QM-2-L8-1404-2

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ترکیب اسپین ها

$$\vec S = \vec S_1 + \vec S_2$$

$$\{ \vec S_1^{\,2}, S_{1z} ; \vec S_2^{\,2}, S_{2z} \} \rightarrow \{ \vec S^{\,2}, S_z ; \vec S_1^{\,2}, \vec S_2^{\,2} \}$$

$$| s, m_s ; s_1, s_2 \rangle = \sum_{\substack{m_{s_1} \\ m_{s_2} = m_s - m_{s_1}}} C \; | s_1, m_{s_1} ; s_2, m_{s_2} \rangle$$

در ترکیب دو اسپین $\frac{1}{2}$ ، سه حالت سه تایی ، و یک حالت تک تایی بوجود میاید

$s = 0, 1$

$s = 0$	$m_s = 0$	حالت تکتا	singlet
$s = 1$	$m_s = \pm 1, 0$	حالت سه تایی	triplet

Triplet
$$\begin{cases} |1, +1\rangle = |\uparrow\uparrow\rangle \\ |1, 0\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right) \\ |1, -1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

Singlet
$$\Big\{ |0, 0\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

$$\vec S^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$
$$S_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

ترکیب $\vec S$ و $\vec L$

$$\{ \vec L^2, L_z ; \vec S^2, S_z \} \rightarrow \{ \vec J^2, J_z ; \vec L^2, \vec S^2 \}$$

$$\vec L^2 |l, m_l\rangle = \hbar^2 l(l+1) |l, m_l\rangle \qquad \vec J^2 |j, m_j\rangle = \hbar^2 j(j+1) |j, m_j\rangle$$
$$L_z |l, m_l\rangle = \hbar m_l |l, m_l\rangle \qquad J_z |j, m_j\rangle = \hbar m_j |j, m_j\rangle$$

$s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$

$$\vec S^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$
$$S_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

سوال : ارتباط بین $(l, m_l ; s, m_s)$ و (j, m_j)

ادعا ۱ :

$$m_j = m_l + \frac{1}{2}$$
$$m_j = m_l - \frac{1}{2}$$

براساس ایدهٔ قضیه ط

Ansatz : $$\boxed{|j, m_j\rangle = \alpha \, |l, m_l\rangle \underbrace{|\uparrow\rangle}_{|s = \frac{1}{2}, m_s = +\frac{1}{2}\rangle} + \beta \, |l, m_l\rangle \underbrace{|\downarrow\rangle}_{|s = \frac{1}{2}, m_s = -\frac{1}{2}\rangle}}$$

$|s=\frac{1}{2}, m_s=+\frac{1}{2}\rangle$ $\qquad$ $|s=\frac{1}{2}, m_s=-\frac{1}{2}\rangle$

اثر J_z بر $|j,m_j\rangle$:

$$J_z = L_z + S_z$$

$$J_z\,|j,m_j\rangle = \hbar m_j\,|j,m_j\rangle$$

طرف راست :
$$\boxed{\hbar m_j\,|j,m_j\rangle = \hbar m_j\left(\alpha\,|l,m_l\rangle\,|\uparrow\rangle + \beta\,|l,m_l\rangle\,|\downarrow\rangle\right)} \quad R$$

طرف چپ
$$J_z\,|j,m_j\rangle = (L_z+S_z)\left(\alpha\,|l,m_l\rangle\,|\uparrow\rangle + \beta\,|l,m_l\rangle\,|\downarrow\rangle\right)$$

$$= \alpha\hbar m_l\,|l,m_l\rangle\,|\uparrow\rangle + \alpha\left(\frac{\hbar}{2}\right)|l,m_l\rangle\,|\uparrow\rangle +$$

$$+\beta\hbar m_l\,|l,m_l\rangle\,|\downarrow\rangle + \alpha\left(-\frac{\hbar}{2}\right)|l,m_l\rangle\,|\downarrow\rangle$$

$$\boxed{= \alpha\hbar\left(m_l+\frac{1}{2}\right)|l,m_l\rangle\,|\uparrow\rangle + \beta\hbar\left(m_l-\frac{1}{2}\right)|l\,m_l\rangle\,|\downarrow\rangle} \quad L$$

مقایسه دو طرف معادله خواهیم داشت :

$$\boxed{\begin{array}{ll} m_j = m_l + \frac{1}{2} & \to \alpha\ldots|\uparrow\rangle \\ m_j = m_l - \frac{1}{2} & \to \beta\ldots|\downarrow\rangle \end{array}}$$

$m_l = m_j - \frac{1}{2}$

$m_l = m_j + \frac{1}{2}$

تعیین j ، α و β ؟

باید رابطه

$$\boxed{\vec{J}^2\,|j,m_j\rangle = \hbar^2 j(j+1)\,|j,m_j\rangle} \; *$$

را از هر دو طرف بررسی کنیم.

$$\vec{J}^2 = (\vec{L}+\vec{S})^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S} = \vec{L}^2 + \vec{S}^2 + 2L_zS_z + L_+S_- + L_-S_+$$

$[\vec{L},\vec{S}]=0$

اثبات شبیه قبل $\qquad L_\pm = L_x \pm iL_y \;,\; S_\pm = S_x \pm iS_y$

$$\vec{J}^2\,|j,m_j\rangle = \left(\vec{L}^2+\vec{S}^2+2L_zS_z+L_+S_-+L_-S_+\right)\left(\alpha\,|l,m_l\rangle\otimes|\uparrow\rangle + \beta\,|l,m_l\rangle\otimes|\downarrow\rangle\right)$$

$m_l = m_j - \frac{1}{2}$ $\qquad$ $m_l = m_j + \frac{1}{2}$

$$= \alpha\hbar^2 l(l+1)\,|l,m_l\rangle\otimes|\uparrow\rangle + \frac{3}{4}\hbar^2\alpha\,|l,m_l\rangle\otimes|\uparrow\rangle + 2\alpha(\hbar m_l)\left(\frac{\hbar}{2}\right)|l,m_l\rangle\otimes|\uparrow\rangle$$

$$+\alpha\hbar^2\sqrt{l(l+1)-m_l(m_l+1)}\;|l,m_l+1\rangle\otimes|\downarrow\rangle + 0$$

$\downarrow m_j-\frac{1}{2}$

$$+\beta\hbar^2 l(l+1)\,|l,m_l\rangle\otimes|\downarrow\rangle + \frac{3}{4}\hbar^2\beta\,|l,m_l\rangle\otimes|\downarrow\rangle + 2\alpha(\hbar m_l)\left(-\frac{\hbar}{2}\right)|l,m_l\rangle\otimes|\downarrow\rangle$$

$$+0+\beta\hbar^2\sqrt{l(l+1)-m_l(m_l-1)}\;|l,m_l-1\rangle\otimes|\uparrow\rangle$$

$\downarrow m_j+\frac{1}{2}$

$m_l = m_j - \frac{1}{2}$ $\qquad$ در تمام جمله‌هایی که ضریب α دارند،

در (۱) جمله‌هایی که ضریب α دارند، $\quad m_\ell = m_j - \frac{1}{2}$

در (۲) جمله‌هایی که ضریب β دارند $\quad m_\ell = m_j + \frac{1}{2}$

* طرف چپ $= \hbar^2 \left\{ \alpha\, \ell(\ell+1) + \frac{3}{4}\alpha + (m_j - \frac{1}{2})\alpha + \beta \sqrt{\ell(\ell+1) - (m_j+\frac{1}{2})(m_j-\frac{1}{2})} \right\} |\ell, m_j - \frac{1}{2}\rangle \otimes |\uparrow\rangle$

$$+ \hbar^2 \left\{ \beta\, \ell(\ell+1) + \frac{3}{4}\beta - (m_j + \frac{1}{2})\beta + \alpha \sqrt{\ell(\ell+1) - (m_j-\frac{1}{2})(m_j+\frac{1}{2})} \right\} |\ell, m_j + \frac{1}{2}\rangle \otimes |\downarrow\rangle$$

* طرف راست

$$\hbar^2 j(j+1)\, |j, m_j\rangle = \hbar^2 j(j+1) \left(\alpha\, |\ell, m_j - \frac{1}{2}\rangle \otimes |\uparrow\rangle + \beta\, |\ell, m_j + \frac{1}{2}\rangle \otimes |\downarrow\rangle \right)$$

مقایسه دو طرف : دو معادله به دست می‌آید

$$(A) \quad \alpha\, \ell(\ell+1) + \frac{3}{4}\alpha + (m_j - \frac{1}{2})\alpha + \beta \sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})} = \alpha\, j(j+1)$$

$$(B) \quad \beta\, \ell(\ell+1) + \frac{3}{4}\beta - (m_j + \frac{1}{2})\beta + \alpha \sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})} = \beta\, j(j+1)$$

عملیات جبری برای یافتن j ؟

$$(A) \quad \alpha \left\{ j(j+1) - \ell(\ell+1) - \frac{3}{4} - (m_j - \frac{1}{2}) \right\} = \beta \sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})}$$

$$(B) \quad \beta \left\{ j(j+1) - \ell(\ell+1) - \frac{3}{4} + (m_j + \frac{1}{2}) \right\} = \alpha \sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})}$$

$(A)\,(B)$ $\qquad$ تعریف $\quad \boxed{j(j+1) - \ell(\ell+1) - \frac{3}{4} \equiv C}$

$$\cancel{\alpha\beta} \left\{ C - (m_j - \frac{1}{2}) \right\} \left\{ C + (m_j + \frac{1}{2}) \right\} = \cancel{\alpha\beta} \left\{ \ell(\ell+1) - (m_j^2 - \frac{1}{4}) \right\}$$

$$C^2 + C\,(\cancel{m_j} + \frac{1}{2} - \cancel{m_j} + \frac{1}{2}) - \cancel{(m_j^2 - \frac{1}{4})} = \ell(\ell+1) - \cancel{(m_j^2 - \frac{1}{4})}$$

$$\boxed{C^2 + C - \ell(\ell+1) = 0 \implies C_\pm = \frac{-1 \pm \sqrt{1 + 4\ell(\ell+1)}}{2}}$$

$$C_\pm = \frac{-1 \pm \sqrt{(2\ell+1)^2}}{2} \qquad \begin{cases} C_+ = \dfrac{-1 + (2\ell+1)}{2} = \ell \\ C_- = \dfrac{-1 - (2\ell+1)}{2} = -\ell - 1 \end{cases}$$

a) $\quad C_+ = \ell = j(j+1) - \ell(\ell+1) - \frac{3}{4} \implies \boxed{j = \ell + \frac{1}{2}}$

b) $\quad C_- = -\ell - 1 = j(j+1) - \ell(\ell+1) - \frac{3}{4} \implies \boxed{j = \ell - \frac{1}{2}}$

b) $C_- = -\ell - 1 = j(j+1) - \ell(\ell+1) - \frac{3}{4} \Longrightarrow \boxed{j = \ell - \frac{1}{2}}$

با توجه به رابطه بین j و m_j داریم:

$$\boxed{\begin{array}{lll} a) & j = \ell + \frac{1}{2} & \curvearrowright\ -(\ell+\frac{1}{2}) \leq m_j \leq \ell + \frac{1}{2} \\ b) & j = \ell - \frac{1}{2} & \curvearrowright\ -(\ell-\frac{1}{2}) \leq m_j \leq \ell - \frac{1}{2} \end{array}}$$

(Ansatz)'
$$\boxed{|j, m_j\rangle = \alpha\ |\ell, m_\ell = m_j - \tfrac{1}{2}\rangle \otimes |\uparrow\rangle + \beta\ |\ell, m_\ell = m_j + \tfrac{1}{2}\rangle \otimes |\downarrow\rangle}$$

قدم بعد: تعیین α و β

دو معادله داشتیم

(A) $\alpha\left(j(j+1) - \ell(\ell+1) - \frac{3}{4} - (m_j - \frac{1}{2})\right) = \beta\sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})}$

(B) $\beta\left(j(j+1) - \ell(\ell+1) - \frac{3}{4} + (m_j + \frac{1}{2})\right) = \alpha\sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})}$

(A) در $\frac{\beta}{\alpha} = \frac{j(j+1) - \ell(\ell+1) - \frac{3}{4} - (m_j - \frac{1}{2})}{\sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})}} \underset{\boxed{j = \ell + \frac{1}{2}}}{=} \frac{\ell - m_j + \frac{1}{2}}{\sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})}}$

$$\frac{\beta}{\alpha} = \sqrt{\frac{\ell - m_j + \frac{1}{2}}{\ell + m_j + \frac{1}{2}}} = \frac{\beta_+}{\alpha_+}$$

$$\alpha_+ \equiv \sqrt{\frac{\ell + m_j + \frac{1}{2}}{\underset{\sim}{2\ell + 1}}}, \quad \beta_+ \equiv \sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell + 1}}$$

خودمان اضافه کردیم

نتیجه: $|j = \ell + \frac{1}{2}, m_j\rangle = \alpha_+\ |\ell, m_j - \frac{1}{2}\rangle \otimes |\uparrow\rangle + \beta_+\ |\ell, m_j + \frac{1}{2}\rangle \otimes |\downarrow\rangle$

$-(\ell + \frac{1}{2}) \leq m_j \leq \ell + \frac{1}{2}$

(A) $\frac{\beta}{\alpha} = \frac{j(j+1) - \ell(\ell+1) - \frac{3}{4} - (m_j - \frac{1}{2})}{\sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})}} \underset{\boxed{j = \ell - \frac{1}{2}}}{=} \frac{-(\ell + m_j + \frac{1}{2})}{\sqrt{\ell(\ell+1) - (m_j^2 - \frac{1}{4})}}$

$\frac{\beta}{\alpha} = \sqrt{\frac{\ell + m_j + \frac{1}{2}}{\ldots}} = \frac{\beta_-}{\ldots}$

$$\frac{\beta}{\alpha} = -\sqrt{\frac{l+m_j+\frac{1}{2}}{l-m_j+\frac{1}{2}}} \equiv \frac{\beta_-}{\alpha_-}$$

نتیجه

$$\boxed{\begin{gathered} \alpha_- = -\sqrt{\frac{l-m_j+\frac{1}{2}}{2l+1}} \quad , \quad \beta_- = \sqrt{\frac{l+m_j+\frac{1}{2}}{2l+1}} \\ |j = l-\tfrac{1}{2}, m_j\rangle = \alpha_- \,|l, m_j-\tfrac{1}{2}\rangle \otimes |\uparrow\rangle + \beta_- \,|l, m_j+\tfrac{1}{2}\rangle \otimes |\downarrow\rangle \\ -(l-\tfrac{1}{2}) \leq m_j \leq l-\tfrac{1}{2} \end{gathered}}$$

$\blacksquare$ همه چیزهایی را که می‌خواستیم بدست بیاوریم محاسبه شد