Introduction to Elementary Particle Physics

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Elementary Particle Physics

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Bosons and Fermions in Relativistic Quantum Mechanics

Particle Classification:

- Bosons: Particles with integer spin

  They obey **Bose-Einstein** statistics

- Fermions: Particles with half-integer spin

  They obey **Fermi-Dirac** statistics

Def.: The statistics obeyed by a particle $\rightarrow$
How does the wave function describing an ensemble of identical particles behave under interchange of any pair of particles?

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In QM: N noninteracting and identical particles:

Hamiltonian= Sum of N identical one-particle Hamiltonian H_i

$$H = \sum_{i=1}^{N} H_i$$

Eigenvalue equation for one-particle states $\varphi_{\alpha_i}(i)$, $i = 1, \cdots, N$

$$H_i \varphi_{\alpha_i}(i) = E_{\alpha_i} \varphi_{\alpha_i}(i).$$

For $H\Phi = E\Phi$, we obtain the energy eigenvalue

$$E = E_{\alpha_1} + \cdots + E_{\alpha_N}$$

For eigenfunctions Φ, bosons and fermions are to be considered separately:

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- **Bosons:**

$$\Phi(1,\cdots,N) = \varphi_{\alpha_1}(1)\cdots\varphi_{\alpha_N}(N)$$

- **Fermions:**

$$\begin{aligned}\Phi(1,\cdots,N) &= \frac{1}{\sqrt{N}}\sum_P (-1)^P P\varphi_{\alpha_1}(1)\cdots\varphi_{\alpha_N}(N) \\ &= \frac{1}{\sqrt{N}}\begin{vmatrix} \varphi_{\alpha_1}(1) & \cdots & \varphi_{\alpha_1}(N) \\ \cdot & \cdots & \cdot \\ \varphi_{\alpha_N}(1) & \cdots & \varphi_{\alpha_N}(N) \end{vmatrix}\end{aligned}$$

  - Slater-determinant
    (interchange of any two columns leads to a -1)
  - For even (odd) permutations $(-1)^P = \pm 1$
  - For $\varphi_{\alpha_i}(i) = \varphi_{\alpha_j}(j)$, we have $\Phi(1,\cdots,N) = 0$
    [Pauli exclusion principle]

Bosons in relativistic QM (Klein-Gordon equation):

- Schrödinger equation in non-relativistic QM (in one space dimension):

$$E = \frac{p_x^2}{2m} + V(x)$$

Using ($c = \hbar = 1$)

$$\boxed{E \to i\partial_0, \qquad p_x \to -i\partial_x}$$

we obtain the non-relativistic Schrödinger equation

$$i\partial_0 \psi(x,t) = \left(-\frac{\partial_x^2}{2m} + V(x)\right)\psi(x,t)$$

Bosons in relativistic QM (Klein-Gordon equation):

- To arrive at the "relativistic Schrödinger equation" (the KG equation), we start with (one-dimensional) **relativistic energy dispersion relation** ($c = \hbar = 1$)

$$E^2 = p_x^2 + m^2$$

Using $E \to i\partial_0$ and $p_x \to -i\partial_x$, we obtain

$$(\partial_0^2 - \partial_x^2 + m^2)\varphi(x, t) = 0$$

or in general $\boxed{(\Box + m^2)\varphi(\boldsymbol{x}, t) = 0}$ with $\Box \equiv \partial_0^2 - \boldsymbol{\nabla}^2$

- $\varphi(\boldsymbol{x}, t)$ is a scalar field and in QFT it describes a spin-0 and electrically neutral particle
- In contrast to Schrödinger equation, the KG equation is second order in space **and** time $\to$ **Relativistic covariance**

Fermions in relativistic QM (Dirac equation):

- Dirac equation is first order in space and time
- For massless (Weyl) fermions (spinors), it reads

$$\partial_0 \psi(\boldsymbol{x}, t) = \pm \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \psi(\boldsymbol{x}, t)$$

$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are three Pauli-matrices with $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ and $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ [$SU(2)$ algebra]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\psi(\boldsymbol{x}, t)$ is a two component spinor field: $\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$

$$\partial_0 \psi_\alpha = (\sigma_i)_{\alpha\beta} \partial_i \psi_\beta, \qquad \alpha, \beta = 1, 2 \tag{1}$$

Fermions in relativistic QM (Dirac equation):

- ψ satisfies at the same time a three-dimensional massless KG equation

$$\Box\psi(\boldsymbol{x}, t) = 0$$

Proof: We start with $\partial_0\psi = \pm\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi$ and build

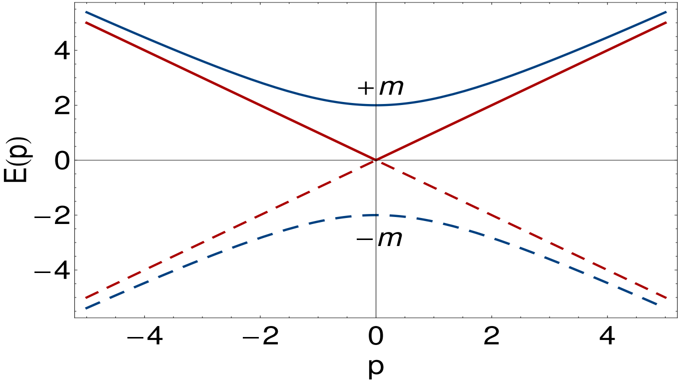
$$\begin{aligned}
\partial_0^2\psi &= \pm\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\partial_0\psi = \sigma_i\sigma_j\partial_i\partial_j\psi \\
&= \frac{1}{2}\left(\{\sigma_i,\sigma_j\} + \left[\sigma_i,\sigma_j\right]\right)\partial_i\partial_j\psi = \frac{1}{2}(2\delta_{ij})\partial_i\partial_j\psi \\
\partial_0^2\psi &= \partial_i^2\psi
\end{aligned}$$

which means $\Box\psi = 0$ q.e.d.

- This is important, because it shows that (massless) Dirac particles satisfy the same dispersion relation as KG particles

$$p^2 = p_0^2 - \boldsymbol{p}^2 = 0, \qquad \text{or} \qquad E^2 = \boldsymbol{p}^2$$

Spectrum of Dirac particles



$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$$

$$E = \pm|\boldsymbol{p}|$$

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Dirac Sea (Hole theory)

$$\gamma + \text{(Full) Negative E particle} \rightarrow \text{Positive E particle}$$

$$\gamma \rightarrow e^+ + e^-$$