نظریه میدان کوانتومی غیرآبلی (میدان پیمانه‌ای غیرآبلی):

Abelian Gauge theory:

Global $U(1)$: $\mathcal{L}_0 = \bar{\psi}(i\not{\partial} - m)\psi$ is invariant under "global" $U(1)$ gauge transformation.

$$\psi \to e^{i\alpha}\psi \quad \& \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi}$$

local $U(1)$: $\psi \to e^{i\alpha(x)}\psi \quad \& \quad \bar{\psi} \to e^{-i\alpha(x)}\bar{\psi}$

$$\delta_\alpha \psi = i\alpha(x)\psi(x) \quad ; \quad \delta_\alpha \bar{\psi} = -i\alpha(x)\bar{\psi}(x)$$

Minimal Coupling: Interaction with EM - Field (A_μ photon)

$$\mathcal{L}_f = \bar{\psi}(x)(i\not{D} - m)\psi(x)$$

$$D_\mu \psi(x) = (\partial_\mu - ig A_\mu(x))\psi(x)$$

$\mathcal{L}_f$ is invariant under $\delta_\alpha \psi = i\alpha(x)\psi(x)$ & $\delta_\alpha A_\mu = \frac{-1}{g}\partial_\mu \alpha(x)$

$$\delta_\alpha \bar{\psi} = -i\alpha\bar{\psi}(x)$$

$\mathcal{L}_g = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is invariant under $U(1)$ local gauge transformation.

$\longrightarrow$ To determine the photon propagator $\longrightarrow$ Gauge fixing term $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ is to be added to $\mathcal{L}$

$$\psi^\alpha(x) = \begin{pmatrix} \psi_1^\alpha(x) \\ \psi_2^\alpha(x) \end{pmatrix}$$

$\psi_{\alpha, f}(x)$

Dirac index $\alpha = 1, 2, 3, 4$

flavor index $f = 1, 2$

$$\Psi(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}$$

Global $SU(2)$ transformation: $\Psi(x) = e^{i\alpha_i \tau_i}\Psi(x)$

$\alpha_i =$ Real numbers $\quad i = 1, 2, 3$

$\tau_i \equiv \frac{\sigma_i}{2} \quad i = 1, 2, 3 \quad \sigma_i$ Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We construct a theory which is invariant under $SU(2)$ global (flavor) symmetry

Lorentz Scalar

$$\mathcal{L} = tr\left(\bar{\Psi}(x)\,(i\partial\!\!\!/-m)\,\Psi(x)\right) = \bar{\Psi}_\alpha{}^{a}(x)\left(i\,(\gamma^\mu)_{\alpha\beta}\,\partial_\mu - m\,\delta_{\alpha\beta}\right)\Psi_\beta{}^{a}$$

<u>Note:</u> $m_1 = m_2 = m$ (isospin symmetry) (SU(2) flavor symmetry)

$\alpha, \beta = 1, \dots 4$ Dirac indices

$a = 1,2$ flavor indices

Global SU(2) transformation:

$$\delta\psi^a = (i\alpha^i \tau_i \psi)^a = i\alpha^i (\tau_i)^{ab}\psi^b$$

$$\Psi \rightarrow \Psi' = e^{i\vec{\alpha}.\vec{\tau}}\,\Psi$$

$$\delta\bar{\Psi}^a = -i\,(\bar{\Psi}\alpha^i\tau_i)^a = -i\alpha^i\,\bar{\Psi}^b\,(\tau_i)^{ba}$$

$$\bar{\Psi} \rightarrow \bar{\Psi}' = \bar{\Psi}\,e^{-i\vec{\alpha}.\vec{\tau}}$$

$\vec{\alpha}.\vec{\tau} = \alpha_1\tau_1 + \alpha_2\tau_2 + \alpha_3\tau_3$ در اینجا به تعداد مولدهای گروه SU(2) پارامتر تبدیل وجود دارد

$\tau_i = \frac{\sigma_i}{2}$ as previously defined.

$\mathcal{L}$ is invariant under these global SU(2) transformation

$$\mathcal{L}' = tr\left(\bar{\Psi}'(i\partial\!\!\!/-m)\Psi\right) = tr\left(\bar{\Psi}e^{-i\vec{\alpha}.\vec{\tau}}(i\partial\!\!\!/-m)e^{i\vec{\alpha}.\vec{\tau}}\Psi\right)$$

$$\partial_\mu\vec{\alpha} = 0 \Rightarrow$$

using $e^{-i\vec{\alpha}.\vec{\tau}}\,e^{+i\vec{\alpha}.\vec{\tau}} = 1 \Rightarrow \delta_{\alpha}\mathcal{L} = 0$

or $\mathcal{L}$ is invariant under global SU(2) transf.

Local SU(2) "gauge" transformation:

$$\mathcal{L} = \bar{\Psi}^a\,(i\partial\!\!\!/-m)\,\psi^a \quad \text{or} \quad \mathcal{L} = tr\left(\bar{\Psi}(i\partial\!\!\!/-m)\Psi\right)$$

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + igA_\mu$$

$$V(x) = e^{i\vec{\alpha}.\vec{\tau}}$$

$$V^{+}(x) = e^{-i\vec{\alpha}.\vec{\tau}} \qquad \vec{\sigma}^{+} = \vec{\sigma}$$

$$\psi^b(x) \rightarrow (V(x)\psi(x))^b = V^{ba}(x)\,\psi^a(x) = \left(e^{i\vec{\alpha}.\vec{\tau}}\right)^{ba}\psi^a(x)$$

$$\bar{\Psi}^b(x) \rightarrow \left(\bar{\Psi}(x)V^{+}(x)\right)^b = \bar{\Psi}^a(x)\left(V^{+}(x)\right)^{ab} = \bar{\Psi}^a(x)\left(e^{-i\vec{\alpha}.\vec{\tau}}\right)^{ab}$$

ادعا: اگر $\mathcal{L} = tr\left(\bar{\Psi}(i\not{D}-m)\Psi\right)$ بخواهد تحت تبدیل موضعی $SU_f(2)$ ناوردا باشد

$$A_\mu \longrightarrow V(x) A_\mu(x) V^+(x) - \frac{i}{g} V(x)\partial_\mu V^+(x)$$

<u>Proof:</u>

$$\mathcal{L} = \bar{\Psi}_\alpha{}^a (i\not{\partial}-m)_{\alpha\beta} \Psi_\beta{}^a - g\,\bar{\Psi}_\alpha{}^a A_\mu{}^i (\tau^i)_{ab} \Psi_\beta{}^b (\gamma^\mu)^{\alpha\beta}$$

$$= tr\left(\bar{\Psi}(i\not{\partial}-m)\Psi - g\,\bar{\Psi}\not{A}\Psi\right)$$

$$\mathcal{L}' = tr\left(\bar{\Psi}'(i\not{\partial}-m)\Psi' - g\,\bar{\Psi}'\not{A}'\Psi'\right)$$

$$= tr\left\{(\bar{\Psi}(x)V^+(x))(i\not{\partial}-m)(V(x)\Psi(x))\right\}$$

$$- g\,tr\left\{(\bar{\Psi}(x)V^+(x)\ \not{A}_\mu'(x)\ (V(x)\Psi(x))\right\}$$

$$= tr\Big\{\bar{\Psi}(x)\left(V^+(x)\ i\not{\partial}V(x)\right)\Psi(x) + \bar{\Psi}(x)(i\not{\partial}-m)\overbrace{V^+(x)V(x)}^{=1}\Psi(x)$$

$$- g\,\bar{\Psi}(x)\left(V^+(x)\not{A}_\mu'(x)V(x)\right)\Psi(x)\Big\}$$

$$\overset{!}{=} tr\left\{\bar{\Psi}(x)(i\not{\partial}-m)\Psi(x) - g\,\bar{\Psi}(x)\not{A}(x)\Psi(x)\right\}$$

$$\mathcal{L}' = +\,tr\left\{\bar{\Psi}(x)\left[V^+(x)i\not{\partial}V(x) - g\,V^+(x)\not{A}'V(x)\right]\Psi(x)\right\}$$

$$+\,tr\left(\bar{\Psi}(x)(i\not{\partial}-m)\Psi(x)\right) =$$

$$-g\,A_\mu = V^+(x)\,i\partial_\mu V(x) - g\,V^+(x)A_\mu'(x)V(x)$$

$$-g\,V(x)A_\mu(x)V^+(x) - \left(i\partial_\mu V(x)\right)V^+(x) = -g\,A_\mu'(x)$$

$$\curvearrowright\quad A_\mu'(x) = V(x)A_\mu(x)V^+(x) + \frac{i}{g}\left(\partial_\mu V(x)\right)V^+(x)$$

now use $VV^+ = 1 \longrightarrow (\partial_\mu V)V^+ = -V\partial_\mu V^+$

$$\boxed{A_\mu'(x) = V(x)A_\mu(x)V^+(x) - \frac{i}{g}V(x)\partial_\mu V^+(x)}$$

<u>Infinitesimal gauge transformation:</u>

$$\Psi'(x) = e^{i\vec{\alpha}\cdot\vec{\tau}}\Psi(x) \qquad \delta\Psi(x) = (i\vec{\alpha}(x)\cdot\vec{\tau})\Psi(x)$$

$$\bar{\Psi}'(x) = \bar{\Psi}(x)e^{-i\vec{\alpha}\cdot\vec{\tau}} \qquad \delta\bar{\Psi}(x) = \bar{\Psi}(x)(-i\vec{\alpha}(x)\cdot\vec{\tau})$$

$$A_\mu'(x) = \left(e^{i\vec{\alpha}(x)\cdot\vec{\tau}}\right)A_\mu(x)\left(e^{-i\vec{\alpha}(x)\cdot\vec{\tau}}\right) - \frac{i}{g}\left(e^{i\vec{\alpha}(x)\cdot\vec{\tau}}\right)\partial_\mu\left(e^{-i\vec{\alpha}(x)\cdot\vec{\tau}}\right)$$

$$\sim (1+i\vec{\alpha}\cdot\vec{\tau})A_\mu(x)(1-i\vec{\alpha}\cdot\vec{\tau}) - \frac{i}{g}(1+i\vec{\alpha}\cdot\vec{\tau})\partial_\mu(1-i\vec{\alpha}(x)\cdot\vec{\tau})$$

$$= A_\mu(x) + i\,\alpha_j\tau_j A_\mu(x) - i\,\alpha_j A_\mu(x)\tau_j - \frac{i}{g}(-i\,\partial_\mu\alpha_j)\tau_j + O(\alpha^2)$$

$$= A_\mu(x) + i\,\alpha_j(x)\left[\tau_j, A_\mu(x)\right] - \frac{1}{g}\partial_\mu\vec{\alpha}\cdot\vec{\tau}$$

Now use $A_\mu(x) = A_\mu{}^k\tau_k$

$$A'^{k}_{\mu}\tau_k = A_\mu{}^k \tau_K + i\alpha_j A_\mu{}^i \, [\tau_j, \tau_i] - \frac{1}{g}\, \partial_\mu \alpha^k \tau^k$$

$$[\tau_j, \tau_i] = i f_{jik} \tau_k \qquad\qquad f_{jik} = - f_{ijk}$$

$$A'^{k}_{\mu} = A_\mu{}^k + i\alpha_j \, i f_{jik} A_\mu{}^i - \frac{1}{g}\partial_\mu \alpha^k$$

$$\boxed{A'_{\mu i k} \quad A_{\mu i k} + \alpha_j \; f_{ijk} A_{\mu i} - \frac{1}{g}\partial_\mu \alpha_k}$$

$$A'_{\mu i k}\,\tau_k = A'_\mu$$

نتیجه اینکه

$$\delta\psi^a = (i\vec{\alpha}(x).\vec{\tau})^{ab}\,\psi^b(x)$$

$$\delta\bar{\psi}^a = \bar{\psi}^b(x)\,(-i\vec{\alpha}(x).\vec{\tau})^{ba}$$

$$\delta A_\mu{}^k = f^{ijk} A_\mu{}^i \alpha^j(x) - \frac{1}{g}\partial_\mu \alpha^k$$

infinitesimal $SU(2)$ local-gauge transformation.

سؤال: $D_\mu\psi$ چگونه تبدیل می‌شود؟

$$D_\mu\psi(x) = (\partial_\mu + igA_\mu(x))\,\psi(x)$$

$$D'_\mu \psi'(x) = \partial_\mu\psi'(x) + ig A'_\mu(x)\psi'(x)$$

$$= \partial_\mu\left(V(x)\psi(x)\right) + ig\left(V(x)A_\mu(x)V^+(x) \underbrace{- \frac{i}{g} V(x)\,\partial_\mu V^+(x)}_{-\partial_\mu V(x) V^+(x)}\right)V(x)\psi(x)$$

$$= (\partial_\mu V(x))\psi(x) + V(x)\,\partial_\mu\psi(x) + ig\, V(x) A_\mu(x)\underbrace{V^+(x)V(x)}_{=1}\psi(x)$$

$$- \partial_\mu V(x)\,\psi(x)$$

$$= V(x)\left(\partial_\mu\psi(x) + ig\, A_\mu(x)\psi(x)\right) = V(x)\, D_\mu\psi(x)$$

$$\rightarrow \boxed{D'_\mu \psi'(x) = V(x)\, D_\mu\psi(x)}$$

Remember $\psi'(x) = V(x)\psi(x)$

D_μ (the covariant derivative) acting on $\psi(x)$, transforms exactly like $\psi(x)$ (in the fundamental representation)

سؤال: چگونه از D_μ، $F_{\mu\nu}$ را بسازیم؟

In QED:
Abelian Gauge Theory

$$[D_\mu, D_\nu] = [(\partial_\mu + igA_\mu), (\partial_\nu + igA_\nu)] =$$

$$= (\partial_\mu\partial_\nu + ig\,\partial_\mu A_\nu + ig\, A_\nu\partial_\mu + ig A_\mu\partial_\nu - g^2 A_\mu A_\nu)$$

$$- (\partial_\nu\partial_\mu + ig\,\partial_\nu A_\mu + ig\, A_\mu\partial_\nu + ig\, A_\nu\partial_\mu - g^2 A_\nu A_\mu)$$

$$= ig\,(\partial_\mu A_\nu - \partial_\nu A_\mu) = ig\, F_{\mu\nu}$$

Non Abelian Gauge Theories: ادعا

$$[D_\mu, D_\nu] = ig\, F_{\mu\nu}(x)$$

$$\boxed{\text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu]}$$

اثبات مانند قبل فقط جمله $- g^2 A_\mu A_\nu + g^2 A_\nu A_\mu$ حذف نمی‌شود، زیرا A_μ ماتریس هستند در فضای $SU(2)$

$A_\mu \in SU(2)$

$$[D_\mu, D_\nu] = ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2 (A_\mu A_\nu - A_\nu A_\mu)$$

$$= ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\right) = ig F_{\mu\nu} \checkmark \quad \text{q.e.d}$$

In components: $\quad F_{\mu\nu}{}^a \tau^a = \partial_\mu A_\nu{}^a \tau^a - \partial_\nu A_\mu{}^a \tau^a + ig \underbrace{[\tau^b, \tau^c]}_{if^{abc}\tau^a} A_\mu{}^b A_\nu{}^c$

$F_{\mu\nu} \in SU(N)$

$$\Rightarrow \boxed{F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a - g f^{abc} A_\mu{}^b A_\nu{}^c}$$

$f^{abc} = \epsilon^{abc}$ for $SU(2)$

سوال: $F_{\mu\nu}$ چگونه تحت تبدیلات پیمانه‌ای تبدیل می‌شود؟

$\psi \to V(x)\psi(x)$

$[D_\mu, D_\nu]\psi(x) = ig F_{\mu\nu}(x)\psi(x)$

We know $\quad D'_\mu \psi'(x) = V(x) D_\mu \psi(x)$

$D'_\mu (D'_\nu \psi'(x)) = V(x) D_\mu D_\nu \psi(x)$

$D'_\nu (D'_\mu \psi'(x)) = V(x) D_\nu D_\mu \psi(x)$

$[D_\mu', D_\nu']\psi'(x) = V(x)[D_\mu, D_\nu]\psi(x)$

$ig F_{\mu\nu}'(x)\psi'(x) = V(x)\, ig F_{\mu\nu}(x)\psi(x)$

$\longrightarrow F_{\mu\nu}'(x) V(x)\psi(x) = V(x) F_{\mu\nu}(x)\psi(x)$

$$\longrightarrow \boxed{F_{\mu\nu}'(x) = V(x) F_{\mu\nu}(x) V^+(x)}$$

گویی $F_{\mu\nu}$ در نمایش adjoint (الحاقی) تبدیل شده است.

Gauge kinetic Term:

$A_\mu = A_\mu{}^a \tau^a \qquad F_{\mu\nu} = F_{\mu\nu}{}^a \tau^a$

$$\mathcal{L}_{gauge} = -\frac{1}{2} tr(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{2} F_{\mu\nu}{}^a F^{\mu\nu, b} \underbrace{tr(\tau^a \tau^b)}_{=\frac{1}{2}\delta^{ab}} = -\frac{1}{4} F_{\mu\nu}{}^a F^{\mu\nu, a}$$

$$\mathcal{L}'_{gauge} = -\frac{1}{2} tr(F_{\mu\nu}' F^{\mu\nu\prime}) = -\frac{1}{2} tr\left(V(x) F_{\mu\nu} V^+(x) V(x) F^{\mu\nu} V^+(x)\right)$$

$$= -\frac{1}{2} tr\left(V(x) F_{\mu\nu} F^{\mu\nu} V^+(x)\right) = -\frac{1}{2} tr\left(F_{\mu\nu}(x) F^{\mu\nu}(x)\right) = \mathcal{L}_{gauge}$$

$$\delta_\alpha \mathcal{L}_{gauge} = 0$$

In Summary:

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_g + \mathcal{L}_{g.f.} + \underbrace{\mathcal{L}_{FPGhost.}}_{}$$

later

$$\mathcal{L}_f = tr\left(\bar{\Psi}\,(i\not{D} - m)\,\Psi\right)$$

$$\mathcal{L}_g = -\frac{1}{2}\,tr\left(F_{\mu\nu}F^{\mu\nu}\right)$$

$$D_\mu \psi^a = \partial_\mu \psi^a + ig\,A_\mu^i\,(\tau^i)^{ab}\,\psi^b \qquad a,b = 1,\dots N_f$$

$$F_{\mu\nu}^{\ i} = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i - g f^{ijk} A_\mu^j A_\nu^k \qquad k,i,j = 1,\dots,N_f^2 - 1$$

for $SU(2)$ $\qquad f^{ijk} = \epsilon^{ijk}$

Equation of Motion (EOM):

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu^d)} - \frac{\partial \mathcal{L}}{\partial A_\mu^d} = 0 \xrightarrow{\text{ادعا}} D_\mu F^{\mu\nu,i} = J^{\nu,i}$$

$$J^{\nu,i} = g\,\bar{\Psi}\,\gamma^\nu \tau^i\,\Psi$$

a) $$\frac{\partial \mathcal{L}}{\partial A_\mu^d} = -g\,\bar{\Psi}^i\,\gamma^\mu(\tau^d)_{ij}\,\Psi_j - \underbrace{\frac{\partial F_{\rho\sigma}^{\ a}}{\partial A_\mu^d}}_{?}\ \underbrace{\frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}^{\ a}}}_{=F^{\rho\sigma,a}\left(-\frac{2}{4}\right)}$$

$$\frac{\partial}{\partial A_\mu^d} F_{\rho\sigma}^{\ a} = \frac{\partial}{\partial A_\mu^d}\left(\partial_\rho A_\sigma^a - \partial_\sigma A_\rho^a - g\epsilon^{abc} A_\rho^b A_\sigma^c\right)$$

$$= -g\epsilon^{abc}\,\delta^{db}\,\delta_\rho^{\ \mu} A_\sigma^c - g\epsilon^{abc}\,\delta^{dc}\,\delta_\sigma^{\ \mu} A_\rho^b$$

$$= -g\epsilon^{adc}\,\delta_\rho^{\ \mu} A_\sigma^c - g\epsilon^{abd}\,\delta_\sigma^{\ \mu} A_\rho^b$$

$$\longrightarrow \frac{\partial F_{\rho\sigma}^{\ a}}{\partial A_\mu^d}\,\frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}^{\ a}} = \left(-g\epsilon^{adc}\,\delta_\rho^{\ \mu} A_\sigma^c - g\epsilon^{abd}\,\delta_\sigma^{\ \mu} A_\rho^b\right) F^{\rho\sigma,a}$$

$$= -g\epsilon^{adc}\,F^{\mu\sigma,a} A_\sigma^c - g\epsilon^{abd}\,F^{\rho\mu,a} A_\rho^b$$

$$= -g\epsilon^{acd}\,F^{\sigma\mu,a} A_\sigma^c - g\epsilon^{abd}\,F^{\rho\mu,a} A_\rho^b$$

$c \to b$
$\rho \to \sigma$ $\qquad$ نامگذاری

$$= -g\epsilon^{abd}\,F^{\rho\mu,a} A_\rho^b - g\epsilon^{abd}\,F^{\rho\mu,a} A_\rho^b$$

$$= -2g\,\epsilon^{abd}\,F^{\rho\mu,a} A_\rho^b = 2g\,\epsilon^{abd}\,F^{\mu\rho,a} A_\rho^b \qquad (1)$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial A_\mu^d} = -g\,\bar{\Psi}^i\,(\tau^d)_{ij}\,\gamma^\mu \Psi^j - g\,\epsilon^{abd}\,F^{\mu\rho,a} A_\rho^b} \qquad (*1)$$

b) $$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu^d)} = \frac{\partial F_{\rho\sigma}^{\ a}}{\partial(\partial_\nu A_\mu^d)}\ \underbrace{\frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}^{\ a}}}_{=\left(-\frac{2}{4}\right) F^{\rho\sigma,a}}$$

$$= -\frac{1}{2} F^{\rho\sigma,a}\,\frac{\partial}{\partial(\partial_\nu A_\mu^d)}\left(\partial_\rho A_\sigma^a - \partial_\sigma A_\rho^a - g\epsilon^{abc} A_\rho^b A_\sigma^c\right)$$

$$= -\frac{1}{2} F^{\rho\sigma,a}\left(\delta^\nu_{\ \rho}\,\delta^{da}\,\delta^\mu_{\ \sigma} - \delta^\mu_{\ \sigma}\,\delta^{da}\,\delta^\nu_{\ \rho}\right)$$

$$= -\frac{1}{2}\left(F^{\nu\mu,d} - F^{\mu\nu,d}\right) = +F^{\mu\nu,d} = -F^{\nu\mu,d}$$

$$\boxed{\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu{}^d)} = -\,\partial_\nu F^{\nu\mu,d}}$$

Together: $\partial_\nu F^{\nu\mu,d} = g\,\bar{\Psi}^i \gamma^\mu (\tau^d)_{ij} \Psi^j + g\,\epsilon^{abd} A_\rho{}^b F^{\mu\rho,a}$

$$\partial_\nu F^{\nu\mu,d} - g\,\epsilon^{bad} A_\nu{}^b F^{\nu\mu,a} = g\,\bar{\Psi}^i \gamma^\mu (\tau^d)_{ij} \Psi^j$$

$$\boxed{\text{EOM}\qquad D_\nu F^{\nu\mu,d} = J^{\mu,d}} \qquad \text{with} \quad J^{\mu,d} = g\,\bar{\Psi}^i \gamma^\mu (\tau^d)_{ij} \Psi^j$$

$$\boxed{\begin{aligned} D_\nu F^{\nu\mu,d} &= \left(\partial_\nu F^{\nu\mu,d} - g\,\epsilon^{bad} A_\nu{}^b F^{\nu\mu,a}\right) \\ &= \left(\partial_\nu \delta^{ad} - g\,\epsilon^{bad} A_\nu{}^b\right) F^{\nu\mu,a} \end{aligned}}$$

معادله حرکت بصورت ماتریسی :

$$D_\nu F^{\nu\mu,d}\tau^d = \left(\partial_\nu F^{\nu\mu,d}\tau^d - g\,\epsilon^{bad} A_\nu{}^b F^{\nu\mu,a}\tau^d\right)$$

$$[\tau^b, \tau^a] = i\epsilon^{bad}\tau^d$$

$$= \left(\partial_\nu F^{\nu\mu,d}\,\tau^d + ig\,[\tau^b, \tau^a]\, A_\nu{}^b F^{\nu\mu,a}\right)$$

$$\boxed{D_\nu F^{\nu\mu} = \partial_\nu F^{\nu\mu} + ig\,[A_\nu, F^{\nu\mu}]}$$

این D_ν جوری روی $F^{\nu\mu}$ در نمایش adjoint است.

$$\rightarrow \quad \underline{\underline{D_\nu F^{\nu\mu} = J^\mu}} \qquad J^\mu = J^{\mu,d}\tau^d$$

معادله حرکت را بصورت دیگری هم می‌توان نوشت :

$$D_\nu F^{\nu\mu,d} = J^{\mu,d} \;\rightarrow\; \partial_\nu F^{\nu\mu,d} - g\,\epsilon^{bad} A_\nu{}^b F^{\nu\mu,a} = J^{\mu,d}$$

$$\rightarrow\; \partial_\nu F^{\nu\mu,d} = \underbrace{g\,\epsilon^{bad} A_\nu{}^b F^{\nu\mu,a} + g\,\bar{\Psi}^a (\gamma^\mu)(\tau^d)_{ab}\Psi^b}_{\equiv\, \mathcal{J}^{\mu,d}}$$

$$\boxed{\partial_\nu F^{\nu\mu,d} = \mathcal{J}^{\mu,d}} \qquad \text{with} \quad \mathcal{J}^{\mu,d} = \mathcal{J}^{\mu,d}_{\text{gluons}} + \mathcal{J}^{\mu,d}_{\Psi}$$

$$\mathcal{J}^{\mu,d}_{gl.} \equiv g\,\epsilon^{bad} A_\nu{}^b F^{\nu\mu,a}$$

• خاصیت معادلات حرکت (جریان پایسته) :

$$\underbrace{D_\mu\left(D_\nu F^{\mu\nu,d}\right)}_{0\,=\,\text{بعداً نشان می‌دهیم}} = D_\mu \mathcal{J}^{\mu,d} \;\Rightarrow\; D_\mu J^{\mu,d} = 0$$

$J^{\mu,d}$ = covariant current.

$$D_\mu J^{\mu,d} = \left(\partial_\mu J^\mu + ig\,[A_\mu, J^\mu]\right)^d = \partial_\mu J^{\mu,d} + ig\left(i\,\epsilon^{abd} A_\mu{}^a J^{\mu,b}\right)$$

$$= \partial_\mu J^{\mu,d} - g\,\epsilon^{abd} A_\mu{}^a J^{\mu,b}$$

Invariant current:

$$\partial_\nu F^{\nu\mu,d} = J^{\mu,d}$$

$$\underbrace{\partial_\mu \partial_\nu F^{\nu\mu,d}}_{=0} = \partial_\mu J^{\mu,d} = 0$$

$$\partial_\mu J^{\mu,d} = 0 = \partial_0 J^{0,d} + \partial_i J^{i,0}$$

$$\int d^3x\ \partial_\mu J^{\mu,d} = 0 = \int d^3x\ \partial_0 J^{0,d} + \underbrace{\int \partial_i J^{i,0}\, d^3x}_{=0}$$

$$\curvearrowright \partial_0 \int d^3x\ J^{0,d} = 0$$

$$\partial_0 Q^d = 0$$ به تعداد مولدهای گروه SU(N) بار پایسته وجود دارد.

$$Q^d = \int J^{0,d} d^3x = \int \left(J^{0,d} + g\,\epsilon^{bad} A_\nu{}^b F^{\nu 0,a}\right) d^3x$$

$$= g \int \left(\bar\psi \gamma^0 \tau^d \psi + \epsilon^{bad} A_i^b F^{i0,a}\right) d^3x$$

$SU(N) \ni F^{i0} = -E^i = E_i$ بصورت ماتریسی (Chromoelectric field)

$$Q^d = g \int \left(\psi^+ \tau^d \psi + \epsilon^{bad} A_i{}^b E_i^a\right) d^3x$$

کل بار رنگ (کل)

$$Q^d = g \int \left[\psi^+ \tau^d \psi + (\vec{A} \times \vec{E})^d\right] d^3x$$

سهم میدان مادی به بار رنگ — سهم میدان پیمانه‌ای به بار رنگ

Appendix: ادّعا $D_\mu D_\nu F^{\mu\nu,d} = 0$

Proof: $D_\mu D_\nu F^{\nu\mu} \underset{\text{we show (1)}}{=} \frac{1}{2}[D_\mu, D_\nu] F^{\nu\mu} \underset{\text{we show (2)}}{\overset{!}{=}} \frac{ig}{2}[F_{\mu\nu}, F^{\nu\mu}] = 0$

$$D_\mu D_\nu F^{\nu\mu} \overset{(1)}{=} \frac{1}{2}[D_\mu, D_\nu] F^{\nu\mu} + \cancel{\frac{1}{2} \underbrace{\{D_\mu, D_\nu\}}_{\text{متقارن } \mu\leftrightarrow\nu} \underbrace{F^{\nu\mu}}_{\text{پادمتقارن } \mu\leftrightarrow\nu}}$$

$$\underset{(2)}{=} \frac{ig}{2}[F_{\mu\nu}, F^{\nu\mu}]$$

می‌خواهیم نشان دهیم $[D_\mu, D_\nu] F^{\nu\mu} = ig[F_{\mu\nu}, F^{\nu\mu}]$

یادآوری: $\psi \to V(x)\psi(x)$ $\qquad D_\mu \psi(x) = \partial_\mu \psi + ig A_\mu$

$D_\mu \psi \to V(x) D_\mu \psi(x)$

می‌توان نشان داد که اگر $F_{\mu\nu} \longrightarrow V(x) F_{\mu\nu} V^+(x)$ یعنی $F_{\mu\nu}$ در نمایش الحاقی تبدیل می‌شود

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + ig[A_\mu, F^{\mu\nu}]$$

دارای خاصیت $[D_\mu, D_\nu] F^{\mu\nu} = ig[F_{\mu\nu}, F^{\mu\nu}]$ است.

باید عبارت کلی تر برای هر "عملگر" (ماتریس) $\mathcal{O}$ که در نمایش الحاقی تبدیل شود

$$\mathcal{O} \to V(x)\,\mathcal{O}\,V^{+}(x) \quad \curvearrowright \quad [D_\mu, D_\nu]\mathcal{O} = ig\,[F_{\mu\nu}, \mathcal{O}]$$

اثبات :

$$[D_\mu, D_\nu]\mathcal{O} = D_\mu D_\nu \mathcal{O} - D_\nu D_\mu \mathcal{O}$$

$$= D_\mu\left(\partial_\nu \mathcal{O} + ig\,[A_\nu, \mathcal{O}]\right) - D_\nu\left(\partial_\mu \mathcal{O} + ig\,[A_\mu, \mathcal{O}]\right)$$

$$= \partial_\mu\left(\partial_\nu \mathcal{O} + ig\,[A_\nu, \mathcal{O}]\right) + ig\,[A_\mu, \partial_\nu \mathcal{O} + ig\,[A_\nu, \mathcal{O}]]$$

$$- \partial_\nu\left(\partial_\mu \mathcal{O} + ig\,[A_\mu, \mathcal{O}]\right) - ig\,[A_\nu, \partial_\mu \mathcal{O} + ig\,[A_\mu, \mathcal{O}]]$$

$$= \cancel{\partial_\mu\partial_\nu \mathcal{O}} + ig\,[\partial_\mu A_\nu, \mathcal{O}] + \cancel{ig\,[A_\nu, \partial_\mu \mathcal{O}]}$$

$$+ \cancel{ig\,[A_\mu, \partial_\nu \mathcal{O}]} - g^2\,[A_\mu, [A_\nu, \mathcal{O}]]$$

$$- \cancel{\partial_\nu\partial_\mu \mathcal{O}} - ig\,[\partial_\nu A_\mu, \mathcal{O}] - \cancel{ig\,[A_\mu, \partial_\nu \mathcal{O}]}$$

$$- \cancel{ig\,[A_\nu, \partial_\mu \mathcal{O}]} + g^2\,[A_\nu, [A_\mu, \mathcal{O}]]$$

$$= ig\,[\partial_\mu A_\nu - \partial_\nu A_\mu, \mathcal{O}] \underbrace{- g^2\left\{[A_\mu, [A_\nu, \mathcal{O}]] - [A_\nu, [A_\mu, \mathcal{O}]]\right\}}_{\text{می‌توان نشان داد } * \; = -g^2\,[[A_\mu, A_\nu], \mathcal{O}]}$$

$$\curvearrowright \quad \boxed{[D_\mu, D_\nu]\mathcal{O} = ig\,[\partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu], \mathcal{O}] = ig\,[F_{\mu\nu}, \mathcal{O}]}$$

$*$ برای اثبات $\longrightarrow \quad [A_\mu, [A_\nu, \mathcal{O}]] + [\mathcal{O}, [A_\mu, A_\nu]] + [A_\nu, [\mathcal{O}, A_\mu]] = 0$

$$\longrightarrow [A_\mu, [A_\nu, \mathcal{O}]] - [A_\nu, [A_\mu, \mathcal{O}]] = [[A_\mu, A_\nu], \mathcal{O}] \quad \checkmark$$

نتیجه

پس اثبات شد که

اگر

$$\mathcal{O} \to V(x)\,\mathcal{O}\,V^{+}(x)$$

$$[D_\mu, D_\nu]\mathcal{O} = ig\,[F_{\mu\nu}, \mathcal{O}]$$

به این ترتیب

$$[D_\mu, D_\nu]F^{\mu\nu} = ig\,[F_{\mu\nu}, F^{\mu\nu}] = 0$$

و چون :

$$\frac{1}{2}[D_\mu, D_\nu]F^{\mu\nu} = D_\mu D_\nu F^{\mu\nu} \Rightarrow$$

$$D_\mu D_\nu F^{\mu\nu} = 0 \quad \curvearrowright \quad D_\mu D_\nu F^{\mu\nu} = D_\mu J^\mu = 0$$

covariant derivative.