Exploring the convergence properties of the Relativistic Hydrodynamics

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Motivation and abstract

In this short report, we are going to study the convergence characters in the Relativistic Hydrodynamics (RH). We have seen before in my earlier talks that RH possesses a divergent series as solutions for its equation. It seems that gradient expansion never works for the initial time after the collision and it takes while to restore the hydrodynamic behavior for corresponding quantities. We talk of asymptotic series rather than convergent series for RH solutions and RH arises when the non-hydro modes start to cease. This where is the birth of RH which we call it as the hydrodynamization time. It is about $\tau_{hyd} \sim \frac{3}{T}$ and after it we are allowed to use at least second-order or causal hydro. This perception is acceptable and it originates from seeing the collective behavior for small collision systems. However, recently some people [1, 3] are talking about the possibility of having convergent series for RH equations. Their study either hydrodynamically or using the fluid/gravity correspondence shows that under some circumstances we are able to have a convergent series solution and dispersion relations might have finite radius of convergence. They argued that there exists a maximum bound for spatial momentum which beyond that the RH gradient expansion never works, while below that critical value RH might have a convergent series solution. This critical momentum is a model-dependent parameter but it sounds that its existence is universal. Historically, having this critical value backs to the seminal paper of Romatschke[4], which he derived the modes of retarded correlators in a weak coupling theory, i.e. kinetic theory with RTA and there he discussed the possibility of having hydro behavior for weakly interacting particles by going to the details of playing hydro and non-hydro modes. He showed that passing a critical value of spatial momentum, the non-hydro modes overwhelm the hydro modes and we don't have hydro modes, but below it, the hydro modes arise in the principal Riemann sheet and they are detectable. In this talk, I try to give you the baseline of this new stream in RH without going further into details and only make you familiar with this new trend.

1 Divergence of Gradient Expansion in hydro

RH as an effective field theory based upon two concepts: i) having stable and local equilibrium, ii) gradient expansion. The first one leads to the definitions of some local DoF and the second assumption is because of the effective nature of RH. We denote the macroscopic DoF collectively as ϕ, such as temperature, chemical potential and fluid's velocity. These are necessary to express the conserved currents perturbatively

$$\mathcal{G} = \mathcal{O}(\phi) + \mathcal{O}(\nabla\phi) + \mathcal{O}(\nabla^2\phi) + \dots . \tag{1}$$

Here, $\mathcal{G}$ stands for the general form of a conserved currents or tensors. Dynamics is given by the conservation equations

$$\nabla\mathcal{G} = 0. \tag{2}$$

Putting the constitutive relations (1) into the equation (2), we get the RH equations. The $\mathcal{O}(\phi)$ gives rise to the ideal fluid and Euler equation and higher order contributions such as $\mathcal{O}(\nabla^n\phi)$ with $n \geq 1$ leads to the Navier-Stokes equations and so on.

To review the divergence features in RH, we back to our previous talking. For an expanding plasma in the $1+1$ dimensions with Bjorken symmetry and in the Muller -Israel-Stewart(MIS) framework, we have seen that EOM are

$$\begin{aligned} \tau\frac{\partial\epsilon}{\partial\tau} &= -\frac{4}{3}\epsilon + \phi, \\ \tau_\pi\frac{\partial\phi}{\partial\tau} &= \frac{4\eta}{3\tau} - \frac{4\tau_\pi}{3\tau}\phi - \phi, \end{aligned} \tag{3}$$

which ϵ represents the energy and ϕ stands for the shear stress field. τ_π is the Relaxation time and η is the shear viscosity transport. After some algebraic manipulation, we simplify the equation (3) into the one non-linear differential equation

$$\begin{aligned} &C_{\tau_\pi} w f(w) f'(w) + 4C_{\tau_\pi} f(w)^2 + \left(w - \frac{16}{3}C_{\tau_\pi}\right) f(w) - \frac{4C_\eta}{9} + \frac{16C_{\tau_\pi}}{9} - \frac{2w}{3} = 0, \\ &w = \tau T, \qquad f(w) = \tau\frac{\dot{w}}{w}. \end{aligned} \tag{4}$$

More detailed calculations are given in the paper [5]. Seeking a series solution gives us

$$f(w) = \sum_{n=0}^{\infty} \frac{a_n}{w^{n+1}}, \quad a_0 = \frac{2}{3}, \quad a_1 = \frac{4C_\eta}{9},$$
$$a_{n+1} = C_{\tau_\pi} \left(\frac{16}{3} a_n - (4 - \frac{n}{2}) \sum_{k=0}^{n} a_k a_{n-k} \right), \tag{5}$$

which $a_n \sim C_{\tau_\pi}^n n!$. Therefore, a series solution is not a convergent one, but it is Borel resummable. In the Borel plane ξ, the Borel transformed function is

$$\mathcal{B}(f)(\xi) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \xi^n. \tag{6}$$

It has poles which make ambiguous the resummation process when passing through the poles either going upward or downward of them. In the Fig 1, a schematic version of resummation of the original function in the Borel plane is shown. The ambiguity of

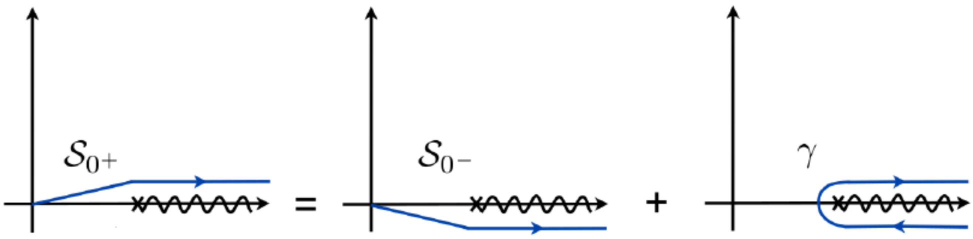

Figure 1: Resumming the original function in the case of having some poles or branch cut. Going up or down of the real axis is not identical and they differ by imaginary contributions, reminiscent of those poles. In the RH language these poles are non-hydro modes.

resummation is purely imaginary and comes from the many poles sitting on the real axis and can be evaluated by Cauchy theorem

$$Im(\mathcal{S}f(-w)) \sim \pm \pi e^{-\frac{w}{C_{\tau_\pi}}}. \tag{7}$$

For the hydro equation (5) the poles are located on the real axis in the form of $\frac{3k}{2C_{\tau_\pi}}$ with $k = 1, 2, \ldots$. It is suggested that solution to the equation (4) is a trans-series solution incorporated both the perturbative and non-perturbative ones

$$f(w) = \sum_{n=0}^{\infty} \left(\sigma e^{-\frac{3w}{2C_{\tau_\pi}}} w^{-\frac{C_\eta}{C_{\tau_\pi}}} \right)^n \phi_n(w) = \underbrace{\phi_0(w)}_{\text{perturbative}} + \underbrace{\sum_{n=1}^{\infty} \left(\sigma e^{-\frac{3w}{2C_{\tau_\pi}}} w^{-\frac{C_\eta}{C_{\tau_\pi}}} \right)^n \phi_n(w)}_{\text{non-perturbative}}. \tag{8}$$

σ is the trans-series parameter and resembles the role of initial conditions and somehow it is fixed by the reality condition. Relation between the non-perturbative and perturbative sections make the above series to be a trans-series one.
The $\phi_0(w)$ encodes the derivative expansion information and higher $\phi_n(w)$ are because of the presence of non-hydro modes (poles in the Borel plane) which spoils the convergence of gradient expansion. In this language, hydrodynamic is valid when the non-hydro mode sector is off or died which translate to a late time such as $\tau > \tau_c \sim \frac{1}{T}$. After this time it is supposed that RH have finite radius of convergence and importance of non-hydro modes vanishes. In what follows, we go to give a different scenario and claim that:
There is a maximum spatial momentum limit which we cal it as q_c, in such a way that for $k < q_c$ the RH might have convergent series solution or finite radius of convergence, while for $k > q_c$ the above mentioned scenario of trans-series and late time appearance of hydro occurs.

2 Convergence in the RH

Before going into the details of the paper [3], it is useful to briefly address the main points of the paper [2], since the Heller work mainly follows the Grozdanov's job to approve the existence of convergence in RH. Difference between these two works is that in the paper [2] or in the parent one [1], the authors try to do the works from gauge/gravity duality and motivate people to look for the convergence in the original RH without being in the strong coupling regime.
In the gauge/gravity duality the boundary hydrodynamics equations relate to the perturbed Einstein equation. This formalism is nicely described in the seminal paper of Minwalla, et al [6]. We have to perturb the metric components and solve the Einstein equation perturbatively. In the asymptotically ADS space-time (we take here it is five dimensions), the "boosted black brane" metric is

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 \Delta_{\mu\nu} dx^\mu dx^\nu, \tag{9}$$

with

$$f(r) = 1 - \frac{1}{r^4}, \quad u^i = \frac{\beta_i}{\sqrt{1-\beta^2}}, \quad u^v = \frac{1}{\sqrt{1-\beta^2}}, \quad T = \frac{1}{\pi b}. \tag{10}$$

which have to satisfy the Einstein equation with negative cosmological constant

$$R_{MN} - \frac{1}{2} g_{MN} R - 6 g_{MN} = 0, \quad R = -20. \tag{11}$$

In simple words, the fluid/gravity correspondence works as it follows. Promote the u_μ and b field to local ones, say $(u_\mu(x), b(x))$. It is evident that Einstein equation (11) doesn't hold anymore. Therefore, we have to add to the metric components given in (9) a local ones $g_{MN}(x)$ such that Einstein equation again does hold. This adding procedure is done perturbatively, namely as a Taylor series in x^μ. This Taylor series is equivalent to the gradient expansion of RH. A main point is that some parts of Einstein equations, namely those at the $r = c(c \to \infty)$ surface are exactly the hydrodynamics equations of the boundary theory. Therefore, solving Einstein equation perturbatively in this manner translates to the gradient expansion of RH equation boundary theory. For example, ideal fluid corresponds to the no "x" dependence in the fields and the derived energy-momentum tensor of the boundary theory is

$$T^{\mu\nu} = \frac{1}{b_0^4} \left(\eta^{\mu\nu} + 4u^\mu u^\nu \right). \tag{12}$$

The conservation equation $\partial_\mu T^{\mu\nu} = 0$ is satisfied trivially. In the first order of expansions such as

$$\begin{aligned} &u^i = u_0^i + \epsilon u_1^i + \mathcal{O}(\epsilon^2), \qquad b = b_0 + \epsilon b_1 \mathcal{O}(\epsilon^2), \\ &g = g_0 + \epsilon g_1 + \mathcal{O}(\epsilon^2), \end{aligned} \tag{13}$$

we can show that the energy-momentum tensor of the boundary theory is

$$T^{\mu\nu} = \frac{1}{b_0^4} \left(\eta^{\mu\nu} + 4u^\mu u^\nu \right) - \frac{2}{b^3} \sigma^{\mu\nu}, \tag{14}$$

and the conservation equation $\partial_\mu T^{\mu\nu} = 0$ is nothing but the Navier-Stokes equation. This process can be continued to the infinity and is equal to the gradient expansion of the boundary RH equations in a small parameter $\frac{\ell_{mic}}{\ell_{mac}} < 1$.

In the paper [2] the above mentioned procedure is applied for special channels, namely the shear and sound channel

$$\delta g(x^\mu) = 2 \underbrace{h_{xy}(x^\mu)}_{\text{shear channel}} dxdy + 2 \underbrace{h_{tz}(x^\mu)}_{\text{sound channel}} dtdz, \tag{15}$$

and try to solve the Einstein equation (11) perturbatively. The equations leads to [2]

$$
\begin{aligned}
&\text{Shear channel,} \qquad Z_1'' - \frac{f(\mathfrak{w}^2 - \mathfrak{q}^2 f) - u\mathfrak{w}^2 f'}{uf(\mathfrak{w}^2 - \mathfrak{q}^2 f)} Z_1' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2} Z_1 = 0, \\
&\text{Sound channel,} \qquad Z_2'' - \mathcal{A} Z_2' + \mathcal{C} Z_2 = 0, \\
&\mathcal{A} = \frac{3\mathfrak{w}^2(1+u^2) + \mathfrak{q}^2(2u^2 - 3u^4 - 3)}{uf(3\mathfrak{w}^2 + \mathfrak{q}^2(u^2-3))}, \quad \mathcal{C} = \frac{3\mathfrak{w}^4 + \mathfrak{q}^4(u^4 - 4u^2 + 3) + \mathfrak{q}^2(4u^5 - 4u^3 + 4u^2\mathfrak{w}^2 - 6\mathfrak{w}^2)}{uf^2(3\mathfrak{w}^2 + \mathfrak{q}^2(u^2-3))}.
\end{aligned}
\tag{16}
$$

Here, $f = 1 - u^2$ and $\mathfrak{w} = \frac{\omega}{2\pi T}$ and $\mathfrak{q} = \frac{q}{2\pi T}$. Special way has to be taken to solve these equations, consistent with the physics of black hole and the boundary theory[2]. It is worthwhile to mention that both of $\mathfrak{q}$ and $\mathfrak{w}$ are taken to be in the $\mathbb{C}^2$ plane. The $\mathfrak{w}$ is called the quasinormal modes of the black hole. The shear and sound channel have the follwing expansion for quasinormal modes

$$
\begin{aligned}
\mathfrak{w}_{shear} &= -i \sum_{n=1}^{\infty} c_n \mathfrak{q}^{2n} = -ic_1 \mathfrak{q}^2 + \dots, \\
\mathfrak{w}^{\pm}_{sound} &= -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} (\mathfrak{q}^2)^{\frac{n}{2}} = \pm a_1 \mathfrak{q} + ia_2 \mathfrak{q}^2 \dots.
\end{aligned}
\tag{17}
$$

Giving the $\mathfrak{q}$ will enable us to get the $\mathfrak{w}$s. The author argued in first pages that what are the nature of equations which we derive modes from them. Suppose that the general form of equation of modes is $P(q^2, \omega) = 0$. By definition, a regular point is a point where

$$
P(q_r^2, \omega_r) = 0, \qquad \left.\frac{\partial P(q^2, \omega)}{\partial \omega}\right|_{\omega_r} \neq 0. \tag{18}
$$

If a point such as this one to be exist, then we could run the normal Taylor expansion around the (q_r^2, ω_r)

$$
\omega = \sum_{n=1}^{\infty} a_n (q^2 - q_r^2)^n = a_1(q^2 - q_r^2) + \dots. \tag{19}
$$

In other side, a critical point of order "p" is a point which

$$
P(q_c^2, \omega_c) = 0, \quad \left.\frac{\partial P(q^2, \omega)}{\partial \omega}\right|_{\omega_c} = 0, \dots \left.\frac{\partial^{p-1} P(q^2, \omega)}{\partial \omega^{p-1}}\right|_{\omega_c} = 0, \ \left.\frac{\partial^{p} P(q^2, \omega)}{\partial \omega^{p}}\right|_{\omega_c} \neq 0. \tag{20}
$$

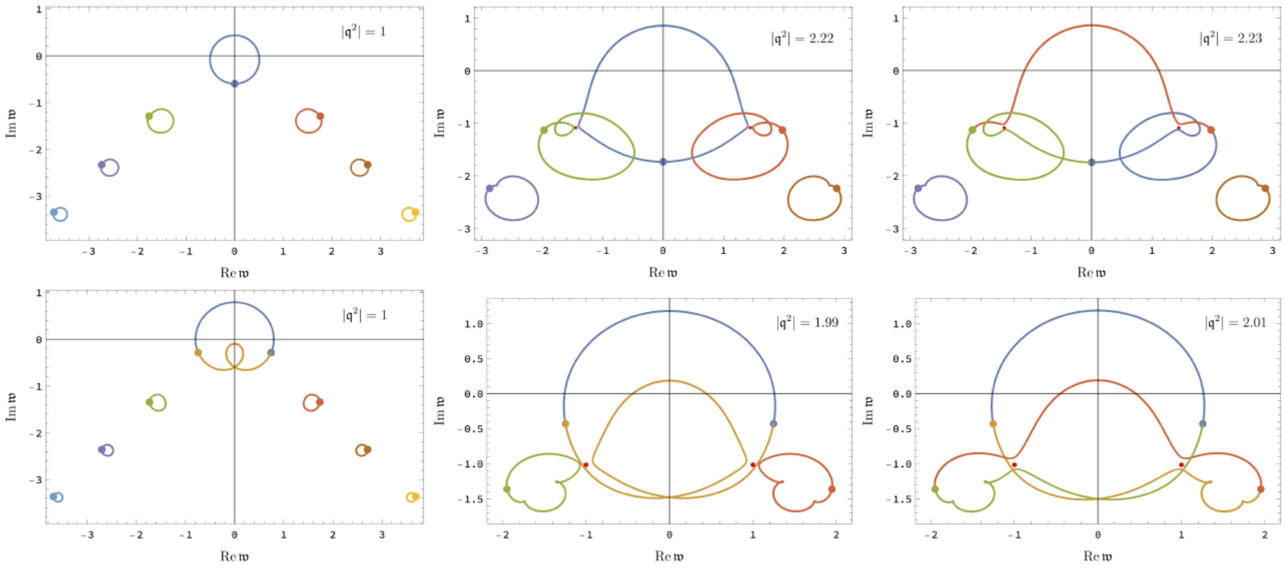

Figure 2: Solutions for quasinormal modes in a special $\mathfrak{q}$. The above row is for shear channel and bottom row is for sound channel. Turning around each mode is represented by varying the θ angle, $\mathfrak{q}^2 = |\mathfrak{q}|^2 e^{i\theta}$. When the modes start to collide with each other, we start to lose the convergence.

If (q_c^2, ω_c) is such a point, then we could not do the regular Taylor expansion, but we have p branches for $(\omega_i(q^2),\ i = 1, \dots p)$ which have fractional expansion around q_c^2

$$\omega_i = \sum_{n=1}^{\infty} a_n^i (q^2 - q_c^2)^{\frac{n}{b_i}}. \tag{21}$$

This called the Puiseux series. In the Puiseux series, radius of convergence is the distance from origin to the first critical point. In this work, this distance is the location where the quasinormal modes start to collide with each other. In the Fig. 2 the real and imaginary parts of qusinormal modes for special $\mathfrak{q}$ is sketched As you see by increasing the $|\mathfrak{q}|^2$, the modes start to collide. For shear and sound channel the critical value of this momentum is

$$\begin{aligned} &\text{Shear channel}, \qquad |\mathfrak{q}|_c^2 = 1.49, \\ &\text{Sound channel}, \qquad |\mathfrak{q}|_c^2 = 1.41. \end{aligned} \tag{22}$$

Therefore, for shear modes if $\mathfrak{q} < 1.49$ we have finite radius of convergence and gradient expansion of RH works and if $\mathfrak{q} < 1.41$ for sound modes, we have again finite radius of convergence for gradient expansion and it works. This means that at small momentum limit, we restore gradient expansion formalism without any need to trans-series or such stuffs, but at higher momentum we have to trust to RH at later times when the hydrodynamization time arises. It has to be mentioned that these critical values corresponds to large momenta. For example, if $T = 300MeV$, then we can have finite radii of convergence up to the $k \sim 2.63GeV$.

Now which we would familiar with the topic, let us to address the points of the paper [3]. As we see, the expanding plasma with say Bjorken symmetry in $1+1$ dimension have zero radius of convergence, but the strong regime calculations show that the gradient series in dispersion relation could have finite radius of convergence. Does this tension is real or not? In this paper [3], the authors try to resolve the apparent inconsistency and they find that

The real space hydrodynamic gradient expansion diverges if initial data support in momentum space exceeding a critical value, and convergences otherwise. This critical value is an intrinsic property of the microscopic theory and corresponds to a branch point of the spectrum where hydrodynamic and nonhydrodynamic modes first collide.

Their work is under completion and the final and more detailed version of their calculation is supposed to release soon. Just in this paper, they try to clarify the idea.

Without going further into the details, I just highlight the main points. For uncharged system, the general form of energy-momentum tensor can be written as

$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P})u^\mu u^\nu + \mathcal{P}\eta^{\mu\nu} + \Pi^{\mu\nu}. \tag{23}$$

They focus on the conformal fluid which possesses some constrains over the quantities such as $\Pi^\mu_\mu = 0$ and $\mathcal{P} = \frac{\mathcal{E}}{d-1}$ and the general form of the viscous tensor can be written as

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \tau_\pi \eta \mathcal{D}\sigma^{\mu\nu} - \frac{1}{2}\theta_1 \mathcal{D}_\alpha \mathcal{D}^\alpha \sigma^{\mu\nu} - \theta_2 \mathcal{D}^{<\mu}\mathcal{D}^{\nu>}\mathcal{D}_\alpha u^\alpha + \dots. \tag{24}$$

Here, $\mathcal{D} \equiv u^\mu \partial_\mu, \mathcal{D}^\mu = \Delta^{\mu\nu}\partial_\nu$ and $\sigma^{\mu\nu} = 2\mathcal{D}^{<\mu}u^{\nu>}$. Ellipsis denotes the higher than third order terms of gradients. We can evaluate the hydro modes by the method described vastly in our previous talks

$$\begin{aligned} \omega_{shear} &= -i\frac{\eta}{sT}k^2 - i(\frac{\eta^2\tau_\pi}{s^2T^2} - \frac{\theta_1}{2sT})k^4 + \dots, \\ \omega^\pm_{sound} &= \pm c_s (k^2)^{\frac{1}{2}} - i\Gamma k^2 \mp \frac{\Gamma}{2c_s}(\Gamma - 2c_s^2\tau_\pi)(k^2)^{\frac{3}{2}} + \dots. \end{aligned} \tag{25}$$

The main properties of the paper [3] is to understand the properties of gradient expansion in relation (24) in the linearized regime in real space that facilitate comparison with earlier studies. They propose a novel way to parametrize $\Pi^{\mu\nu}$ which is constructed from three tensorial structures

$$\sigma_{ij} = \left(\partial_i u_j + \partial_j u_i - \frac{2}{d-1}\delta_{ij}\partial_k u^k\right),$$
$$\pi^{\epsilon}_{ij} = \left(\partial_i\partial_j - \frac{1}{d-1}\delta_{ij}\partial^2\right)\epsilon,$$
$$\pi^{u}_{ij} = \left(\partial_i\partial_j - \frac{1}{d-1}\delta_{ij}\partial^2\right)\partial_k u^k. \tag{26}$$

We take $\mathcal{E} = \epsilon_0 + \epsilon$. In this manner, we can write the general of viscous tensor as

$$\Pi_{ij} = -A(\partial^2)\sigma_{ij} - B(\partial^2)\pi^{\epsilon}_{ij} - C(\partial^2)\pi^{u}_{ij}, \tag{27}$$

where A, B and C are infinite series in spatial derivatives

$$(A, B, C) = \sum_{n=0}^{\infty}(a_n, b_n, c_n)\,(-\partial^2)^n. \tag{28}$$

The (a_n, b_n, c_n) coefficients are nothing but the transport parameters. For some simple choices, we could derive the hydro modes. If we take $\mathbf{u} = (u_1, 0, \ldots, 0)$ and $\mathbf{k} = (0, \ldots, k)$ we can derive the shear modes

$$\omega_{shear} = -i\frac{1}{sT}\sum_{n=0}^{\infty} a_n (k^2)^{n+1}. \tag{29}$$

If we take $\mathbf{u} = (0, \ldots, u_{d-1})$ and $\mathbf{k} = (0, \ldots, k)$, by solving the following matrix valued equation, we are able to derive the sound modes

$$\begin{aligned}
&-i\omega\epsilon + iksTu_{d-1} = 0,\\
&-i\omega sTu_{d-1} + \frac{1}{d-1}ik\epsilon + \frac{d-2}{d-1}\sum_{n=0}^{\infty}(2a_n - b_{n-1})(k^2)^{n+1}u_{d-1} + \frac{d-2}{d-1}\sum_{n=0}^{\infty} ic_n(k^2)^{n+\frac{3}{2}}\epsilon = 0.
\end{aligned} \tag{30}$$

Important thing to look for the convergence properties, is to know about the growth of transport parameters. We could say that (a_n, b_n, c_n) grow in a manner controlled by the position of branch points closest to the origin $k = 0$[7]

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = |k^A_\star|^{-2}. \tag{31}$$

Similar expression holds for b_n and c_n, say $|k^B_\star|^{-2}$ and $|k^C_\star|^{-2}$. Therefore, the convergence properties of viscous tensor in the relation (27) firmly depends on the transport parameters

(a_n, b_n, c_n) and their kind of growth as well as the particular solutions ϵ and $\mathbf{u}$. The support in momentum space plays a crucial role to determine the radius of convergence. According to the Paley-Wiener theorem [8], the Fourier transform of a square-integrable function $\hat{f}(k)$ supported in $|k| < |k_{max}|$ is an entire function of exponential type $|k_{max}|$ and it follows

$$\lim_{n\to\infty} \sup|f^{(n)}(x)|^{\frac{1}{n}} = |k_{max}|. \tag{32}$$

Combining the equations (31) and (32), we reach to

$$\lim_{n\to\infty} \sup|a_n \partial_x^{2n} \sigma_{ij}|^{\frac{1}{n}} = \frac{|k_{max}|^2}{|k_\star^A|^2}. \tag{33}$$

For convergence criterium we should have $\frac{|k_{max}|^2}{|k_\star^A|^2} < 1$. In a simple example, they try to derive the $|k_{max}|$. For $\epsilon = 0$ and $\mathbf{u} = (u_1, 0, \ldots)$, we have just the following non-vanishing component

$$\Pi_{1,d-1} = -\sum_{n=0}^{\infty} a_n \partial_x^{2n+1} u_1(t,x). \tag{34}$$

This channel contribute to the shear mode and the general structure of shear mode is

$$\omega_{shear} = i\frac{-1+\sqrt{1-4D\tau_\pi k^2}}{2\tau_\pi}, \qquad D = \frac{\eta}{sT}. \tag{35}$$

Comparing the latter equation with equation (29), we could derive the a_n such as

$$a_n = sT\mathcal{C}_n \tau_\pi^n D^{n+1}. \tag{36}$$

Therefore, we have

$$|k_\star^A| = \left(\lim_{n\to\infty} \sup|a_n|^{\frac{1}{n}}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{4D\tau_\pi}}, \tag{37}$$

which is also the location of branch point in equation (35). In the Fig 3, they look for the ratios of successive contributions for various $|k_{max}|$. The figure was drawn for $(t,x) = (1, 0.5)$ with $(s = T = \eta = \tau_\pi = 1)$ which leads to the $|k_\star^A| = \frac{1}{2}$. If $k_{max} < \frac{1}{2}$ we lead to the convergent series, while diverges otherwise. In the inset picture, you see the factorial growth of δ_n when $k_{max} \to \infty$, the well known realization of divergence in the RH addressed in lots of papers.

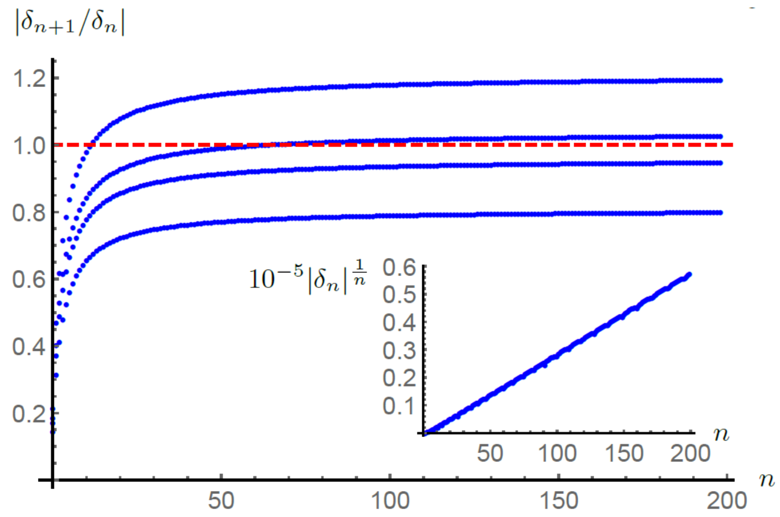

Figure 3: Ratios of successive contributions for the given values described in text. From top to bottom, $k_{max} = 0.55, 0.51, 0.49, 0.45$. The gradient expansion is convergent for $k_{max} < |k_\star^A| = \frac{1}{2}$ and diverges otherwise. Inset: when $k_{max} \to \infty$ the δ_n was sketched. As you see, factorial growth manifests itself.

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