$$e^{i\Gamma[A]} = \int \mathcal{D}A_\mu \, \mathcal{D}c \, \mathcal{D}\bar{c} \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \; e^{i(S_{gauge} + S_{g.f.} + S_{ghost} + \underbrace{S_{ferm.}}_{\text{Matter}}) + i\,\delta S}$$

$$= e^{-\frac{i}{4g^2}\int d^4x \, (F^{\mu\nu}[A])^2 + i\int d^4x \, \delta\mathcal{L}[A]}$$

$$\times \int \mathcal{D}A_\mu \, \mathcal{D}c \, \mathcal{D}\bar{c} \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \; e^{i\int (\mathcal{L}_A + \mathcal{L}_\psi + \mathcal{L}_c)_{quadr.} d^4x}$$

$$= \exp\left(\frac{-i}{4g^2}\int d^4x \, (F^{\mu\nu}[A])^2 + i\int d^4x \, \delta\mathcal{L}[A]\right)$$

$$\times \underbrace{\left(\det \Delta_{G,1}\right)^{-1/2}}_{\substack{\text{Gauge fields in the} \\ \text{adjoint representation} \\ \text{and spin} = 1}} \; \underbrace{\left(\det \Delta_{r,\frac{1}{2}}\right)^{N_f/2}}_{\substack{\text{Fermions in the} \\ r-\text{representation} \\ \& \text{ spin } \frac{1}{2}}} \; \underbrace{\left(\det \Delta_{G,0}\right)^{+1}}_{\substack{\text{Ghost (Grassmann)} \\ \text{Spin 0 -- in the adjoint} \\ \text{representation}}}$$

$$\Gamma[A] = -\frac{1}{4g^2}\int d^4x \, \left(F_{\mu\nu}^a[A]\right)^2 - i\sum_j \alpha_j \log\det \Delta_{r,j} + \int \delta\mathcal{L} \, d^4x$$

$\alpha_j = -\frac{1}{2}$ for gauge fields in the adjoint representation

$\alpha_j = \frac{N_f}{2}$ for (Dirac) fields in the fundamental representation

$\alpha_j = +1$ for ghosts in the adjoint representation.

این دترمینان‌ها باید برای تک تک موجودات محاسبه شوند :

$$\log\det \Delta_{r,j} = \log\det\left(-D^2 + F_{\rho\sigma}^a \, \gamma^{\rho\sigma} t^a\right)$$

$$= \log\det\left(-\partial^2 + \Delta^{(1)} + \Delta^{(2)} + \Delta^{(J)}\right)$$

$-D^2$ را باز کرده‌ام (البته $-\partial^2$ جدا شده است)

$$\begin{cases} \Delta^{(1)} = i\,(\partial_\mu A^{\mu a})t^a + 2i\,A_\mu^a \, t^a \partial^\mu \\ \Delta^{(2)} = A_\mu^a \, A^{\mu b} \; t^a t^b \end{cases}$$

$$\Delta^{(J)} = \left(F_{\rho\sigma}^a \, \gamma^{\rho\sigma}\right) t^a = \Big[\underbrace{\left(\partial_\rho A_\sigma^a - \partial_\sigma A_\rho^a\right) t^a}_{\Delta^J_{(1)}} + \underbrace{f^{abc} A_\rho^b A_\sigma^c}_{\Delta^J_{(2)}}\Big] \gamma^{\rho\sigma} t^a$$

$$\to \log\det\left(-\partial^2 + \Delta^{(1)} + \Delta^{(2)} + \Delta^{(J)}\right) = \ln\det\left(\left(-\partial^2\right)\left(1 + \frac{\Delta^{(1)} + \Delta^{(2)} + \Delta^{(J)}}{(-\partial^2)}\right)\right)$$

$$\simeq \log\det\left(1 + (-\partial^2)^{-1}\left(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(J)}\right)\right)$$

$$= \mathrm{Tr}\,\log(\quad) = \mathrm{Tr}(\quad) - \frac{1}{2}\mathrm{Tr}(\quad)^2 + \cdots$$

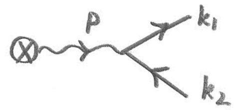
نشان می‌دهیم

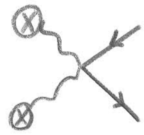
برای مقاصد ما فقط کافی است جملات مربعی در A_μ را نگه داریم:

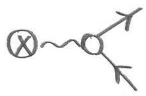
$$\Gamma[A] = \frac{-1}{2g^2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}^a_\mu(k) \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \tilde{A}^a_\nu(-k) + \text{corrections}$$

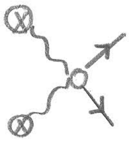
$$\Longrightarrow \quad \frac{1}{g^2_{eff}} = \frac{1}{g^2} + \text{corrections}$$

نکته: (ما در فرمیون به صورت تقریبی نداریم $\not{D}^2$)

a)  $\Delta^{(1)}: \quad -i(p+2k_2)_\mu t^a (2\pi)^4 \delta^4(p+k_1+k_2)$

b)  $\Delta^{(2)}: \quad i g^{\mu\nu} t^a t^b$

c)  $\Delta^{(J)}_{(1)}:$

d)  $\Delta^{(J)}_{(2)}:$

1)  $\sim \frac{-1}{2} \mathrm{Tr}\left((-\partial^2)^{-1} \Delta^{(1)} (-\partial^2)^{-1} \Delta^{(1)}\right)$

2)  $\sim \mathrm{Tr}\left((-\partial^2)^{-1} \Delta^{(2)}\right)$

3)  $\sim -\frac{1}{2} \mathrm{Tr}\left((-\partial^2)^{-1} \Delta^{J}_{(1)} (-\partial^2)^{-1} \Delta^{(J)}_{(1)}\right)$

Use:

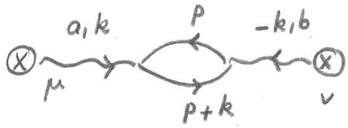
$$\mathrm{tr}(t^a) = 0$$

$$\mathrm{tr}(t^a t^b) = C(r)\, d(j)\, \delta^{ab}$$

$$\mathrm{tr}(\mathcal{J}^{\rho\sigma} \mathcal{J}^{\alpha\beta}) = C(j) \left(g^{\rho\alpha} g^{\sigma\beta} - g^{\rho\beta} g^{\sigma\alpha}\right)$$

$d(j)$ = # of spin components

$C(j)$ = trace over spin indices ($\to$ see Table)

(1) 

$$-\frac{1}{2}\,\mathrm{Tr}\left((-\partial^2)^{-1}\,\Delta^{(1)}\,(-\partial^2)^{-1}\,\Delta^{(1)}\right)$$

$$= \frac{-1}{2}\,\mathrm{tr}(t^a t^b)\int\frac{d^4k}{(2\pi)^4}\,\tilde{A}^{\mu,a}(k)\,\tilde{A}^{\nu,b}(-k)$$

$$\times\int\frac{d^4p}{(2\pi)^4}\,\frac{(k+2p)_\mu\,(2p+2k-k)_\nu}{p^2\,(p+k)^2}$$

$$\boxed{= \frac{-1}{2}\,\mathrm{tr}(t^a t^b)\int\frac{d^4k}{(2\pi)^4}\,\tilde{A}^{\mu,a}(k)\,\tilde{A}^{\nu,b}(-k)\int\frac{d^4p}{(2\pi)^4}\,\frac{(k+2p)_\mu\,(2p+k)_\nu}{p^2\,(p+k)^2}}$$

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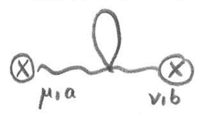
(2) 

$$\mathrm{Tr}\left((-\partial^2)^{-1}\,\Delta^{(2)}\right) = \int d^4x\,A_\mu^a(x)\,A_\nu^b(x)\,g^{\mu\nu}\,\mathrm{tr}(t^a t^b)\,\varphi^*(x)\varphi(x)$$

φ می‌تواند هر یک از میدان‌های A_μ ، c ، $\bar{\psi}$ ، ψ یا $\bar{c}$ باشد

$$= \int_{p_1, p_2, k_1, k_2} d^4x\; e^{i(p_1+p_2-k_1+k_2)x}\,\underbrace{\tilde{\varphi}^*(k_1)\tilde{\varphi}(k_2)}\,\tilde{A}_\mu^a(p_1)\,\tilde{A}_\nu^b(p_2)$$

$$\times\, g^{\mu\nu}\,\mathrm{tr}(t^a t^b)\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2}$$

$$p_1+p_2 = k_1-k_2$$

$$\left.\begin{array}{r} k_1 = p_1+p_2+k_2 \\ \underbrace{\varphi^*(k_1)\varphi(k_2)} \longrightarrow k_1 = k_2 \end{array}\right|\; p_1 = -p_2 \equiv k$$

نامگذاری

$$\boxed{\int\frac{d^4k}{(2\pi)^4}\,\tilde{A}_\mu^a(k)\,\tilde{A}^b_\nu(-k)\,g^{\mu\nu}\,\mathrm{tr}(t^a t^b)\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2}}$$

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(3) 

$$\Delta^{(J)}_{(1)} = (\partial_\rho A_\sigma^a - \partial_\sigma A_\rho^a)\,t^a\,\gamma^{\rho\sigma}$$

ترکیب vertex

$$\int d^4x\,\left\{i\,\varphi^*(x)\,\partial_\rho A_\sigma^b(x)\,\varphi(x)\,\gamma^{\rho\sigma} - i\,\varphi^*(x)\,(\partial_\sigma A_\rho^b(x))\,\varphi(x)\,\gamma^{\rho\sigma}\right\}t^b$$

$$\overset{\gamma^{\rho\sigma} = -\gamma^{\sigma\rho}}{=} \quad = 2i\int d^4x\;\varphi^*(x)\,\partial_\rho A_\sigma^b\,t^b\,\varphi(x)\,\gamma^{\rho\sigma}$$

از ترکیب عامل رأس

برای کل سطح خواهیم داشت :

$$-\frac{1}{2}\,\mathrm{Tr}\left((-\partial^2)^{-1}\,\Delta^J_{(1)}\,(-\partial^2)\,\Delta^J_{(1)}\right)$$

$$= -\frac{1}{2}(2i)^2\,\mathrm{Tr}\left\{\int\varphi^*(x)\,\partial_\rho A_\sigma^a(x)\,t^a\,\gamma^{\rho\sigma}\,\varphi(x)\,d^4x \times\int\varphi^*(y)\,d^4y\,\partial_\alpha A_\beta^b(y)\,t^b\,\gamma^{\alpha\beta}\,\varphi(y)\right\}$$

$$= -\frac{1}{2}(2i)^2\int d^4x\;e^{i(-p_1+p_2)x}\,\tilde{\varphi}^*(p_1)\,(\partial_\rho e^{ikx})\,\tilde{A}_\sigma^a(k)\,\tilde{\varphi}(p_2)$$

$$\times\int d^4y\;e^{i(-p_1'+p_2')y}\,\tilde{\varphi}^*(p_1')\,(\partial_\alpha e^{ik'y})\,\tilde{A}_\beta^b(k')\,\tilde{\varphi}(p_2')$$

$$\times\left(\mathrm{tr}(\gamma^{\rho\sigma}\gamma^{\alpha\beta})\right)\,\mathrm{tr}(t^a t^b)$$

$$= -\frac{1}{2}(2i)^2 (i)^2 \int d^4x \, e^{i(-p_1+p_2+k)} \; k_\rho \; \tilde{\varphi}^*(p_1)\tilde{A}_\sigma^{\,a}(k)\,\tilde{\varphi}(p_2)$$

$$\times \int d^4x \, e^{i(-p_1'+p_2'+k')} \; k'_\alpha \; \tilde{\varphi}^*(p_1')\,\tilde{A}_\beta^{\,b}(k')\,\tilde{\varphi}(p_2')$$

$$\times \operatorname{tr}(t^a t^b) \quad \operatorname{tr}(\mathcal{J}^{\rho\sigma}\mathcal{J}^{\alpha\beta})$$

$$\rightarrow \quad \tilde{\varphi}^*(p_1)\,\tilde{\varphi}(p_2)\;\;\tilde{\varphi}^*(p_1')\,\tilde{\varphi}(p_2') = \frac{1}{p_1'^{\,2}}\,\frac{1}{p_2'^{\,2}} = \frac{1}{(p_1-k)^2}\,\frac{1}{p_1^{\,2}}$$

$$\text{or} \quad \frac{1}{p_2^{\,2}\,(k+p_2)^2}$$

$p_2 = p_1' \quad , \quad p_1 = p_2' \quad *$

$k = p_1 - p_2$

$k' = p_1' - p_2' \overset{*}{=} p_2 - p_1 = -k$

$$= -2 \int \frac{d^4k}{(2\pi)^4} \, \tilde{A}_\sigma^{\,a}(k)\,\tilde{A}_\beta^{\,b}(-k)$$

$$\times \int \frac{k_\rho k_\alpha \operatorname{tr}(\mathcal{J}^{\rho\sigma}\mathcal{J}^{\alpha\beta})\operatorname{tr}(t^a t^b)}{p_2^{\,2}\,(k+p_2)^2}\,\frac{d^4p_2}{(2\pi)^4} \qquad \sigma\to\mu \quad \beta\to\nu$$

$\operatorname{tr}(t^a t^b) = C(r)\,d(j)\,\delta^{ab}$

$$\operatorname{tr}(\mathcal{J}^{\rho\mu}\mathcal{J}^{\alpha\nu}) = (g^{\rho\alpha}g^{\mu\nu} - g^{\rho\nu}g^{\mu\alpha})\,C(j)$$

$$k_\rho k_\alpha \operatorname{tr}(\mathcal{J}^{\rho\mu}\mathcal{J}^{\alpha\nu}) = (k^2 g^{\mu\nu} - k^\nu k^\mu)$$

$$= -\frac{1}{2}\, 4C(j)\operatorname{tr}(t^a t^b) \int \frac{d^4k}{(2\pi)^4}\,\tilde{A}^a_\mu(k)\,\tilde{A}^b_\nu(-k)\,(k^2 g^{\mu\nu} - k^\mu k^\nu)$$

$$\times \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2(k+p)^2}$$

به این ترتیب :

1) 

$$= \frac{1}{2}\operatorname{tr}(t^a t^b) \int \frac{d^4k}{(2\pi)^4}\,\tilde{A}^{\mu,a}(k)\,\tilde{A}^{\nu,b}(-k) \int \frac{d^4p}{(2\pi)^4}\,\frac{(2p+k)_\mu(2p+k)_\nu}{p^2(p+k)^2}$$

2) 

$$= g^{\mu\nu}\operatorname{tr}(t^a t^b) \int \frac{d^4k}{(2\pi)^4}\,\tilde{A}^{\mu,a}(k)\,\tilde{A}^{\nu,b}(-k) \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2}$$

3) 

$$= -\frac{1}{2}\operatorname{tr}(t^a t^b)\,4C(j) \int \frac{d^4k}{(2\pi)^4}\,\tilde{A}^{\mu,a}(k)\,\tilde{A}^{\nu,b}(-k)\,(k^2 g_{\mu\nu} - k_\mu k_\nu)$$

$$\times \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2(p+k)^2}$$

Use:

$$\int_0^1 dx \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2-\Delta)^2} = \int_0^1 dx \frac{i(-1)^2}{(4\pi)^{\frac{d}{2}}} \Gamma\left(2-\frac{d}{2}\right) \frac{1}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$= \frac{i}{(4\pi)^2} \int_0^1 dx \left(\frac{4\pi}{\Delta}\right)^{\epsilon/2} \left(\frac{2}{\epsilon} - \gamma_E\right) = \frac{i}{16\pi^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \gamma_E\right)\left(1 - \frac{\epsilon}{2} \ln \frac{\Delta}{4\pi}\right)$$

$$= \int_0^1 dx \left(\frac{2}{\epsilon} - \gamma_E - \ln \frac{\Delta}{4\pi}\right) \xrightarrow[\text{MS - scheme}]{} \frac{i}{16\pi^2} \ln \frac{\mu^2}{k^2}$$

$$\int \frac{d^4p}{(2\pi)^4} \frac{p_\mu p_\nu}{(p^2-\Delta)^2} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{g_{\mu\nu}}{2} \frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

نتیجه

$$(1+2) = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}^{\mu,a}(k)\, \tilde{A}^{\nu,a}(-k)\, (k^2 g_{\mu\nu} - k_\mu k_\nu) \left\{ \frac{C(r) d(j)}{3(4\pi)^2} \frac{2}{\epsilon} + \cdots \right\}$$

$$(3) = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}^{\mu,a}(k)\, \tilde{A}^{\nu,a}(-k)\, (k^2 g_{\mu\nu} - k_\mu k_\nu) \left\{ \frac{-4C(j)C(r)}{(4\pi)^2} \frac{2}{\epsilon} + \cdots \right\}$$

Together:

$$\boxed{\log \det \Delta_{r,j} \Big|_{\substack{\text{quadratic} \\ \text{in } A}} = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}^{\mu,a}(k)\, \tilde{A}^{\nu,a}(-k)\, (k^2 g_{\mu\nu} - k_\mu k_\nu) \times \left\{ \frac{1}{3} d(j) - 4C(j) \right\} C(r) \left(\frac{2}{\epsilon} + \cdots\right)}$$

$$\Gamma[A] = \frac{-1}{4g^2} \int d^4x\, (F_{\mu\nu}{}^a)^2 - i \sum_j d_j \log\det \Delta_{r,j} + \int d^4x\, \delta\mathcal{L}$$

$$= -\frac{1}{2g^2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu{}^a(k)\, (k^2 g^{\mu\nu} - k^\mu k^\nu)\, \tilde{A}_\nu{}^a(-k)$$

$$+ \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu{}^a(k)\, (k^2 g^{\mu\nu} - k^\mu k^\nu)\, \tilde{A}_\nu^a(-k) \left[\sum_j \frac{\alpha(j)}{16\pi^2} \left(\frac{1}{3} d(j) - 4C(j)\right) C(r) \right] \times \left(\frac{2}{\epsilon} + \text{finite}\right) + \int \delta\mathcal{L}\, d^4x$$

$$\stackrel{!}{=} \frac{-1}{2 g_{\text{eff}}^2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu{}^a(k)\, (k^2 g_{\mu\nu} - k_\mu k_\nu)\, \tilde{A}_\nu{}^a(-k)$$

$$\circ\ \frac{2}{\epsilon} \longrightarrow \ln \frac{k^2}{\mu^2}$$

$$-\frac{1}{g_{\text{eff}}^2} = -\frac{1}{g^2} + \sum_j \frac{\alpha(j)}{16\pi^2} \left(\frac{1}{3} d(j) - 4C(j)\right) C(r) \ln \frac{k^2}{\mu^2}$$

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{g^2} - A \ln \frac{k^2}{\mu^2} \longrightarrow g_{\text{eff}}^2 = \frac{g^2}{1 - g^2 A \ln \frac{k^2}{\mu^2}}$$

$$\rightarrow \quad g_{\text{eff}} = g\left(1 + \frac{g^2 A}{2} \ln \frac{k^2}{\mu^2}\right) \longrightarrow \mu \frac{d}{d\mu} g_{\text{eff}} = \frac{g^3 A}{2} \frac{\left(\frac{-2\mu k^2}{\mu^3}\right)}{\frac{k^2}{\mu^2}} =$$

$$= -g^3 A$$

$$\boxed{\beta(g) = \mu \frac{d}{d\mu} g_{eff} = -g^3 A = \frac{-g^3}{16\pi^2} \sum_j \alpha(j) \left(\frac{1}{3} d(j) - 4C(j) \right) C(r)}$$

	for SU(3) $C_2(G) = N_c$		$C(r) = \frac{1}{2}$		$C(s) = \frac{1}{2}$
	ghosts (scalars)	gauge fields	Dirac fields	Weyl ferm.	Complex scalars
$d(j)$	1	4	4	2	1
$C(j)$	0	2	1	1	0
$\alpha(j)$	1	$-\frac{1}{2}$	$\frac{N_f}{2}$	$\frac{N_f}{2}$	$(-\frac{1}{2} \times 2) N_s$
$\alpha(j)\left(\frac{1}{3} d(j) - 4C(j)\right)$	$\frac{1}{3}$	$\frac{10}{3}$	$-\frac{4N_f}{3}$	$-\frac{2}{3} N_f$	$-\frac{N_s}{3}$

$$A = \sum_j \frac{\alpha(j)}{16\pi^2} \left(\frac{1}{3} d(j) - 4C(j) \right) C(r) =$$

$$= \frac{1}{16\pi^2} \left[\left(\frac{1}{3} + \frac{10}{3} \right) C_2(G) - \frac{4}{3} N_f\, C(r) - \frac{N_s}{3} C(s) \right]$$

$$\boxed{A = \frac{1}{16\pi^2} \left(\frac{11}{3} C_2(G) - \frac{4}{3} N_f\, C(r) \right)}$$

$C_2(G) = N_c = 3 \qquad C(r) = \frac{1}{2} \qquad \left(C(s) = \frac{1}{2} \right.$ که در اینجا لازم نداریم $\left. \right)$

$$\boxed{\beta(g) = -A g^3 = \frac{-g^3}{16\pi^2} \left(11 - \frac{2}{3} N_f \right)}$$

$\mathcal{N} = 1$ SUSY QCD: $\quad V =$ Vector multiplet (4vectors v_μ, Weyl gaugino λ)

$Q =$ Matter multiplet (Weyl quark q, squarks φ)

$f = 1, \dots, N_f$

	Ghost	Gauge Multiplet		Matter Multiplet	
		Vector	Gaugino	quarks	squarks
$d(j)$	1	4	2	2	1
$C(j)$	0	2	$\frac{1}{2}$	$\frac{1}{2}$	0
$\alpha(j)$	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{n_f}{2}$	$-n_f = -n_s$
$\alpha(j)\left(\frac{1}{3} d(j) - 4C(j)\right)$	$\frac{1}{3}$	$\frac{10}{3}$	$-\frac{2}{3}$	$-\frac{2}{3} n_f$	$-\frac{n_f}{3}$
		$C_2(G) = N_c$		$C_f = \frac{1}{2}$	

$2 \times \frac{1}{2} = 1$

$$\beta(g) = -Ag^3$$

$$A = \left[\frac{(1+10-2)}{3} N_c + \frac{(-2-1)}{3} n_f \frac{1}{2} \right] = 3N_c - N_f$$

$$\frac{1}{2} n_f = N_f$$

SUSY خصوصیات

$$\sum_j \alpha_j\, d(j) = 0$$

$1 - 2 + 1 = 0$ ghost - gauge - gaugino

$n_f - n_f = 0$ Matter multiplet

$$A = \sum_j \alpha_j \left(\frac{1}{3} d(j) - 4C(j) \right) = -4 \sum_j \alpha_j C(j) = 0 \curvearrowright \beta(g) = 0$$

$N=4$ Susy "

No running in $N=4$ SUSY