Sharif University of Technology - Department of Physics

Quantum Field Theory 1 - Midterm Exam - Take Home - Fall 2022

Due: Saturday Azar 19, 1401

Submit your responses to sadooghi@physics.sharif.ir in a PDF format. Be sure that your scans are readable. Do not forget your names on the answer sheets

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Exercise 1 = Problem 12.2 Peskin-Schroeder (5 pts):

Compute the one-loop $\beta(g)$ in the two-dimensional Gross-Neveu model

$$\mathcal{L} = \bar{\psi}_i(i\gamma \cdot \partial)\psi_i + \frac{1}{g^2}(\bar{\psi}_i\psi_i)^2, \tag{1}$$

with $i = 1, \cdots, N$.

Exercise 2 = Part of Problem 10.1 Peskin-Schroeder (10 pts)

Exercise 3 = Part of Problem 10.2 Peskin-Schroeder (10 pts)

Exercise 4 = Part of Problem 11.2 Peskin-Schroeder (10 pts):

Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2 + \bar{\psi}(i\gamma \cdot \partial)\psi - g\bar{\psi}\left(\phi^1 + i\gamma^5\phi^2\right)\psi, \tag{2}$$

where ϕ^i is a two-component field i=1,2.

a) Denote the vacuum expection value (VEV) of ϕ^i by v and make the change of variables

$$\phi^i(x) = (v + \sigma(x), \pi(x)). \tag{3}$$

Write out the Lagrangian in these new variables, and show that the fermion aquires a mass given by

$$m_f = gv. \tag{4}$$

b) Compute the one-loop radiative correction to m_f, choosing renormalization conditions so that v and g (defined as the vertex $\bar{\psi}\psi\pi$ vertex at zero momentum transfer) receive no radiative corrections. Show that relation (4) receives nonzero corrections, but these corrections are finite.

Exercise 5 [See Pokorski Chapter 2] (15 pts):

a) Prove the relation

$$\int dz_1 dz_1^\star \cdots dz_n dz_n^\star \exp\left(i(z^\star, Az)\right) = \frac{(2\pi)^n i^n}{\det A}, \tag{5}$$

for complex vector $\mathbf{z} = (z_1, \cdots, z_n)$ and a generic $(n \times n)$ dimensional complex matrix A. Here, $(z^\star, Az) \equiv \sum_{i,j} z_i^\star A_{ij} z_j$.

b) For a real Grassmann variables η_i prove the formula

$$\int d\eta_1 \cdots d\eta_n \exp\left(\frac{1}{2}(\eta, A\eta)\right) = (\det A)^{1/2}\,, \tag{6}$$

for an antisymmetric matrix A, where $(\eta, A\eta) = \sum_{i,j} \eta_i A_{ij} \eta_j$.

c) Show that for complex Grassmann variables, Eq. (9) is generalized by

$$\int d\eta_1 d\eta_1^\star \cdots d\eta_n d\eta_n^\star \exp\left((\eta^\star, A\eta)\right) = \det A, \tag{7}$$

where $(\eta^\star, A\eta) = \sum_{i,j} \eta_i^\star A_{ij} \eta_j$.

Exercise 6 (25 pts):

Consider the derivative expansion of the effective action $\Gamma[\Phi]$ for $\Phi = (\varphi_0, \cdots, \varphi_{N-1})$

$$\Gamma[\Phi] = \Gamma[\Phi_0] + \int d^dx \frac{\delta\Gamma[\Phi_0]}{\delta\varphi_i(x)} \bar{\varphi}_i(x) + \frac{1}{2}\int d^dx d^dy \frac{\delta^2\Gamma[\Phi_0]}{\delta\varphi_i(x)\delta\varphi_j(y)} \bar{\varphi}_i(x)\bar{\varphi}_j(y) + \cdots, \tag{8}$$

where $\Phi(x) = \Phi_0 + \bar{\Phi}(x)$ is used.

a) Assuming that Φ_0 describes a configuration that minimizes the effective action, and using the Taylor expansion $\Phi(y) = \Phi(x) + z^\mu\partial_\mu\Phi(x) + \frac{1}{2}z^\mu z^\nu \partial_\mu\partial_\mu\Phi(x) + \cdots$, with $z = y - x$, show that $\Gamma[\Phi]$ can be written as

$$\Gamma[\Phi] = \Gamma[\Phi_0] - \frac{1}{2}\int d^dx \mathcal{M}^2_{ij}[\Phi_0]\bar{\varphi}_i(x)\bar{\varphi}_j(x) + \frac{1}{2}\int d^dx\, \chi^{\mu\nu}_{ij}[\Phi_0]\partial_\mu\bar{\varphi}_i(x)\partial_\nu\bar{\varphi}_j(x) + \cdots, \tag{9}$$

with

$$\mathcal{M}^2_{ij}[\Phi_0] = -\int d^dz \frac{\delta^2\Gamma[\Phi_0]}{\delta\varphi_i(0)\delta\varphi_j(z)}, \qquad \text{and} \qquad \chi^{\mu\nu}_{ij}[\Phi_0] = -\frac{1}{2}\int d^dz\, z^\mu z^\nu \frac{\delta^2\Gamma[\Phi_0]}{\delta\varphi_i(0)\delta\varphi_j(z)}. \tag{10}$$

As we have mentioned in the class, the above derivative expansion of $\Gamma[\Phi]$ can alternatively be given as

$$\Gamma[\Phi] = \int d^dx \left(-V[\Phi] + \frac{1}{2}\chi^{\mu\nu}_{ij}[\Phi]\partial_\mu\varphi_i\partial_\nu\varphi_j(x) + \cdots\right). \tag{11}$$

b) Let us now break the $O(N)$ symmetry of the original action by choosing a constant field configuration for $\Phi_0 = (\sigma_0, 0, \cdots, 0)$. To determine the kinetic part of the effective action, we use the ansatz

$$\chi^{\mu\nu}_{ij}[\Phi] = (F_1^{\mu\nu})_{ij} + 2F_2^{\mu\nu}\frac{\varphi_i\varphi_j}{\Phi^2}, \tag{12}$$

where $i, j = 0, \cdots, N-1$ and $\Phi^2 = \sum_{i=0}^{N-1}\varphi_i^2$. Show that the "kinetic" part of the effective Lagrangian density including two derivatives is then given by

$$\mathcal{L}_k = \frac{1}{2}(F_1^{\mu\nu})_{ij}\partial_\mu\varphi_i\partial_\nu\varphi_j + \frac{F_2^{\mu\nu}}{\Phi^2}(\varphi_i\partial_\mu\varphi_i)(\varphi_j\partial_\nu\varphi_j). \tag{13}$$

c) To determine the "form factors" $F_1^{\mu\nu}$ and $F_2^{\mu\nu}$, or at least a combination of them, let us use the definition $\Gamma_k \equiv \int d^dx\mathcal{L}_k$ as a part of the effective action including only two derivatives. Show that

$$\begin{aligned} \frac{\delta^2\Gamma_k[\phi_0]}{\delta\varphi_0(x)\delta\varphi_0(0)} &= -\mathcal{G}^{\mu\nu}[\Phi_0]\partial_\mu\partial_\nu\delta^d(x), \\ \frac{\delta^2\Gamma_k[\phi_0]}{\delta\varphi_\ell(x)\delta\varphi_m(0)} &= -\mathcal{F}^{\mu\nu}_{\ell m}[\Phi_0]\partial_\mu\partial_\nu\delta^d(x), \qquad \forall\ell, m \geq 1, \end{aligned} \tag{14}$$

with $\mathcal{G}_{\mu\nu} = [(F_1^{\mu\nu})_{00} + 2F_2^{\mu\nu}]$ and $\mathcal{F}^{\mu\nu}_{\ell m} = \frac{1}{2}[(F_1^{\mu\nu})_{\ell m} + (F_1^{\mu\nu})_{\ell m}]$.

d) Assuming that $\mathcal{M}^2_{\ell m} = -\mathcal{M}^2_{m\ell}$ and $\mathcal{F}^{\mu\nu}_{\ell m} = -\mathcal{F}^{\mu\nu}_{m\ell}$, for all $\ell \neq m$ and $\ell, m \geq 1$, and denoting $\mathcal{M}^2_{aa}$ by M_a^2 for all $a = 0, \cdots, N-1$, show that the effective action is given by

$$\begin{aligned} \Gamma[\Phi] &= \Gamma[\Phi_0] - \frac{1}{2}\int d^dx\, \bar{\varphi}_0(x)(M_0^2 + \mathcal{G}^{\mu\mu}\partial_\mu^2)\bar{\varphi}_0(x) \\ &\quad -\frac{1}{2}\sum_{\ell=1}^{N-1}\int d^dx\, \bar{\varphi}_\ell(x)(M_\ell^2 + \mathcal{F}^{\mu\mu}_{\ell\ell}\partial_\mu^2)\bar{\varphi}_\ell(x), \end{aligned} \tag{15}$$

where $\mathcal{G}^{\mu\nu} = \mathcal{G}^{\mu\mu}g^{\mu\nu}$ and $\mathcal{F}^{\mu\nu}_{\ell m} = \mathcal{F}^{\mu\mu}_{\ell m}g^{\mu\nu}$ is also assumed.

e) Show finally that the energy dispersion relations for φ_0 and $\varphi_\ell, \ell = 1, \cdots, N-1$ are given by

$$\begin{aligned} \omega_0^2 &= u_0^{(1)2}p_1^2 + u_0^{(2)2}p_1^2 + u_0^{(2)2}p_1^2 + m_0^2, \\ \omega_\ell^2 &= u_\ell^{(1)2}p_1^2 + u_\ell^{(2)2}p_1^2 + u_\ell^{(2)2}p_1^2 + m_\ell^2, \qquad \forall\ell \geq 1, \end{aligned} \tag{16}$$

with the pole masses

$$m_0^2 = \frac{M_0^2}{\mathcal{G}^{00}}, \qquad \text{and} \qquad m_\ell^2 = \frac{M_\ell^2}{\mathcal{F}^{00}_{\ell\ell}} \tag{17}$$

and the refraction indices

$$u_0^{(i)2} = \frac{\mathcal{G}^{ii}}{\mathcal{G}^{00}}, \qquad \text{and} \qquad u_\ell^{(i)2} = \frac{\mathcal{F}^{ii}_{\ell\ell}}{\mathcal{F}^{00}_{\ell\ell}} \tag{18}$$