}{3}} \left|1, \tfrac{-1}{2}\right\rangle + \frac{1}{\sqrt{3}} \left|0, \tfrac{1}{2}\right\rangle$$

For $\left|\tfrac{1}{2}, \tfrac{-1}{2}\right\rangle$ we can find the state orthogonal to $\left|\tfrac{3}{2}, \tfrac{-1}{2}\right\rangle$ or apply J_- to $\left|\tfrac{1}{2}, \tfrac{-1}{2}\right\rangle$ which gives

$$\left|\tfrac{1}{2}, \tfrac{-1}{2}\right\rangle = \sqrt{\tfrac{2}{3}} \left|-1, \tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}} \left|0, \tfrac{-1}{2}\right\rangle$$

Note: When we are constructing $|j\,m\rangle$ states there is always a phase freedom. It is important to use it consistently. i.e.

$$\left|\tfrac{1}{2}, \tfrac{-1}{2}\right\rangle = J_- \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle.$$

Orbital & spin AM :

For the electron in the Hydrogen atom, it has both an Orbital AM l and a spin AM, S.

When measured, they are not distinguished & we measure one AM for the electron that is the combied angular momentums. $l \otimes \frac{1}{2} = (l + 1/2) \oplus (l - \frac{1}{2})$.

For the ground state, $l=0$ so we see only $j = 1/2$.

But for higher levels, we get multiple j values.

Operators under rotation

A operator transforms as $A' = \mathcal{U}^\dagger A \mathcal{U}$ under transformation $\mathcal{U}$. For a rotation, $\mathcal{U} = D(R_n(\theta))$.

For small rotation angles $\delta\theta \ll 1$,

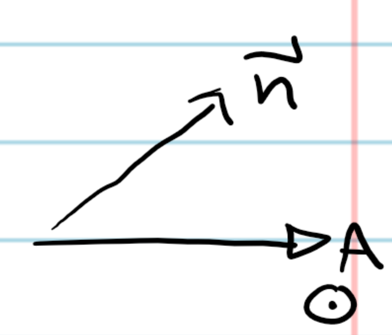
Ⓐ Show that $D_n^\dagger(\delta\theta)\, A\, D_n(\delta\theta) = A - \frac{i\delta\theta}{\hbar}[A, \vec{n}\cdot\vec{J}]$.

Based on how operators transform under rotation, we can distinguish them:

- **Scalar Operators:** $A' = D_n^\dagger(\delta\theta)\, A\, D_n(\theta) = A$ → Does not change.

  $\Rightarrow [A, n\cdot\vec{J}] = 0 \quad \forall \vec{n}$

- **Vector Operators:** $\vec{A}' = \vec{A} + \delta\theta\, \vec{n} \times \vec{A}$

  $\Rightarrow [\vec{A}, n\cdot\vec{J}] = i\hbar\, \vec{n} \times \vec{A}$

Ⓐ Check this for yourselves!

- **Tensor Operators:** $A_{ij\cdots} = B_i\, C_j \cdots \quad i,j \in \{x,y,z\}$

  Where $B, C, \ldots$ are vector operators.