Report on Circular and Spherical Structuring

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In this paper we are searching for a circular or spherical structuring scheme in human mental operation of counting. Does it happen naturally, that in order to count a set of objects we develop a strategy, which isn't linear? Our research indicates that circular structuring is usually reduced to the idea of dividing in half and spherical structures are used very often, once they are internalized. We describe various levels of sophistication in human's spherical structuring abilities.

Spatial structuring is defined to be the mental operation of constructing an organization or form for an object or set of objects (Battista & Clemens 1996). Before we understand mental process of two-dimensional spatial structuring completely, it is found too complicated to deal with quantitative three-dimensional spatial situations. The situation of 2D rectangular arrays of squares is dealt with in detail (Batista & Clemens & Aaron & Battista & Borrow 1998). But other 2D structures like spherical structures are not examined yet. Here, we try to understand different counting algorithms for sets of objects with circular or spherical symmetry.

Circular & Spherical Structuring

The idea is to implement circular and spherical symmetry in the process of counting. We intend to find a better understanding of how natural these symmetries appear to our minds. We intend to see if our brain tries to generalize the row-column organizations, or it tries to initialize new ways of counting. It is known that spatial structuring precedes meaningful enumeration (Battista & Clemens 1996). Without organization of objects one often skip-counts or multiplies to find the total. Several levels of sophistication in 2D arrays is recognized. Students with complete lack of row- or column-structuring try to find a linear pattern which covers all the objects, even when there is an obvious array structure. They often fail to do this properly. Their linear pattern often skips a few objects or passes through an object several times. This is because the geometry of a linear pattern is often too complicated to be memorized. Students who have little familiarity with arrays try to give a partition of objects to sections which are parts of rows or columns. More sophisticated students can visualize both row and column structures in an array, but they still use addition to count the total number of objects. And at last, the students who completely understand the array structures multiply in order to count (Battista & Clemens & Arnoff & Battista & Borrow 1998). Our research on circular and spherical symmetries will recognize more delicate levels of sophistication in the 2D array structures.

Spherical symmetries are subject of the first part of our research on counting. We start with platonic solids. We make students count their faces edges and corners. We explain duality between platonic solids. Then we try other polygons which are regular with respect to corners. Spherical counting has the two-fold advantage of trying to internalize consideration of spherical symmetries in counting, and at the same time measuring the recognition that an array is just division into branches with equal cardinality. So one must multiply in order to count when the spherical symmetry is notices in the process of counting.

Motivated by investigations on spherical counting, we go back to simple circular patterns and try to show that only very limited circular symmetries are recognized by our brains. This fact shall influence methods of teaching elementary mathematics.

Counting on Platonic solids

As Euclid proved in the last proposition of the Elements, the Tetrahedron, Cube, Octahedron, Dodecahedron, and Icosahedron are the only regular solids with equivalent faces composed of congruent regular convex polygons. We asked a number of students of different ages to count the number of their faces, edges and corners. Several levels of sophistication were recognized.

Here is a list of student differences in the skill of counting objects with spherical symmetry:

Counting on tetrahedron

Students showed 3 different levels of sophistication in counting the corners of tetrahedron:

Level 1- Student tries to count the number of corners by considering them on a line passing through all of them only once (linear counting).

Level 2- One corner on up and three underneath: 1+3=4 (dividing into simple subsets).

Level 3- Student knows the number of corners by heart.

Students showed 3 different levels of sophistication in counting the edges of tetrahedron:

Level 1- Student counts the edges in linear order (linear counting).

Level 2- three edges on top and three underneath: 3+3=6 (dividing into simple subsets).

Level 3- Student knows the number of edges by heart.

Students showed 3 different levels of sophistication in counting the faces of tetrahedron:

Level 1- Student counts the faces in linear order (linear counting).

Level 2- Three faces on top and one underneath: 3+1=4 (dividing into simple subsets).

Level 3- Student knows the number of faces by heart.

Notice that the duality between faces and corners of tetrahedron holds even in the levels of sophistication in counting them. Some of the students put the faces and corners of a tetrahedron in one to one correspondence.

Counting on cube

Students showed 5 different levels of sophistication in counting the corners of cube:

Level 1- Linear counting

Level 2- Linear counting of corners of the same height and continuing by corners of another height (Grouping by height).

Level 3- Dividing into simple subsets:

4(top)+4(underneath)=8

1(top)+(3+3)(vest)+1(underneath)=8

Level 4- Four corners on top and as much underneath (Symmetry):

2*4=8

2*(1+3)=8

Level 5- Student knows the number of corners by heart.

Students showed 3 different levels of sophistication in counting the edges of cube:

Level 1- Linear counting

Level 2- Grouping by height and dividing into simple subsets:

4(top)+4(vest)+4(underneath)=12

3(top)+6(vest)+3(underneath)=12

Level 3- Student knows the number of edges by heart.

Students showed 3 different levels of sophistication in counting the faces of cube:

Level 1- Linear counting

Level 2- Grouping by height and dividing into simple subsets:

1(top)+4(vest)+1(underneath)=6

3(top)+3(underneath)=6

Level 3- Student knows the number of faces by heart.

Some of the students relate the number of faces, edges and corners of cube in order to count them.

Counting on octahedron

Students showed 3 different levels of sophistication in counting the corners of octahedron:

Level 1- Linear counting

Level 2- Grouping by height and dividing into simple subsets:

1(top)+4(vest)+1(underneath)=6.

3(top)+3(underneath)=6.

Level 3- Student knows the number of corners by heart.

Students showed 3 different levels of sophistication in counting the edges of octahedron:

Level 1- Linear counting

Level 2- Grouping by height and dividing into simple subsets:

4(top)+4(vest)+4(underneath)=12

3(top)+6(vest)+3(underneath)=12

Level 3- Student knows the number of edges by heart.

Students showed 5 different levels of sophistication in counting the faces of octahedron:

Level 1- Linear counting

Level 2- Grouping by height

Level 3- Dividing into simple subsets:

4(top)+4(underneath)=8

1(top)+(3+3)(vest)+1(underneath)=8

Level 4- Symmetry:

2*4=8

2*(1+3)=8

Level 5- Student knows the number of faces by heart.

Some of the students relate the number of faces, edges and corners of octahedron in order to count them.

Notice that the duality between faces of cube and corners of octahedron and the duality between the corners of cube and the faces of octahedron and also the duality between their edges hold again in the levels of sophistication in counting them. Some of the students have used these dualities to count the number of faces, edges and corners.

Counting on Dodecahedron

Students showed 5 different levels of sophistication in counting the corners of dodecahedron:

Level 1- Linear counting

Level 2- Linear counting and grouping by height.

Level 3- Dividing into simple subsets:

5(top)+(5+5)(vest)+5(underneath)=20

(1+3)(top)+(6+6)(vest)+(1+3)(underneath)=20

Level 4- Symmetry:

2*(5+5)=20

2*(1+3+6)=20

Level 5- Student knows the number of corners by heart.

Students showed 5 different levels of sophistication in counting the

edges of dodecahedron:
Level 1- Linear counting
Level 2- Grouping by height
Level 3- Dividing into simple subsets:

5(top)+(5+10+5)(vest)+5(underneath)=30
3(top)+(6+12+6)(vest)+3(underneath)=30

Level 4-Symmetry:

2*(5+5)+10(vest)=30
2*(3+6)+12(vest)=30

Level 5- Student knows the number of edges by heart.

Students showed 3 different levels of sophistication in counting the faces of dodecahedron:
Level 1- Linear counting
Level 2- Grouping by height and dividing into simple subsets:

1(top)+(5+5)(vest)+1(underneath)=12
3(top)+6(vest)+3(underneath)=12

Level 3- Student knows the number of faces by heart.

Some of the students relate the number of faces, edges and corners of dodecahedron in order to count them.

Counting on Icosahedron

Students showed 3 different levels of sophistication in counting the corners of icosahedron:
Level 1- Linear counting
Level 2- Grouping by height and dividing into simple subsets:

1(top)+(5+5)(vest)+1(underneath)=12
3(top)+6(vest)+3(underneath)=12

Level 3- Student knows the number of corners by heart.

Students showed 5 different levels of sophistication in counting the edges of icosahedron:
Level 1- Linear counting
Level 2- Grouping by height
Level 3- Dividing into simple subsets:

5(top)+(5+10+5)(vest)+5(underneath)=30
3(top)+(6+12+6)(vest)+3(underneath)=30

Level 4-Symmetry:

2*(5+5)+10(vest)=30
2*(3+6)+12(vest)=30

Level 5- Student knows the number of edges by heart.

Students showed 5 different levels of sophistication in counting the faces of icosahedron:

Level 1- Linear counting
Level 2- Grouping by height
Level 3- Dividing into simple subsets:

5(top)+(5+5)(vest)+5(underneath)=20
(1+3)(top)+(6+6)(vest)+(1+3)(underneath)=20

Level 4- Symmetry:

2*(5+5)=20
2*(1+3+6)=20

Level 5- Student knows the number of faces by heart.

Some of the students relate the number of faces, edges and corners of icosahedron in order to count them.

Notice that the duality between faces of dodecahedron and corners of icosahedron and the duality between the corners of dodecahedron and the faces of icosahedron and also the duality between their edges hold again in the levels of sophistication in counting them. Some of the students have used these dualities to count the number of faces, edges and corners.

Counting on objects with spherical symmetry

After becoming familiar with platonic solids and counting on them, students are asked to count the number of faces, edges and corners of several objects with spherical symmetry. Here is the list of symmetric objects in the order of counting complexity: truncated tetrahedron, truncated cube, truncated octahedron, cuboctahedron, small rhombicuboctahedron, truncated dodecahedron, truncated icosahedron, icosidodecahedron and small rhombicosidodecahedron. Trying to count on each object, many students started by choosing their counting strategy before starting to count. They were able to predict, if the counting task using each counting technique was simple enough for them to be performed smoothly. Many changed their counting technique in the middle of the process of counting. Many found new techniques for counting some objects and forgot their techniques while they were trying to count other objects.

As a result several different levels of sophistication were recognized, some of which had appeared in counting on Platonic solids:

Level 1-Linear counting
Level 2-Grouping by height
Level 3-Dividing into simple subsets
Level 4-Symmetry
Level 5-Dividing objects into meaningful subsets with spherical symmetry (introducing correspondence between each subset and a set of objects on platonic solids).
Level 6-Counting faces, edges or corners by Euler's formula.

Level 7-Counting faces, edges or corners knowing only one of the other two.

The more complicated the objects become, the more students tend to more sophisticated counting techniques. All of the students show progress in counting skills they use. Also, the more they count, the more they tend to use correspondence with objects on Platonic solids. Therefore, our mind does naturally understand spherical symmetry and does use it as a counting skill.

Counting on objects with circular symmetry

Through the process of counting many students, were grouping the objects by height into circular patterns. We found out that, understanding levels of sophistication in counting circular patterns precedes understanding counting skills for objects with spherical symmetry. We performed similar counting experiments to understand different levels of student skills in counting objects with circular symmetry.

We made a number of coloured signs with equal distances on a wooden torus and asked the students to count them. Here comes the list of different levels of sophistication of students in counting the signs:

Level 1-Linear counting

Level 2-Linear counting, marking the first sign they started to count from

Level 3-Linear skip counting, marking the first sign

Level 4-Dividing into simple subsets

Level 5-Counting using mirror symmetry

Level 6-Students were able to count 6 and 8 signs using order 6 and order 8 symmetries respectively, but none of them was able to count 9 signs using order their 3 symmetry.

These experiments show that contrary to the case of spherical symmetries, our brains only recognize very simple circular symmetries. This implies that in counting circular objects, one shall use different colours or provide other means of finding correspondence with very simple circular symmetries. For example, one shall at least mark the hours 3, 6, 9 and 12 on a hand watch or a clock to make reading the time fluent for people.

References

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