QM I _ Fall 2019
Sadegh Raeisi

Lecture 4: A physical theory to make sense of these experiments

October 5, 2019 7:48 PM

What are the essential ingredients of a physical theory?

Classical physics

$\{x, p\} \longrightarrow$ Description of the system.

$E, F, a, \omega, \vec{L} \longrightarrow$ functions of $\{x, p\}$

$f(x, p)$

$X(t), P(t) \longrightarrow$ Newton's law : $F = ma$

or

Hamiltonian eq.

$$\dot{q} = -\frac{\partial H}{\partial t}, \quad \dot{p} = \frac{\partial H}{\partial t}$$

So, it seems that a description of the physical world is at least expected to

* Give a description of the state of the system
* " " " " different measurable quantities
* " " " " how the system evolves in time.

→ A description of what happens in a measurement.

* what the outcomes are
* With how much certainty
* The average / Expected value

Try to map this picture to E&M & thermodynamics.

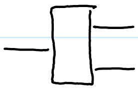
$\boxed{\text{Homework}}$ Find a mathematical description that works for the SGE (Don't need to worry about the evolution).

The classical description is $\vec{L} = (l_x, l_y, l_z)$

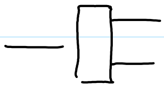
$$\Rightarrow \vec{\mu} = (\mu_x, \mu_y, \mu_z).$$

Why doesn't this work?

Think of an alternative mathematical structure that is compatible with the experiment.

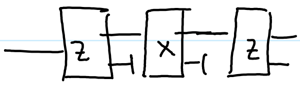


f_1 Continous function $\longrightarrow$ $\tilde{f}_1 \in \{1, -1\}$



f deterministic or probabilistic

$\tilde{f}$ probabilistic

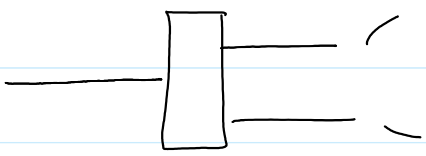


$f_1 \quad f_2 \quad f_1$

classically both f_1 & f_2 can be specified

If f_2 is known, f_1 could be unspecified.

Mathematical Structure that could work for SGE



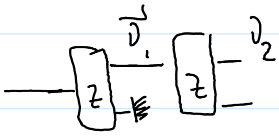
$\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \vec{z}_+$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \vec{z}_-$

State $\longrightarrow$ A vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

Input state $\vec{v}_{in} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

Input state $v_{in} = (\beta)$



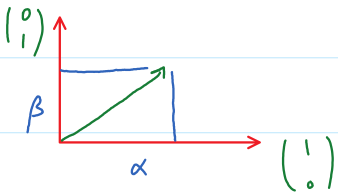
$\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix} \longrightarrow$ Outcome $\begin{cases} Pr(0) = 1 \to \begin{pmatrix}1\\0\end{pmatrix} \\ Pr(1) = 0 \to \text{No } \begin{pmatrix}0\\1\end{pmatrix} \end{cases}$

Similarly $\quad \vec{v}_1 = \begin{pmatrix}0\\1\end{pmatrix} \to$ Outcome $\begin{cases} Pr(0) = 0 \\ Pr(1) = 1 \end{cases}$

What if $\quad Pr(z+) = Pr(z-)$?

$\alpha = \beta \longrightarrow$ Symmetry

$$\Rightarrow \vec{v}_{in} = \begin{pmatrix}\alpha\\ \alpha\end{pmatrix}$$



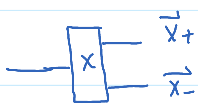
$\longrightarrow Pr(z+) = f(\alpha)$

$Pr(z-) = f(\beta)$

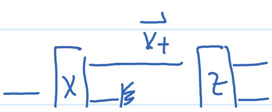
Expectations: $\quad f(\alpha) = \vec{v}_{in}^{\dagger} \cdot \vec{z}_+ = (\alpha, \beta) \cdot \begin{pmatrix}1\\0\end{pmatrix} = \alpha$

$f(\beta) = \vec{v}_{in}^{\dagger} \cdot \vec{z}_- = (\alpha, \beta) \cdot \begin{pmatrix}0\\1\end{pmatrix} = \beta$

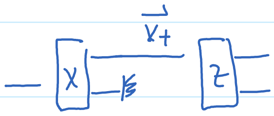
Does it work and how?



What's the description of $\vec{x}_+$ & $\vec{x}_-$?



$\vec{v}_+ \to \{ Pr(z_+) = 1/2$

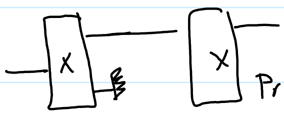


$$\vec{x}_+ \to \begin{cases} P_r(Z_+) = 1/2 \\ P_r(Z_-) = 1/2 \end{cases}$$

$$\Rightarrow \vec{x}_+ = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \quad \to \text{There's a freedom here.}$$

But it is the same for $\vec{x}_- = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$.

But

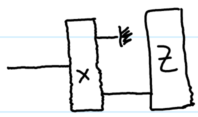


$P_r(x_-) = 0$

$$P_r(\vec{x}_-) = \left(\vec{x}_+^{\dagger}\right) \cdot \left(\vec{x}_-\right) = 0$$

$$\Rightarrow \vec{x}_+ = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \qquad \vec{x}_- = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$$

But then



$P_r(Z_-) = \vec{x}_-^{\dagger} \cdot \vec{Z}_- = -\alpha$

But probability cannot be negative.

<u>But</u> again inspired by polarization

Is this the only choice?

Will ... come back

$$P_r(\vec{Z}_+) = \left| \vec{D}_{in}^{\dagger} \cdot \vec{Z}_+ \right|^2 = |\alpha|^2 = \left(\vec{D}_{in}^{\dagger} \cdot \vec{Z}_+\right)\left(\vec{Z}_+^{\dagger} \cdot \vec{D}_{in}\right)$$

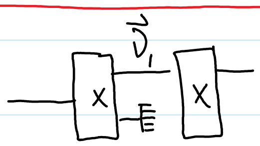
Will come back to this

$$Pr(\vec{z}_+) = |\vec{v}_{in}' \cdot \vec{z}_+|^2 = |\alpha|^2 = (\vec{v}_{in}^{\dagger} \cdot \vec{z}_+)(\vec{z}_+^{\dagger} \cdot \vec{v}_{in})$$

A9 Show that $(\vec{v}_{in}^{\dagger} \cdot \vec{z}_+)(\vec{z}_+^{\dagger}\, \vec{v}_{in}) =$

$$\vec{v}_{in}^{\dagger} \cdot \Pi_{z_+} \cdot v_{in} \quad \text{with } \Pi_{z_+} = \vec{z}_+ \cdot \vec{z}_+^{\dagger}$$

Normalization of the state



$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Pr(\vec{x}_+) = |\vec{v}_1^{\dagger} \cdot \vec{x}_+|^2 = 2$$

For a general state $\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow Pr(\vec{x}_+) = |\alpha|^2$

$Pr(\vec{x}_-) = |\beta|^2$

What do we expect for the $Pr(\vec{x}_+) + Pr(\vec{x}_-)$?

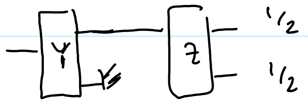
Should be 1. $\{\vec{x}_+, \vec{x}_-\}$ are the only two options so the probabilities of the two should add up to one.

$$\hookrightarrow \quad \vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow \vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Big/ (|\alpha|^2 + |\beta|^2)$$

A10 $\longrightarrow$ Check that this preserves the probability in all bases. (For all different measurements)

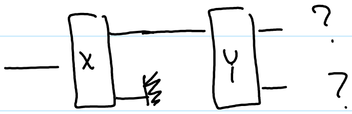
Is that all?

Think about this!
Is it easy to do all three directions of SGE?



$$\rightarrow \quad \vec{Y}_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} / \sqrt{2}$$

but then what does it mean for



If $Y_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2}$ then

$$Pr(\vec{Y}_+) = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$$

$$\vec{Y}_+ = \vec{X}_+ \Rightarrow \begin{array}{l} Pr_+ = 1 \\ Pr_- = 0 \end{array}$$

→ Not what we see in the exp. gives 50/50 outcomes.

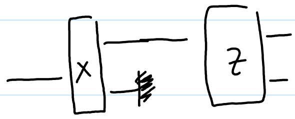
$$\Rightarrow \begin{cases} \vec{X}_\pm^\dagger \cdot \vec{Z}_+ = \dfrac{1}{\sqrt{2}} \\ \vec{Y}_\pm^\dagger \cdot \vec{Z}_+ = 1/\sqrt{2} \end{cases}$$

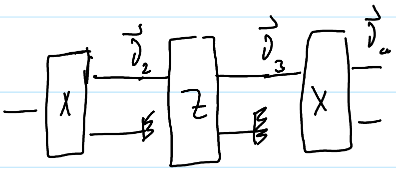
→ How is this possible?

α, β could be complex.

$$Z_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad X_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} / \sqrt{2} \qquad Y_\pm = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} / \sqrt{2}$$

$$Z_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$  ✓



$$\vec{v}_{in} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{cases} Pr(+) = |\alpha|^2 \\ \vec{v}_2 = \vec{X}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2} \end{cases}$$

$$\rightarrow \begin{cases} Pr(+) = 1/2 \\ \vec{v}_3 = \vec{Z}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

$$\rightarrow \begin{cases} Pr(+) = 1/2 \\ \vec{v}_4 = \vec{X}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2} \end{cases}$$

* Different representation of the state

We picked the outcomes of SGE in z direction to be $\vec{Z}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\vec{Z}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

But we could take them to be any other two orthogonal vectors.

A11 Redo the model above & take the $\vec{X}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\vec{X}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

* So far we have

* State $\rightarrow \vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$ complex

* State $\longrightarrow \vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$ complex

* Measurement $\Pr(\vec{i}) = \left| \vec{v}^{\dagger} \cdot \vec{i} \right|^2$ probability of getting $\vec{i}$

$\vec{v}_{out} = \vec{i}$ State after measurement

What about expectation values?

$\longrightarrow$ What's magnetization in certain direction?

SG z

$\alpha \longrightarrow \vec{z}_+ \longrightarrow m_z = \mu$

$\beta \longrightarrow \vec{z}_- \longrightarrow m_z = -\mu$

$$\Longrightarrow \langle m_z \rangle = \alpha \mu + \beta(-\mu)$$

Note that

$$\alpha = \left(\vec{v}_{in}^{\,\dagger} \cdot \vec{z}_+\right)\left(\vec{z}_+^{\,\dagger} \cdot \vec{v}_{in}\right)$$

$$\beta = \left(\vec{v}_{in}^{\,\dagger} \cdot \vec{z}_-\right)\left(\vec{z}_-^{\,\dagger} \cdot \vec{v}_{in}\right)$$

A12 $$\Longrightarrow \langle m \rangle = \vec{v}_{in}^{\,\dagger} \cdot \Big(\underbrace{\mu \vec{z}_+ \vec{z}_+^{\,\dagger} - \mu \vec{z}_- \vec{z}_-^{\,\dagger}}_{S_z}\Big) \cdot \vec{v}_{in}$$

$S_z \downarrow$ An operator

* Should be Hermitian.

Summary

① $\{\vec{v}_1, \vec{v}_2\} \rightarrow$ Outcomes are assigned orthonormal vectors that form a basis for a complex vector space with some inner product.

② $\{\vec{v}_1, \vec{v}_2\}$

$\{\vec{w}_1, \vec{w}_2\}$

Different measurements give different outcomes that are represented with different bases.

e.g. $\boxed{Z}$ $\left\{ \vec{z}_+ = \begin{pmatrix}1\\0\end{pmatrix},\ \vec{z}_- = \begin{pmatrix}0\\1\end{pmatrix} \right\}$

$\boxed{x}$ $\left\{ \vec{x}_+ = \begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2},\ \vec{x}_- = \begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2} \right\}$

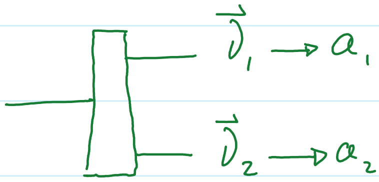
③ Probabilities of outcomes

$\vec{v}_{in}$ → $\vec{v}_1$, $\vec{v}_2$

$$Pr(\vec{v}_i) = |v_{in}^{\dagger} \cdot \vec{v}_1|^2$$

$$= \vec{v}_{in}^{\dagger} \cdot \Pi_i \cdot \vec{v}_{in} \quad \text{where } \Pi_1 = \vec{v}_1 \cdot \vec{v}_1^{\dagger}$$

④ Expected value for measurement of a quantity. (Let's call it A)



$\{a_1, a_2\}$ are the possible values of the quantity.

$$\langle A \rangle = a_1 \Pr(\vec{v}_1) + a_2 \Pr(\vec{v}_2)$$

$$= \vec{v}_{in}^{\,\dagger} \cdot \underbrace{\left(a_1 \Pi_1 + a_2 \Pi_2\right)}_{\text{Use this as a representation of } A} \cdot \vec{v}_{in}$$

$$\hat{A} = a_1 \Pi_1 + a_2 \Pi_2$$

$$= a_1 \vec{v}_1 \cdot \vec{v}_1^{\,\dagger} + a_2 \vec{v}_2 \cdot \vec{v}_2^{\,\dagger}$$

(A13) Take an experiment with 3 outcomes and build a similar model?