Lecture 24-QM_II (22/09/00)

Monday, December 13, 2021   2:58 PM

$$H = \sum_i \frac{1}{2m}\left(\vec{P}_i - \frac{e}{c}\vec{A}(\vec{x}_i,t)\right)^2 + e\,\varphi(\vec{x}_i,t) + V(\vec{x}_i)$$

هر نوع پتانسیل به غیر از پتانسیل الکترومغناطیسی

$$H = H_0 + V(t)$$

$$H_0 = \sum_i \left(\frac{P_i^2}{2m} + V(x_i)\right)$$

$$\rightarrow V(t) = \sum_i \left(\frac{-e}{2mc}\left\{\vec{P}_i, \vec{A}(\vec{x}_i,t)\right\} + \frac{e^2}{2mc^2}\vec{A}^2(x_i,t) + e\,\varphi(\vec{x}_i,t)\right)$$

هامیلتونی تابش $H_{rad} = \dots$ (radiation)

ادعا

$$\boxed{V(t) = \int d^3x \left(-\frac{e}{c}\vec{J}(\vec{x})\cdot\vec{A}(\vec{x},t) + \frac{e^2}{2mc^2}\rho(\vec{x})\,\vec{A}^2(\vec{x},t) + e\,\rho(\vec{x})\,\varphi(\vec{x},t)\right)}$$

$$\rho(\vec{x}) \equiv \sum_i \delta(\vec{x}-\vec{x}_i)$$

$$\vec{J}(\vec{x}) \equiv \sum_i \left\{\frac{\vec{P}_i}{2m}, \delta(\vec{x}-\vec{x}_i)\right\}$$

$$\rightarrow \int d^3x\; e\,\rho(\vec{x})\,\varphi(\vec{x},t) = \int d^3x\; e\sum_i \delta(\vec{x}-\vec{x}_i)\,\varphi(\vec{x},t) = \sum_i e\,\varphi(\vec{x}_i,t)$$

$$\int d^3x \left(-\frac{e}{c}\vec{J}(\vec{x})\cdot\vec{A}(\vec{x},t)\right) = \int d^3x \left(\frac{-e}{c}\right)\sum_i \left\{\frac{\vec{P}_i}{2m}, \delta(\vec{x}-\vec{x}_i)\right\}\cdot\vec{A}(\vec{x},t)$$

یادآوری:

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = 0 \qquad \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right) = 0 \rightarrow \vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{J}$$

حالت خاص: پیمانه کولن

$$\rho = 0 \quad \varphi = 0 \qquad \vec{J} = 0 \quad \vec{\nabla}\cdot\vec{A} = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right) = 0$$

$\vec{A}(\vec{x},t)$ معادله موج

$$\boxed{\Delta\vec{A} - \frac{1}{c}\frac{\partial^2}{\partial t^2}\vec{A} = 0}$$

$$\vec{A}(\vec{x},t) = \sum_{\vec{k}} \vec{A}_k(t)\,e^{i\vec{k}\cdot\vec{x}}$$

H_{rad} برحسب $\vec{A}$.

$$H_{rad} = \frac{1}{8\pi}\int d^3x\,(\vec{E}^2 + \vec{B}^2) \qquad \begin{cases}\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{x},t)\\ \vec{B} = \vec{\nabla}\times\vec{A}\end{cases}$$

ادعا

$$H_{rad} = \frac{V}{8\pi}\sum_{\vec{k}}\left(\frac{1}{c^2}\left|\dot{\vec{A}}_k(t)\right|^2 + \left|\vec{k}\times\vec{A}_k(t)\right|^2\right)$$

$$H = \sum_k \frac{1}{2}m\,\dot{q}_k^2(t) + \frac{1}{2}m\omega^2 q_k^2(t)$$

اثبات:

$$H_{rad} = \frac{1}{8\pi}\int d^3x\left\{\left|-\frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{x},t)\right|^2 + \left|\vec{\nabla}\times\vec{A}(\vec{x},t)\right|^2\right\}$$

جمله اول

$$\text{جمله اول} = +\frac{1}{} \int d^3x \sum \dot{\vec{A}}_k(t)\,e^{i\vec{k}\cdot\vec{x}}\;\dot{\vec{A}}^*_{k'}(t)\,e^{-i\vec{k}'\cdot\vec{x}}$$

$$\text{جمله اول} = +\frac{1}{c^2 8\pi}\int d^3x \sum_{\vec{k},\vec{k}'} \dot{\vec{A}}_k(t)\, e^{i\vec{k}\cdot\vec{x}}\, \dot{\vec{A}}^*_{k'}(t)\, e^{-i\vec{k}'\cdot\vec{x}}$$

جمله اول

$$= +\frac{1}{8\pi c^2}\sum_{k,\vec{k}'} \dot{\vec{A}}_k(t)\, \dot{\vec{A}}^*_{k'}(t)\, \underbrace{\int d^3x\; e^{i\vec{x}(\vec{k}-\vec{k}')}}_{=V\,\delta_{\vec{k},\vec{k}'}}$$

$$= \frac{+V}{8\pi c^2}\sum_{\vec{k}} |\dot{\vec{A}}_k(t)|^2$$

$$\text{جمله دوم} = \frac{1}{8\pi}\int d^3x \sum_{\vec{k},\vec{k}'}\left(i\vec{k}\times\vec{A}_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}}\right)\left(-i\vec{k}'\times\vec{A}^*_{k'}(t)\, e^{-i\vec{k}'\cdot\vec{x}}\right)$$

$$= \frac{1}{8\pi}\sum_{\vec{k},\vec{k}'}\left(\vec{k}\times\vec{A}_k(t)\right)\left(\vec{k}'\times\vec{A}^*_{k'}(t)\right)\underbrace{\int d^3x\; e^{i\vec{x}\cdot(\vec{k}-\vec{k}')}}_{=V\,\delta_{\vec{k},\vec{k}'}}$$

$$= \frac{V}{8\pi}\sum_{\vec{k}} |\vec{k}\times\vec{A}_k(t)|^2$$

$$H_{rad} = \frac{V}{8\pi}\sum_{\vec{k}}\left(\frac{1}{c^2}|\dot{\vec{A}}_{\vec{k}}(t)|^2 + |\vec{k}\times\vec{A}_k(t)|^2\right)$$

$$H_{HO} = \sum_k\left(\frac{1}{2}m\dot{q}_k(t) + \frac{1}{2}m\omega^2 q_k^2(t)\right)$$

$$q_k(t) \longrightarrow \vec{A}_k(t)$$

$$\frac{m}{2} \longrightarrow \frac{V}{8\pi c^2} \qquad \left(m \to \frac{V}{4\pi c^2}\right)$$

$$\frac{1}{2}m\omega^2 \longrightarrow \frac{V}{8\pi}\vec{k}^2$$

$$\frac{1}{2}\frac{V}{4\pi c^2}\omega^2 \longrightarrow \frac{V}{8\pi}\vec{k}^2 \Rightarrow \frac{\omega^2}{c^2} = \vec{k}^2 \longrightarrow \omega_k^2 = \vec{k}^2 c^2$$

رابطه پاشندگی فوتون

$$E = \sqrt{\vec{p}^2c^2 + \cancel{m^2c^4}}$$

ذره نسبیتی بدون جرم

$$E = |\vec{p}|c$$

$$\cancel{\hbar}\omega = \cancel{\hbar}|\vec{k}|c \qquad \omega^2 = \vec{k}^2c^2$$

$$\hat{q} = \frac{x_0}{\sqrt{2}}(a + a^+) \;;$$

$$\begin{cases} a(t) = e^{+iH_0t/\hbar}\, a\, e^{-iH_0t/\hbar} \\ H_0 = \hbar\omega(a^+a + \frac{1}{2}) \end{cases}$$

$$[a, a^+] = 1$$

$$q_k(t) = \frac{x_0}{\sqrt{2}}\left(a_k e^{-i\omega_k t} + a_k^+ e^{+i\omega_k t}\right)$$

$$\begin{cases} a(t) = a\, e^{-i\omega t} \\ a^+(t) = a^+ e^{+i\omega t} \end{cases}$$

$$q_k(t) \longrightarrow \vec{A}_k(t)$$

$$\vec{A}(\vec{x},t) = \sum_k \vec{A}_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$V(t) = \left(F e^{-i\omega t} + F^+ e^{i\omega t}\right)$$

رابطه پتانسیل میدان الکترومغناطیسی

$$\boxed{\vec{A}(\vec{x},t) = \sum_{k,\lambda=1,2}\sqrt{\frac{2\pi\hbar c}{Vk}}\left(a_{\vec{k},\lambda}\,\vec{\epsilon}_{\vec{k},\lambda}\, e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + a^+_{\vec{k},\lambda}\,\vec{\epsilon}^{\,*}_{\vec{k},\lambda}\, e^{-i\vec{k}\cdot\vec{x} + i\omega_k t}\right)}$$

$$\frac{x_0}{\sqrt{2}} = \sqrt{\frac{\hbar}{2m\omega}} = \sqrt{\frac{\hbar}{2\frac{V}{2\,4\pi c^2}\,kc}} = \sqrt{\frac{2\pi c\hbar}{Vk}}$$

$\vec{\epsilon}_1$

$\vec{\epsilon}_2$

$$\vec{\nabla}\cdot\vec{A} = 0$$

$$\vec{k}\cdot\vec{\epsilon}_{\vec{k},\lambda} = 0 \qquad i=1,2 \qquad \vec{k}\cdot\vec{\epsilon}_i = 0$$

عملگر فنای یک فوتون با تکانه k و جهت پلاریزاسیون λ $\quad a_{\vec{k},\lambda}$ $\qquad$ $a^{+}_{\vec{k},\lambda}$ عملگر خلق فوتون با تکانه k و جهت پلاریزاسیون λ

$$|0\rangle = |0.0..0 \;....0\; ...\rangle$$ فضای Fock

$$a_{\vec{k},\lambda}\,|\,\ldots\, n_{\vec{k},\lambda}\,\ldots\rangle = \sqrt{n_{\vec{k},\lambda}}\,|\,\ldots\, n_{\vec{k},\lambda}-1_{\vec{k},\lambda}\,\ldots\rangle$$

$$a^{+}_{\vec{k},\lambda}\,|\,\ldots\, n_{\vec{k},\lambda}\,\ldots\rangle = |\,\ldots\, n_{k,\lambda}+1_{\vec{k},\lambda}\,\ldots\rangle$$

$$[a, a^{+}] = 1 \quad\longrightarrow\quad [a_{k,\lambda}, a^{+}_{k',\lambda'}] = \delta_{\vec{k},\vec{k}'}\,\delta_{\lambda,\lambda'}$$

$$\hat{n} = a^{+}a \quad\longrightarrow\quad \hat{n}_{\vec{k},\lambda} = a^{+}_{\vec{k},\lambda}\,a_{\vec{k},\lambda}$$

$$\hat{n}\,\underline{\underline{|n\rangle}} = n\,|n\rangle$$

$$\hat{n}_{\vec{k},\lambda}\,|\,\ldots\, n_{\vec{k},\lambda}\,\ldots\rangle = n_{\vec{k},\lambda}\,|\,\ldots\, n_{\vec{k},\lambda}\,\ldots\rangle$$

$$|\,\ldots\, n_{\vec{k},\lambda}\,\ldots\rangle = \frac{1}{\sqrt{n_{\vec{k},\lambda}!}}\left(a^{+}_{\vec{k},\lambda}\right)^{n_{\vec{k},\lambda}}|0\rangle$$

$$H_{rad} = \frac{1}{8\pi}\int d^3x\,\left(\vec{E}^2+\vec{B}^2\right) \underset{\uparrow\,\vec{A}}{=} \frac{V}{8\pi}\sum_{\vec{k}}\left(\frac{1}{c^2}\left|\dot{\vec{A}}_{k}(t)\right|^2 + \left|\vec{k}\times\vec{A}_{\vec{k}}(t)\right|^2\right)$$

$$H_{rad} = \sum_{\vec{k},\lambda}\hbar\omega_{\vec{k}}\left(a^{+}_{\vec{k},\lambda}\,a_{\vec{k},\lambda} + \frac{1}{2}\right)$$

$$E = \sum_{\vec{k},\lambda}\hbar\omega_k\left(n_{\vec{k},\lambda} + \frac{1}{2}\right)$$