Lecture 7

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QM I _ Fall 2019

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Some points about the structure of QM:

☐ Dynamics

Time-indep. Schrodinger's eq.

Symmetry & constants of motion

☐ Conservation of probability

☐ Commutativity of Operators

Compatible measurements

CSCO

Uncertainity relation

Schrodinger's eq.

The dynamics of the state is given by

$$\frac{d}{dt}|\psi(t)\rangle = \frac{-i}{\hbar} H(t) \, |\psi(t)\rangle$$

And we showed that in the position basis, we get

$$\Rightarrow \boxed{\frac{d}{dt}\psi(x,t) = \left(\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2} - \frac{i}{\hbar}V(x)\right)\psi(x,t)}$$

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For a time-independent V, we can do seperation of variables:

$$\psi(x,t) = \psi(x)\,\phi(t)$$

$$\Rightarrow i\hbar \frac{1}{\phi(t)}\frac{d\phi(t)}{dt} = \frac{1}{\psi(x)}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = E$$

$$\Rightarrow \phi(t) = \phi_0\, e^{-iEt/\hbar}$$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(\hat{X})\right)\psi(x) = E\,\psi(x) \longrightarrow \text{Time-indep Sch. eq.}$$

$$\psi(x,t) = \psi_E(x)\, e^{-iEt/\hbar} \longrightarrow \text{Staitionary states}$$

$$P_r = |\psi(x,t)|^2 = |\psi(x)|^2 \longrightarrow \text{indep of time}$$

Symmetry

$$\frac{d\langle \hat{A}(t)\rangle}{dt} = \frac{d}{dt}\langle \psi(t)|\, A\, |\psi(t)\rangle = \frac{d}{dt}\langle \psi(0)|\, U^\dagger(t) A U(t)\, |\psi(0)\rangle$$

(A) Show that if $\frac{\partial A}{\partial t} = 0 \longrightarrow$

$$\frac{d}{dt}\langle \psi(0)|\, U^\dagger(t) A U(t)\, |\psi(0)\rangle = \frac{1}{i\hbar}\langle [\hat{A}, \hat{H}] \rangle$$

If $[\hat{A}, \hat{H}] = 0 \Rightarrow \frac{d\langle A\rangle}{dt} = 0 \Rightarrow \langle A\rangle$ is constant.

Also

$$\left[e^{i\lambda\hat{A}}, \hat{H}\right] = 0 \Rightarrow e^{i\lambda\hat{A}} H e^{-i\lambda\hat{A}} = H$$

$$\Rightarrow H \text{ is symmetic under } V = e^{i\lambda\hat{A}}$$

Example

$$H = P^2/2m \rightarrow \text{Free particle}$$

$$\hat{A} = \hat{P} \Rightarrow \hat{V} = e^{i\lambda\hat{P}} \text{ translation}$$

$$[\hat{H}, \hat{P}] = 0 \Rightarrow \text{It is symmetric under translation}$$

$$\Rightarrow \langle\hat{P}\rangle \text{ is constant}$$

Similarly for any symmetry $\hat{V} = e^{i\lambda\hat{A}} \rightarrow [\hat{V}, \hat{H}] = 0$

(A) Show that $\langle A \rangle$ is constant.

→ Probability current and Conservation of Probability

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|U^\dagger U|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

Equivalently, we can check that: $\frac{d}{dt} \qquad = 0$

$$\frac{d}{dt}\langle\psi(t)|\psi(t)\rangle = \left(\frac{d}{dt}\langle\psi(t)|\right)\left(|\psi(t)\rangle\right) + \langle\psi(t)|\left(\frac{d}{dt}|\psi(t)\rangle\right)$$

We have: $\frac{d}{dt}|\psi(t)\rangle = \frac{-i}{\hbar} H |\psi(t)\rangle$

$$\frac{d}{dt}\langle\psi(t)|\psi(t)\rangle = \dots = 0$$

↳ Check to see that if U was not unitary, this wouldn't be true.

Let's take a closer look at this in the position space:

$$\langle\psi(t)|\psi(t)\rangle = \int dx\ |\psi(x,t)|^2 = 1$$

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \qquad \times\,\psi^*(x,t)$$

$$-\ (\qquad\qquad ''\qquad\qquad)^* \qquad \times\,\psi(x,t)$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\underbrace{\left(\psi^*(x,t)\,\psi(x,t)\right)}_{\rho = |\psi(x,t)|^2 \text{ prob. density}} = \frac{-\hbar^2}{2m}\left(\psi^*(x,t)\frac{\partial^2}{\partial x^2}\psi(x,t) - \psi(x,t)\frac{\partial^2}{\partial x^2}\psi^*(x,t)\right)$$

We also introduce current density

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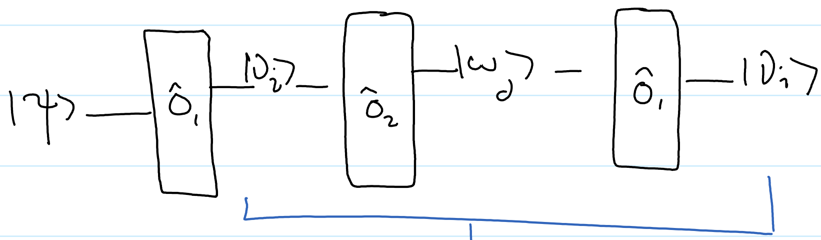
$$J = \frac{i\hbar}{2m}\left[\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t) - \psi(x,t)\frac{\partial}{\partial x}\psi^*(x,t)\right]$$

$$\Rightarrow \quad \frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} J = 0 \qquad (eq\ *)$$

Compatible measurements

These are measurements for which the result of all of them can be identified.



The outcomes are compatible.

What does it imply for $\hat{O}_1$ & $\hat{O}_2$? $\left[\hat{O}_1 = \sum \alpha_i \Pi_i \ ,\ \hat{O}_2 = \sum \alpha_i' \Pi_i'\right]$

$|v_i\rangle = \frac{\Pi_i |\psi\rangle}{Pr(i)} \longrightarrow$ is an eigenstate of $\hat{O}_1$

After $\hat{O}_2$, we get

$$|\tilde{w}_j\rangle = \frac{\Pi_j' |v_i\rangle}{Pr'(j)}$$

And then

$$|0_i\rangle = \Pi_1 |w_j\rangle = \Pi_1 \Pi_j' |0_1\rangle$$

$\Rightarrow \Pi_i = \Pi_j' \rightarrow$ They have a common diagonal basis

$$\Rightarrow [\hat{O}_1, \hat{O}_2] = 0$$

The set of compatible measurements could be

larger $\rightarrow \{\hat{O}_i\} : [\hat{O}_i, \hat{O}_j] = 0 \quad \forall i,j$

CSCO

A set of observables $\{O_i\}$ such that

* $[\hat{O}_i, \hat{O}_j] = 0 \quad \forall i,j$

* Their set of common eigenstate is complete & not degenerate.

(A) $[\hat{A}, \hat{B}] = 0$, $[\hat{B}, \hat{C}] = 0$

Can we conclude that $[\hat{A}, \hat{C}] = 0$?

Prove or make a counter-example.

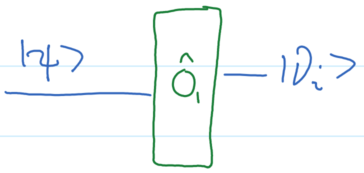
Ⓐ For the SG experiment, find a CSCO?

Is it unique?

Ⓐ For a particle in 3D, find a CSCO?

Uncertainty Relation

Consider the following measurement.



The outcome could be any of $|O_i\rangle$ with $Pr(i)$.

We can calculate $\langle O_1 \rangle = \sum_i \alpha_i Pr(i)$.

But to get a sense of how certain the outcome is we calculate the variance (as we do in statistics)

$$\langle \Delta \hat{O}_1 \rangle = \sqrt{\langle \hat{O}_1^2 \rangle - \langle \hat{O}_1 \rangle^2}.$$

Ⓐ For SGE calculate $\langle \Delta S_z \rangle$ when the input state is

1) $|\psi\rangle = |x+\rangle$

2) $|\psi\rangle = |z+\rangle$

For two compatible measurements, the variance of the second " is zero. → Show this!

But if two measurements do not commute, then the outcome of the 2^{nd} measurement can have some uncertainity.

Ⓐ Prove that for observable $\hat{A}, \hat{B}$

$$\langle \Delta \hat{A} \rangle \langle \Delta \hat{B} \rangle \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|$$

Note that this gives a lower bound

It's also important to note that for two non-commuting operators A & B, there could exist states where ΔA & ΔB are both zero. Take this example

$$A = \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & & & \\ & 3 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix}$$

You can check that $[A, B] \neq 0$

but for $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, they both A & B can be identified with

$\langle \Delta A \rangle = \langle \Delta B \rangle = 0$