QM-2-L7-1404-2

Sunday, April 19, 2026 8:44 AM

جمع تکانه زاویه‌ای

. دو ذره هرکدام با اسپین $\frac{1}{2}$ $\quad \vec{S}_1 \,,\, \vec{S}_2$

. مولفه‌های $\vec{S}_i$ ($i=1,2$) هرکدام جداگانه در رابطه جابجایی صدق می‌کنند

$$[S_{1i}, S_{1j}] = i\hbar\, \epsilon_{ijk}\, S_{1k}$$ برای اسپین ذره اول

$$[S_{2i}, S_{2j}] = i\hbar\, \epsilon_{ijk}\, S_{2k}$$ برای اسپین ذره دوم

نکته

$$[\underset{\text{ذره اول}}{S_{1i}}, \underset{\text{ذره دوم}}{S_{2j}}] = 0$$

بطور کلی : $[\vec{S}_1, \vec{S}_2] = 0$

عملگر اسپین کل : $\vec{S} = \vec{S}_1 + \vec{S}_2$ مولفه اسپین کل

سوال : $[S_i, S_j] = ?$

$$[S_i, S_j] = [S_{1i} + S_{2i}, S_{1j} + S_{2j}] =$$

$$= [S_{1i}, S_{1j}] + \cancel{[S_{1i}, S_{2j}]} + \cancel{[S_{2i}, S_{1j}]} + [S_{2i}, S_{2j}]$$

$$= i\hbar\, \epsilon_{ijk} S_{1k} + i\hbar\, \epsilon_{ijk} S_{2k} = i\hbar\, \epsilon_{ijk} \underbrace{(S_{1k} + S_{2k})}_{S_k}$$

$$= i\hbar\, \epsilon_{ijk} S_k$$

$$\Rightarrow \boxed{[S_i, S_j] = i\hbar\, \epsilon_{ijk} S_k}$$

به این ترتیب $\vec{S}$ (اسپین کل) در جبر اسپین صدق می‌کنند.

$$\left.\begin{aligned} \vec{S}^2 |s, m_s\rangle &= \hbar^2 s(s+1) |s, m_s\rangle \\ S_z |s, m_s\rangle &= \hbar m_s |s, m_s\rangle \end{aligned}\right\}$$ برای اسپین کل

$$|\uparrow\rangle_{(i)} = |s_i = \tfrac{1}{2}, m_{s_i} = +\tfrac{1}{2}\rangle \qquad i = 1,2$$ ذره ۱، ۲

$$|\downarrow\rangle_{(i)} = |s_i = \tfrac{1}{2}, m_{s_i} = -\tfrac{1}{2}\rangle$$

$$\vec{S}_i^2 |\uparrow\rangle_{(i)} = \hbar^2 s_i (s_i + 1) |\uparrow\rangle_{(i)} \overset{s_i = \frac{1}{2}}{=} \frac{3}{4}\hbar^2 |\uparrow\rangle_{(i)}$$

$$\vec{S}_i^2 |\downarrow\rangle_{(i)} = \hbar^2 s_i (s_i + \tfrac{1}{2}) |\downarrow\rangle_{(i)} \underset{s_i = \frac{1}{2}}{=} \frac{3}{4}\hbar^2 |\downarrow\rangle_{(i)}$$

$$S_{iz} |\uparrow\rangle_{(i)} = \hbar m_{s_i} |\uparrow\rangle_{(i)} \underset{m_{s_i} = +\frac{1}{2}}{=} +\frac{\hbar}{2} |\uparrow\rangle_{(i)}$$

$$S_{iz} |\downarrow\rangle_{(i)} = \hbar m_{s_i} |\downarrow\rangle_{(i)} = -\frac{\hbar}{2} |\downarrow\rangle_{(i)}$$

$$m_{s_i} = +\frac{1}{2}$$

$$S_{iz}\,|\downarrow\rangle_{(i)} = \hbar m_{s_i} |\downarrow\rangle_{(i)} = -\frac{\hbar}{2} |\downarrow\rangle_{(i)}$$

سؤال: رابطه بین S با S_1 و S_2 ؟

رابطه بین m_s با m_{s_1}, m_{s_2} ؟

برای یک سیستم از دو الکترون عملاً ۴ حالت داریم:

$$|\uparrow\rangle_{(1)} \otimes |\uparrow\rangle_{(2)} \equiv |\uparrow\uparrow\rangle$$

$$|\uparrow\rangle_{(1)} \otimes |\downarrow\rangle_{(2)} \equiv |\uparrow\downarrow\rangle$$

$$|\downarrow\rangle_{(1)} \otimes |\uparrow\rangle_{(2)} \equiv |\downarrow\uparrow\rangle$$

$$|\downarrow\rangle_{(1)} \otimes |\downarrow\rangle_{(2)} \equiv |\downarrow\downarrow\rangle$$

تعیین m_s برحسب m_{s_1}, m_{s_2} ؟

(a) $$S_z |\uparrow\uparrow\rangle = (S_{1z} + S_{2z}) |\uparrow\uparrow\rangle = (S_{1z}+S_{2z}) \left(|\uparrow\rangle_{(1)} \otimes |\uparrow\rangle_{(2)}\right)$$

$$= \underbrace{\left(S_{1z}|\uparrow\rangle_{(1)}\right)}_{=\frac{\hbar}{2}|\uparrow\rangle_{(1)}} \otimes |\uparrow\rangle + |\uparrow\rangle_{(1)} \otimes \underbrace{S_{2z}|\uparrow\rangle_{(2)}}_{=\frac{\hbar}{2}|\uparrow\rangle_{(2)}}$$

$$= \frac{\hbar}{2} |\uparrow\uparrow\rangle + \frac{\hbar}{2} |\uparrow\uparrow\rangle = \hbar\, |\uparrow\uparrow\rangle$$

$$\Rightarrow \boxed{S_z |\uparrow\uparrow\rangle = +\hbar\, |\uparrow\uparrow\rangle}$$

$\leftarrow$ $|\uparrow\uparrow\rangle$ ویژه حالت S_z به ازای ویژه مقدار $+\hbar$

$$S_z |\uparrow\uparrow\rangle = \hbar m_s |\uparrow\uparrow\rangle \curvearrowright m_s = +1$$

$$m_{s_1} = +\frac{1}{2} \qquad m_{s_2} = +\frac{1}{2}$$

$$\boxed{\begin{aligned} m_s &= m_{s_1} + m_{s_2} \\ 1 &= \frac{1}{2} + \frac{1}{2} \end{aligned}} \ \checkmark$$

پس به نظر می‌رسد درسته.

(b) $$S_z |\uparrow\downarrow\rangle = (S_{1z} + S_{2z}) \left(|\uparrow\rangle_{(1)} \otimes |\downarrow\rangle_{(2)}\right)$$

$$= \frac{\hbar}{2} |\uparrow\downarrow\rangle - \frac{\hbar}{2} |\uparrow\downarrow\rangle = 0 = \hbar m_s |\uparrow\downarrow\rangle$$

$$\Rightarrow m_s = 0 \qquad m_{s_1} = +\frac{1}{2} \qquad m_{s_2} = -\frac{1}{2}$$

$$\boxed{\begin{aligned} m_s &= m_{s_1} + m_{s_2} \\ 0 &= \frac{1}{2} - \frac{1}{2} \end{aligned}}$$

پس دوباره

(c) $$S_z |\downarrow\uparrow\rangle = (S_{1z} + S_{2z}) \left(|\downarrow\rangle_{(1)} \otimes |\uparrow\rangle_{(2)}\right)$$

(c) $$S_z\,|\downarrow\uparrow\rangle = (S_{1z}+S_{2z})\left(|\downarrow\rangle_{(1)}\otimes|\uparrow\rangle_{(2)}\right)$$

$$= -\frac{\hbar}{2}|\downarrow\uparrow\rangle + \frac{\hbar}{2}|\downarrow\uparrow\rangle = 0 = \hbar m_s\,|\downarrow\uparrow\rangle$$

$$\Rightarrow m_s = 0 \qquad m_{s_1} = -\frac{1}{2} \qquad m_{s_2} = +\frac{1}{2}$$

$$\boxed{\begin{array}{l} m_s = m_{s_1}+m_{s_2} \\ 0 = -\frac{1}{2}+\frac{1}{2}\end{array}}$$

(d) $$S_z\,|\downarrow\downarrow\rangle = (S_{1z}+S_{2z})\left(|\downarrow\rangle_{(1)}\otimes|\downarrow\rangle_{(2)}\right)$$

$$= \left(-\frac{\hbar}{2}-\frac{\hbar}{2}\right)|\downarrow\downarrow\rangle = -\hbar\,|\downarrow\downarrow\rangle = \hbar m_s\,|\downarrow\downarrow\rangle$$

$$\Rightarrow m_s = -1 \qquad m_{s_1} = -\frac{1}{2} \qquad m_{s_2} = -\frac{1}{2}$$

$$\boxed{\begin{array}{l} m_s = m_{s_1}+m_{s_2} \\ -1 = -\frac{1}{2}-\frac{1}{2}\end{array}}$$

نتیجه:

$$\left.\begin{array}{l} S_z\,|\uparrow\uparrow\rangle = \hbar\,|\uparrow\uparrow\rangle \\ S_z\,|\uparrow\downarrow\rangle = S_z\,|\downarrow\uparrow\rangle = 0 \\ S_z\,|\downarrow\downarrow\rangle = -\hbar\,|\downarrow\downarrow\rangle \end{array}\right\} \Rightarrow m_s = -1, 0, +1$$

سوال $s = ?$

تعیین s:

$$\vec{S}_1 + \vec{S}_2 = \vec{S}$$

$$S^2\,|s, m_s\rangle = \hbar^2 s(s+1)\,|s, m_s\rangle$$

مطلوب است s ؟

قدم اول:

$$\boxed{\text{ادعا:}\quad \vec{S}^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}}$$

اثبات:

$$\vec{S}^2 = (\vec{S}_1+\vec{S}_2)^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + \vec{S}_1\cdot\vec{S}_2 + \vec{S}_2\cdot\vec{S}_1 \qquad \text{A}$$

از آنجاکه

$$\vec{S}_1\cdot\vec{S}_2 = \vec{S}_2\cdot\vec{S}_1 \leftarrow [\vec{S}_1, \vec{S}_2] = 0$$

$$\vec{S}^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\vec{S}_1\cdot\vec{S}_2$$

$$2\vec{S}_1\cdot\vec{S}_2 = 2(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}) =$$

$$= 2(S_{1x}S_{2x} + S_{1y}S_{2y}) + 2S_{1z}S_{2z}$$

استفاده می‌کنیم از:

$$\boxed{S_\pm = S_x \pm iS_y \Rightarrow S_x = \frac{1}{2}(S_+ + S_-), \quad S_y = \frac{1}{2i}(S_+ - S_-)}$$

خواهیم داشت

$$2\vec{S}_1\cdot\vec{S}_2 = S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z}$$

و خواهیم داشت

$$\boxed{2\vec{S}_1\cdot\vec{S}_2 = S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z}}$$

Ⓐ با جایگذاری در رابطه

$$\boxed{\vec{S}^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}}$$

q.e.d.

• حال برای بدست آوردن S :

$$\boxed{\begin{aligned} \vec{S}^2|\uparrow\uparrow\rangle &= ? \\ \vec{S}^2|\uparrow\downarrow\rangle &= ? \\ \vec{S}^2|\uparrow\downarrow\rangle &= ? \\ \vec{S}^2|\downarrow\downarrow\rangle &= ? \end{aligned}}$$

$|\uparrow\uparrow\rangle = |\uparrow\rangle_{(1)}\otimes|\uparrow\rangle_{(2)}$

a) $\vec{S}^2|\uparrow\uparrow\rangle = (\vec{S}_1^{\,2}+\vec{S}_2^{\,2}+2S_{1z}S_{2z}+S_{1+}S_{2-}+S_{1-}S_{2+})\,|\uparrow\uparrow\rangle$

طبیعتاً عملگرها با اندیس 1 روی $|\uparrow\rangle_{(1)}$ و عملگرها با اندیس 2 روی $|\uparrow\rangle_{(2)}$ اثر می‌کنند:

$$\vec{S}^2|\uparrow\uparrow\rangle = \frac{3}{4}\hbar^2|\uparrow\uparrow\rangle + \frac{3}{4}\hbar^2|\uparrow\uparrow\rangle + 2\left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right)|\uparrow\uparrow\rangle + 0 + 0$$

$S_{1+}|\uparrow\rangle_{(1)} = 0$ $\quad$ $S_{2+}|\uparrow\rangle_{(2)} = 0$

$$= 2\hbar^2|\uparrow\uparrow\rangle$$

$$\Longrightarrow \boxed{\vec{S}^2|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle}$$

b) $\vec{S}^2|\uparrow\downarrow\rangle = (\vec{S}_1^{\,2}+\vec{S}_2^{\,2}+2S_{1z}S_{2z}+S_{1+}S_{2-}+S_{1-}S_{2+})\,|\uparrow\downarrow\rangle$

$$= \frac{3}{4}\hbar^2|\uparrow\downarrow\rangle + \frac{3}{4}\hbar^2|\uparrow\downarrow\rangle + 2\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)|\uparrow\downarrow\rangle + 0 + (\hbar)^2\,\underline{|\downarrow\uparrow\rangle}$$

$S_{1+}|\uparrow\rangle_{(1)} = 0$

$$= \hbar^2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$\Longrightarrow \boxed{\vec{S}^2|\uparrow\downarrow\rangle = \hbar^2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)}$$

$S_{1-}|\uparrow\rangle_{(1)} = \hbar\,|\downarrow\rangle_{(1)}$

$S_{2+}|\downarrow\rangle_{(2)} = \hbar\,|\uparrow\rangle_{(2)}$

c) $\vec{S}^2|\downarrow\uparrow\rangle = (\vec{S}_1^{\,2}+\vec{S}_2^{\,2}+2S_{1z}S_{2z}+S_{1+}S_{2-}+S_{1-}S_{2+})\,|\downarrow\uparrow\rangle$

$$= \frac{3}{4}\hbar^2|\downarrow\uparrow\rangle + \frac{3}{4}\hbar^2|\downarrow\uparrow\rangle + 2\left(-\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right)|\downarrow\uparrow\rangle + \hbar^2|\uparrow\downarrow\rangle + 0$$

$S_{1-}|\downarrow\rangle = 0$

$$\boxed{\vec{S}^2|\uparrow\downarrow\rangle = \hbar^2\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)}$$

$S_{1+}|\downarrow\rangle_{(1)} = \hbar|\uparrow\rangle_{(1)}$

$S_{2-}|\uparrow\rangle_{(2)} = \hbar\,|\downarrow\rangle_{(2)}$

d) $\vec{S}^2|\downarrow\downarrow\rangle = (\vec{S}_1^{\,2}+\vec{S}_2^{\,2}+2S_{1z}S_{2z}+S_{1+}S_{2-}+S_{1-}S_{2+})\,|\downarrow\downarrow\rangle$

$$= \frac{3}{4}\hbar^2|\downarrow\downarrow\rangle + \frac{3}{4}\hbar^2|\downarrow\downarrow\rangle + 2\left(-\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)|\downarrow\downarrow\rangle + 0 + 0$$

$S_{2-}|\downarrow\rangle_{(2)} = 0$ $\quad$ $S_{1-}|\downarrow\rangle_{(1)} = 0$

$$= 2\hbar^2|\downarrow\downarrow\rangle$$

$$\Longrightarrow \boxed{\vec{S}^2|\downarrow\downarrow\rangle = 2\hbar^2|\downarrow\downarrow\rangle}$$

$$\Longrightarrow \boxed{\vec{S}^2|\downarrow\downarrow\rangle = 2\hbar^2|\downarrow\downarrow\rangle}$$

نتیجه : از روابط بالا که مشاهده کردیم که $|\uparrow\uparrow\rangle$ ، $|\downarrow\downarrow\rangle$ ویژه‌حالت $\vec{S}^2$ به ازای ویژه‌مقدار $\hbar^2 s(s+1) = 2\hbar^2$
$\Rightarrow s=1$

ولی $|\uparrow\downarrow\rangle$ ، $|\downarrow\uparrow\rangle$ ویژه‌حالتهای $\vec{S}^2$ نیستند !

ولی ترکیب خطی $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ می‌توانند ویژه‌حالت $\vec{S}^2$ باشد .

ادعا

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ و $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ ویژه‌حالتهای $\vec{S}^2$ هستند:

اثبات .

a') $\vec{S}^2 \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \overset{(c)\,\&\,(d)}{=}$

$$= \frac{1}{\sqrt{2}}\hbar^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) + \frac{1}{\sqrt{2}}\hbar^2(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$$

$$= \frac{2}{\sqrt{2}}\hbar^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \overset{!}{=} \hbar^2 s(s+1)\,(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$\Rightarrow \boxed{s=1}$$

b') $\vec{S}^2 \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \overset{(c)\,\&\,(d)}{=}$

$$= \frac{1}{\sqrt{2}}\hbar^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) - \frac{1}{\sqrt{2}}\hbar^2(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) = 0$$

$$\overset{!}{=} \hbar^2 s(s+1)\,(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$\Rightarrow \boxed{s=0}$$

خلاصه :

ترکیب دو ذره اسپین $\frac{1}{2}$

$|s, m_s\rangle$ $\quad s = 1, 0$

for $s=1$ $\quad m_s = \pm 1, 0$

for $s=0$ $\quad m_s = 0$

$$|s, m_s\rangle = |1, +1\rangle = |\uparrow\uparrow\rangle$$
$$= |1, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$
$$= |1, -1\rangle = |\downarrow\downarrow\rangle$$

حالتهای سه‌تایی
Triplet states

$$|s, m_s\rangle = |0, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

حالت تکتا
Singlet state

صورت مسئله کلی :

وقتی

$$\vec{S}_1^2|s_1, m_{s_1}\rangle = \hbar^2 s_1(s_1+1)|s_1, m_{s_1}\rangle$$
$$S_{1z}|s_1, m_{s_1}\rangle = \hbar m_{s_1}|s_1, m_{s_1}\rangle$$

,

$$\vec{S}_2^2|s_2, m_{s_2}\rangle = \hbar^2 s_2(s_2+1)|s_2, m_{s_2}\rangle$$

$$\vec{S}_2^{\,2}\,|s_2, m_{s_2}\rangle = \hbar^2 s_2(s_2+1)\,|s_2, m_{s_2}\rangle$$
$$S_{2z}\,|s_2, m_{s_2}\rangle = \hbar m_{s_2}\,|s_2, m_{s_2}\rangle$$

با ترکیب

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$
$$\vec{S}^{\,2}\,|s, m_s\rangle = \hbar^2 s(s+1)\,|s, m_s\rangle$$
$$S_z\,|s, m_s\rangle = \hbar m_s\,|s, m_s\rangle$$

$s = ?\quad m_s = ?$, حالت‌ها و روابط = ؟

$$\{\vec{S}_1^{\,2}, S_{1z}\,;\ \vec{S}_2^{\,2}, S_{2z}\} \rightarrow \{\vec{S}^{\,2}, S_z\,;\ \vec{S}_1^{\,2}, \vec{S}_2^{\,2}\}$$

قضیه بسط

$$|s, m_s\,; s_1, s_2\rangle = \sum_{m_{s_1}, m_{s_2}} C\ |s_1, m_{s_1}\rangle \otimes |s_2, m_{s_2}\rangle$$

$m_{s_2} = m_s - m_{s_1}$ با این قید

ضریب بسط
Clebsch-Gordan coefficient

در مثال ما:

$$|s, m_s\,; \tfrac{1}{2}, \tfrac{1}{2}\rangle = \sum C\ |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$$
$$= C_{\uparrow\uparrow}|\uparrow\uparrow\rangle + C_{\uparrow\downarrow}|\uparrow\downarrow\rangle + C_{\downarrow\uparrow}|\downarrow\uparrow\rangle + C_{\downarrow\downarrow}|\downarrow\downarrow\rangle$$

for:

Triplet
$$\begin{cases} |s, m_s\rangle = |1, +1\rangle = |\uparrow\uparrow\rangle \Rightarrow C_{\uparrow\uparrow} = 1\ ,\ C_{\downarrow\uparrow} = C_{\uparrow\downarrow} = C_{\downarrow\downarrow} = 0 \\ |s, m_s\rangle = |1, 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \Rightarrow C_{\uparrow\uparrow} = 0 = C_{\downarrow\downarrow}\ ,\ C_{\uparrow\downarrow} = C_{\downarrow\uparrow} = \frac{1}{\sqrt{2}} \\ |s, m_s\rangle = |1, -1\rangle = |\downarrow\downarrow\rangle \Rightarrow C_{\uparrow\uparrow} = C_{\uparrow\downarrow} = C_{\downarrow\uparrow} = 0\ ,\ C_{\downarrow\downarrow} = 1 \end{cases}$$

$$|s, m_s\rangle = |0, 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \Rightarrow C_{\uparrow\uparrow} = C_{\downarrow\downarrow} = 0\ ,\ C_{\uparrow\downarrow} = \frac{1}{\sqrt{2}}\quad C_{\downarrow\uparrow} = \frac{-1}{\sqrt{2}}$$

در ادامه باید به دنبال یک روش سیستماتیک برای پیدا کردن ضرایب Clebsch-Gordan باشیم. (بعداً)

Note :

حالت‌های triplet, singlet به ترتیب ذکر شده اند

روش حساب کردن این گزاره:

$$|s=1, m_s=+1\rangle = |\uparrow\uparrow\rangle = |\uparrow\rangle_{(1)} \otimes |\uparrow\rangle_{(2)}$$

$$\langle 1, +1 | 1, +1\rangle = \langle \uparrow\uparrow | \uparrow\uparrow\rangle = {}_{(2)}\langle\uparrow| \otimes {}_{(1)}\langle\uparrow|\uparrow\rangle_{(1)} \otimes |\uparrow\rangle_{(2)}$$

$$\langle 1,+1 | 1,+1 \rangle = \langle \uparrow\uparrow | \uparrow\uparrow \rangle = {}_{(2)}\langle \uparrow | \otimes {}_{(1)}\langle \uparrow | \uparrow \rangle_{(1)} \otimes | \uparrow \rangle_{(2)}$$

$$= {}_{(1)}\langle \uparrow | \uparrow \rangle_{(1)} \; {}_{(2)}\langle \uparrow | \uparrow \rangle_{(2)} = 1$$

به همین ترتیب

$$\langle 0,0 | 00 \rangle = \frac{1}{2} \left(\langle \uparrow\downarrow | - \langle \downarrow\uparrow | \right) \left(| \uparrow\downarrow \rangle - | \downarrow\uparrow \rangle \right)$$

$$= \frac{1}{2} \left(\underbrace{\langle \uparrow\downarrow | \uparrow\downarrow \rangle}_{=1} - \underbrace{\langle \uparrow\downarrow | \downarrow\uparrow \rangle}_{=0} - \underbrace{\langle \downarrow\uparrow | \uparrow\downarrow \rangle}_{=0} + \underbrace{\langle \downarrow\uparrow | \downarrow\uparrow \rangle}_{1} \right)$$

$$= \frac{1}{2} (1+1) = 1$$

$$\langle \uparrow\downarrow | \uparrow\downarrow \rangle = {}_{(1)}\langle \uparrow | \uparrow \rangle_{(1)} \; {}_{(2)}\langle \downarrow | \downarrow \rangle_{(2)} = 1$$

$$\langle \uparrow\downarrow | \downarrow\uparrow \rangle = {}_{(1)}\langle \uparrow | \downarrow \rangle_{(1)} \; {}_{(2)}\langle \downarrow | \uparrow \rangle_{(2)} = 0$$

$$\langle \downarrow\uparrow | \uparrow\downarrow \rangle = {}_{(1)}\langle \downarrow | \uparrow \rangle_{(1)} \; {}_{(2)}\langle \uparrow | \downarrow \rangle_{(2)} = 0$$

$$\langle \downarrow\uparrow | \downarrow\uparrow \rangle = {}_{(1)}\langle \downarrow | \downarrow \rangle_{(1)} \; {}_{(2)}\langle \uparrow | \uparrow \rangle_{(2)} = 1$$

وغیره !