Lecture 6-QM_II (17/07/00)

Saturday, October 9, 2021 2:14 PM

$\vec{S} = (S_x, S_y, S_z)$

$[S_i, S_j] = i\epsilon_{ijk} S_k$ $\qquad$ $[L_i, L_j] = i\epsilon_{ijk} L_k$

$\vec{S}^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$ $\qquad$ $L^2 |\ell, m_\ell\rangle$

$S_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$ $\qquad$ $L_z |\ell, m_\ell\rangle$

$S_\pm = S_x \pm i S_y$

$S_\pm |s, m_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s \pm 1)} \; |s, m_s \pm 1\rangle$

$[S_\pm, S_z] = \pm\hbar S_\pm$

$|s = \frac{1}{2}, m_s = +\frac{1}{2}\rangle = |\uparrow\rangle$

$|s = \frac{1}{2}, m_s = -\frac{1}{2}\rangle = |\downarrow\rangle$

$$\begin{cases} S_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \\ S_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle \\ S_x |\uparrow\rangle = \frac{\hbar}{2} |\downarrow\rangle \\ S_x |\downarrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \\ S_y |\uparrow\rangle = +\frac{i}{2}\hbar |\downarrow\rangle \\ S_y |\downarrow\rangle = -\frac{i}{2}\hbar |\uparrow\rangle \end{cases}$$

$S_+ |\uparrow\rangle = 0$

$S_+ |\downarrow\rangle = \hbar |\uparrow\rangle$

$S_- |\uparrow\rangle = \hbar |\downarrow\rangle$

$S_- |\downarrow\rangle = 0$

$S_x = \frac{1}{2}(S_+ + S_-)$

$S_y = \frac{-i}{2}(S_+ - S_-)$

$(S_x, S_y, S_z) = \vec{S}$

$$\boxed{\vec{S} = \frac{\hbar}{2}\vec{\sigma}}$$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ $\quad$ 2×2

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad$ Pauli matrices

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\vec{S} = \begin{pmatrix} \langle\uparrow|\vec{S}|\uparrow\rangle & \langle\uparrow|\vec{S}|\downarrow\rangle \\ \langle\downarrow|\vec{S}|\uparrow\rangle & \langle\downarrow|\vec{S}|\downarrow\rangle \end{pmatrix}$$

* 1) $[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\,\sigma_k$ $\qquad$ $\{A, B\} = AB + BA$

2) $\sigma_i^2 = 1_{2\times 2}$

* 3) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\, \mathbb{1}_{2\times 2}$ $\qquad$ $\mathbb{1}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\frac{**}{1\;3} \rightarrow$ 4) $\sigma_i \sigma_j = \delta_{ij} \mathbb{1}_{2\times 2} + \epsilon_{ijk}\sigma_k$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sigma_i a_i \; \sigma_j b_j = (\delta_{ij} + \epsilon_{ijk}\sigma_k)\, a_i b_j = \vec{a}\cdot\vec{b}\, \mathbb{1}_{2\times 2} + (\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

5) $Tr\, \sigma_i = 0$ $\qquad \forall i = x, y, z$

6) $\det \sigma_i = -1$ $\qquad \forall i = x, y, z$

$|s = \frac{1}{2}, m_s = +\frac{1}{2}\rangle = |\uparrow\rangle \overset{!}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ حمل

$|s = \frac{1}{2}, m_s = -\frac{1}{2}\rangle = |\downarrow\rangle \overset{!}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|s=\tfrac{1}{2},\ m_s=-\tfrac{1}{2}\rangle = |\downarrow\rangle \stackrel{!}{=} \begin{pmatrix}0\\1\end{pmatrix}$$

اثبات :

$$S_z\,|\uparrow\rangle = \frac{\hbar}{2}\,|\uparrow\rangle \qquad\qquad S_z=\frac{\hbar}{2}\sigma_z$$

$$|\uparrow\rangle = \begin{pmatrix}a\\b\end{pmatrix} \qquad\qquad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\cancel{\frac{\hbar}{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = +\cancel{\frac{\hbar}{2}}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\begin{pmatrix}a\\-b\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \Rightarrow \begin{matrix}b=0\\ a\end{matrix}$$ دلخواه

$$\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a\\0\end{pmatrix} \rightarrow |a|^2+|b|^2=1 \Rightarrow |a|^2=1 \rightarrow \underline{a=1}$$

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} = \chi_+$$ اسپینور Spinor

$$S_z\,|\downarrow\rangle = -\frac{\hbar}{2}\,|\downarrow\rangle \rightarrow \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}a\\b\end{pmatrix}$$

$$|\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix} = \chi_-$$

$$a=-a \Rightarrow a=0$$
$$-b=-b \rightarrow b=1$$

$$\vec{a}=(a_x,a_y,a_z)$$

$$U_{\delta\vec\varphi} = e^{\frac{i}{\hbar}\delta\vec\varphi\cdot\vec L} \qquad \delta\vec\varphi=\delta\varphi\,\hat n$$

$\vec L =$ عملگر تکانه زاویه‌ای

$$\vec L = \vec x\times\vec p = \frac{\hbar}{i}\,\vec x\times\vec\nabla$$
$$L_k = \frac{\hbar}{i}\,\epsilon_{ijk}\,x_i\partial_j$$

$$U_{\delta\varphi}\,f(\vec x) = f(\vec x\,')$$
$$\vec x\,' = \vec x + \delta\vec\varphi\times\vec x$$

- $$e^{\frac{i}{\hbar}\delta\vec\varphi\cdot\vec L} f(\vec x) = \left(1+\frac{i}{\hbar}\,\delta\vec\varphi\cdot\vec L\right) f(\vec x) + O(\delta\varphi^2)$$

$$= f(\vec x) + \frac{i}{\hbar}\,\delta\varphi_k\,L_k\,f(\vec x) + O(\delta\varphi^2)$$

$$= f(\vec x) + \cancel{\frac{i}{\hbar}}\,\delta\varphi_k\,\cancel{\frac{\hbar}{i}}\,\epsilon_{ijk}\,x_i\,\partial_j f + O(\delta\varphi^2)$$

$$= f(\vec x) + (\delta\vec\varphi\times\vec x)\,\vec\nabla f(\vec x) + O(\delta\varphi^2)$$

$$= f(\vec x+\delta\vec\varphi\times\vec x) = f(\vec x\,')$$

$$U_{\delta\varphi}\,\psi(\vec x) = \psi(\vec x\,')$$
$$U_{\delta\varphi}\,\langle\vec x|\psi\rangle = \langle\vec x\,'|\psi\rangle$$

$$|\psi\rangle \xrightarrow{\delta\varphi} |\psi\rangle$$

$$\langle\vec x\,'|\psi\rangle$$

$$\begin{pmatrix}\cos\varphi&-\sin\varphi&0\\ \sin\varphi&\cos\varphi&0\\ 0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x'\\y'\\z'\end{pmatrix}$$
$$\mathcal{R}\ \vec x = \vec x\,'$$

$$\vec x\,' = \vec x + \delta\vec\varphi\times\vec x + O(\delta\varphi^2) \qquad\qquad \delta\vec\varphi = \delta\varphi\,\hat e_z$$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}\times\begin{pmatrix}x\\y\end{pmatrix} + O(\delta\varphi^2)$$

$$\vec{x}' = \vec{x} + \delta\vec{\varphi} \times \vec{x} + O(\delta\varphi^2) \qquad \delta\vec{\varphi} = \delta\varphi\, \hat{e}_z$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \delta\varphi \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} + O(\delta\varphi^2)$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x - \delta\varphi\, y \\ \delta\varphi\, x + y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -\delta\varphi & 0 \\ \delta\varphi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + O(\delta\varphi^2)$$

$$= \begin{pmatrix} \cos\delta\varphi & -\sin\delta\varphi & 0 \\ \sin\delta\varphi & \cos\delta\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{x}' = e^{\frac{i}{\hbar}\delta\vec{\varphi}\cdot\vec{L}}\, \vec{x}$$

$$\delta\vec{\varphi} = \delta\varphi\, \hat{e}_z$$
$$\delta\vec{\varphi}\cdot\vec{L} = L_z\, \delta\varphi = \delta\varphi\, \frac{\hbar}{i}(x\partial_y - y\partial_x)$$

$$= \left(1 + \frac{i}{\hbar}\delta\varphi L_z + O(\delta\varphi^2)\right)\vec{x}$$

$$= \left(1 + \delta\varphi\,(x\partial_y - y\partial_x) + O(\delta\varphi^2)\right)\vec{x} \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -y\,\delta\varphi \\ x\,\delta\varphi \\ 0 \end{pmatrix} + O(\delta\varphi^2)$$

$$= \begin{pmatrix} 1 & -\delta\varphi & 0 \\ \delta\varphi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + O(\delta\varphi^2)$$

$\vec{L}$ مولد دوران در فضای مکان

مولد دوران در فضای اسپینور: $\vec{S}$

$\vec{\varphi} = \varphi\,\hat{n}$

$\vec{\varphi}\cdot\vec{S} = \varphi S_z$ اگر دوران حول محور z باشند

$$S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}$$

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = e^{\frac{i}{\hbar}\overbrace{\varphi S_z}^{\vec{\varphi}\cdot\vec{S}}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{\hbar}\varphi S_z\right)^n \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} \overset{!}{=} \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i}{2}\varphi} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$|s = \tfrac{1}{2}, m_s = +\tfrac{1}{2}\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi_+$$

$$|s = \tfrac{1}{2}, m_s = -\tfrac{1}{2}\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi_-$$

$$|s\rangle = \alpha_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_- \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$|\psi\rangle = \sum_n c_n |n\rangle$ قضیه بسط

$$|s\rangle = \alpha_+ |\uparrow\rangle + \alpha_- |\downarrow\rangle$$

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تعبیر احتمال

$|s\rangle = \alpha_+ |\uparrow\rangle + \alpha_- |\downarrow\rangle$

?

تعبیر احتمال

$c_n = \langle n | \psi \rangle$

$\sum_n |c_n|^2 = 1$

$\alpha_+ = \langle \uparrow | s \rangle$

$\alpha_- = \langle \downarrow | s \rangle$

$S_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$

$|\alpha_+|^2$

$$\langle \uparrow | s \rangle = \alpha_+ \langle \uparrow | \uparrow \rangle + \alpha_- \langle \uparrow | \downarrow \rangle$$

$$= \alpha_+ \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{=1} + \alpha_- \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{=0} = \alpha_+$$

$$\langle \downarrow | s \rangle = \alpha_+ \underbrace{\langle \downarrow | \uparrow \rangle}_{=0} + \alpha_- \underbrace{\langle \downarrow | \downarrow \rangle}_{=1} = \alpha_-$$

$S_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle$

$|\alpha_+|^2 + |\alpha_-|^2 = 1$?!

$$\langle s | s \rangle = 1 = \left(\alpha_+^* \langle \uparrow | + \alpha_-^* \langle \downarrow |\right)\left(\alpha_+ |\uparrow\rangle + \alpha_- |\downarrow\rangle\right)$$

$$= |\alpha_+|^2 \underbrace{\langle \uparrow | \uparrow \rangle}_{=1} + |\alpha_-|^2 \underbrace{\langle \downarrow | \downarrow \rangle}_{=1} + 0$$

$$\boxed{1 = |\alpha_+|^2 + |\alpha_-|^2}$$

• رابطه کامل بودن

$$|\psi\rangle = \sum_n c_n |n\rangle \qquad (1)$$

$$\sum_n |n\rangle \langle n| = 1$$

اثبات • $\langle n | \psi \rangle = c_n \quad \xrightarrow{(1)} \quad |\psi\rangle = \sum_n |n\rangle \langle n | \psi \rangle \quad *$

$$\boxed{1 = \sum_n |n\rangle \langle n|}$$

$|s\rangle = \alpha_+ |\uparrow\rangle + \alpha_- |\downarrow\rangle \qquad \alpha_+ = \langle \uparrow | s \rangle$

$\alpha_- = \langle \downarrow | s \rangle$

$$|s\rangle = |\uparrow\rangle\langle \uparrow | s \rangle + |\downarrow\rangle \langle \downarrow | s \rangle$$

$$\boxed{1 = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|}$$

• $\vec{S}$

$\vec{\varphi} = \varphi \hat{n}$

$$e^{\frac{i}{\hbar}\vec{\varphi}\cdot\vec{S}} = U$$

$$[\vec{S}, \vec{x}] = 0$$

$$[\vec{S}, \vec{p}] = 0$$

$$[\vec{S}, \vec{L}] = 0$$

• بدون اسپین :

$$|\psi\rangle = \int d^3x' \, c_{\vec{x}'} \, |\vec{x}'\rangle$$

$$\psi(\vec{x}) = \langle \vec{x} | \psi \rangle = \int d^3x' \, c_{\vec{x}'} \underbrace{\langle \vec{x} | \vec{x}' \rangle}_{\delta(\vec{x}-\vec{x}')} = c_{\vec{x}}$$

$$\boxed{|\psi\rangle = \int d^3x' \, \psi(\vec{x}') \, |\vec{x}'\rangle}$$

قضیه بسط

$$\delta(\vec{x}-\vec{x}')$$

$$\boxed{|\psi\rangle = \int d^3x' \;\; \psi(\vec{x}') \;\; |\vec{x}'\rangle}$$

قضیه بسط

• با در نظر گرفتن اسپین :

$$|\vec{x}'\rangle \otimes |\uparrow\rangle$$

$$|\vec{x}'\rangle \otimes |\downarrow\rangle$$

$$|\Psi\rangle = \int d^3x' \left(\psi_+(\vec{x}') \, \underbrace{|\vec{x}'\rangle \otimes |\uparrow\rangle} + \psi_-(\vec{x}') \, \underbrace{|\vec{x}'\rangle \otimes |\downarrow\rangle} \right)$$

$$\langle \vec{x} | \Psi \rangle = \int d^3x' \left(\psi_+(\vec{x}') \underbrace{\langle \vec{x} | \vec{x}' \rangle}_{\delta(\vec{x}-\vec{x}')} \otimes |\uparrow\rangle + \psi_-(\vec{x}') \underbrace{\langle \vec{x} | \vec{x}' \rangle}_{\delta(\vec{x}-\vec{x}')} \otimes |\downarrow\rangle \right)$$

$$\boxed{\langle \vec{x} | \Psi \rangle = \psi_+(\vec{x}) \, |\uparrow\rangle + \psi_-(\vec{x}) \, |\downarrow\rangle}$$

$$= \psi_+(\vec{x}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_-(\vec{x}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\langle \vec{x} | \Psi \rangle = \begin{pmatrix} \psi_+(\vec{x}) \\ \psi_-(\vec{x}) \end{pmatrix}$$

$$\begin{cases} \langle \uparrow | \langle \vec{x} | \Psi \rangle = \psi_+(\vec{x}) \underbrace{\langle \uparrow | \uparrow \rangle}_{=1} + \cancel{\psi_-(\vec{x}) \underbrace{\langle \uparrow | \downarrow \rangle}_{=0}} = \psi_+(\vec{x}) \\ \langle \downarrow | \langle \vec{x} | \Psi \rangle = \psi_-(\vec{x}) \end{cases}$$

$$\mathcal{H} = \mathcal{L}_2(\mathbb{R}^3) \otimes \mathbb{C}^2$$