Who is a good problem solver?

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It is more than one decade now, that Iran is participating in "International Mathematical Olympiad". Many of Iranian Olympiad contestants have pursued their studies in mathematics up to the PhD level. Students in each generation become the coaches of the next generation. This way, every generation of Olympiad students develops a new culture of problem solving. As a member of this club, I have been through all the different layers of scientific and administrative works involved in this movement, and at the same time, trying to record people's experiences in problem solving, I have been communicating and collaborating with members of different generations of Olympiad club, some of whom are well-experienced mathematicians now. This paper is a gist of what I have been learning from my colleagues and students about the good problem solvers; their personality, habits and preferences. I believe recording of this group-experience, sincerely serves the educational purposes in mathematics.

Introduction

For some mathematicians, problem solving is the main motivation to become involved in this profession and for others problem solving is of less importance with respect to developing rich and beautiful theories. But, for both of these groups, becoming a good problem solver is gaining a skill which could not be replaced. If we study the mathematical life of great mathematicians, we see that their mathematical maturity is correlated to their skillfulness in problem solving. Of course, it is difficult to judge from one's collected works, about one's progress in mathematical maturity and problem solving skills, but recognition of such a correlation is possible. This observation indicates that it is important for everyone interested in mathematics to improve one's problem solving skills. The following article is meant for providing guidance on how to help students in becoming better problem solvers. This note has done well, if it could introduce a few new educational perspectives to coaches of Olympiad contestants and teachers of problem solving skills.

Advices to problem solvers

It is rarely the case that advices are helpful. The only way to make sure that students use them is to make them compare and see the benefit of taking advice. One can compare the performance of students, or make a student redo the problem solving process under teacher's supervision and observe the progress.

1) Writing neat and clean

Make your students write their arguments. Show them their first draft and give them reasons why they would have performed better if they had written a clean draft. Tell them that, what they write represents what goes on in their mind. A clean paper represents a clean mind. Make them compare their drafts on different problems and see for themselves that the clearer they think, the cleaner they write.

2) Writing down the summary of arguments

It happened often that students have the key piece of argument on hand, but they use it in the wrong place. Not writing down the summary of these arguments makes them forget to think of them in the right time. Show them examples of their own first drafts which contain the main ideas needed to complete the solution of a problem which they failed to solve.

3) Clarifying the logical structure

Make students to solve a problem under your supervision while you ask them questions like: what is the nature of what we are trying to prove? What kind of argument do you expect to imply such a result? What kind of information you have on hand? How can one relate this kind of information and such implications? Can one use local or global arguments, or perform estimations to get the result?

4) Drawing big and clean figures

Hand out a nice figure when your students fail to solve a problem. They will see the benefit of drawing nice figures themselves. Ask them to draw figures which suit the arguments better. Ask them to draw figures which show what happens at limiting cases. Ask them to draw figures which show different components of the arguments needed to solve the problem. Ask them to draw an appropriate figure after solving the problem in order to explain the argument to other students.

5) Recording the process of thinking

It is not the case that only the final solution is of any importance to the students. Recording the process of solving the problem could help the students to know their mathematical personality and their personality of problem solving better. Help them understand better what they were trying to do while they were thinking about the problem. Supervise their thinking and ask them constantly of what they are trying to do. Help them how to summarize and record the answer.

6) Deleting irrelevant remarks and explanations

Read their first drafts and delete irrelevant remarks, and explain to them why you think a particular explanation could not be helpful in solving the problem. Make them read other student's drafts and correct them under your supervision.

7) Writing down side results

In the course of problem solving many side results are proven which may or may not be of use in the final argument. It is wise to write down all of these little results. Side results could be forgotten when they are the most useful. Show students drafts in which forgotten side results have failed students to solve the problems.

8) Putting down the full proof after finishing the arguments

Students usually write down the proof while they are still thinking about the solution. This makes them not to think about how to present the proof in written form. This is why one should stop students after solving the problem and give them a break and then ask them to decide many ways to explain the arguments and choose the most appropriate one.

9) Notifying important steps in form of lemmas

Lemmas are the best tools to simplify explanation of a complicated argument. Students usually believe that there is a unique way to divide a proof into lemmas. Show them examples of proofs with many suggestions for important lemmas and ask students to decide which form is the most appropriate.

10) Considering the mind of reader

Students are usually not taught the fact that they do not write for themselves only. They usually write for others to read and understand their results. Read students' solution sheet and let them know about the parts which could be unclear for a reader which is ignorant of the final solution of the problem. Ask the students to write down the proof again minding what would come to the mind of a reader. Read and correct the improved version again.

Decisions to be made

What makes good problem solvers successful is to make right decisions at the right time. It is not the case that students always race against time to solve problems. Sometimes they have to race against depth of vision of other people who thought about the same problems or against their own mathematical understanding and abilities regarding a specific problem. Anyhow, decision making in the process of problem solving keeps students in control of the flow of their mathematical thoughts. This makes them recognize their own mathematical personality is the course of decision making.

11) Where to start

Make your student suggest many ways to start thinking about a problem. The goal is to understand the problem better. Working out illuminating examples, understanding assumptions, considering implications of the statement of the problem are examples of good starting points. Ask students to decide among several starting points and give reasonable indication why their choices are the most efficient.

12) Listing different strategies to attack the problem

There are many strategies which could be chosen by students in the course of problem solving. Drawing a picture, trial and error, organizing data regarding particular cases, reducing to a few simpler problems, solving similar simpler problems, using algebra to translate the problem to symbolic language, are all examples of strategies to attack a problem. Students should decide how to attack the problem after they gain some understanding of the problem.

13) Mathematical modeling in different frameworks

In many cases, there are different possibilities to mathematically formulate a problem. One could think of a problem geometrically or suggest different ways for algebraic coordinatization of the problem, or extract the combinatorial essence of a mathematical situation or translate it to analytical language. It is crucial to choose the best formulation. In some problems different parts are best to be done in different formulations.

14) Using symbols or avoiding symbols

Ask students for introducing different mathematical symbols in order to solve the problem algebraically and make them decide which is better and if it is appropriate to use symbols or not. They must decide if it is appropriate to introduce new symbols or not.

15) Deciding what not to think about

When racing with time, thinking about a problem from particular perspectives could be confusing. There could be mathematical languages which are unsuitable for particular problems, because they do not reveal the infra structure or because they disregard some of the important information. Students should be taught to foresee difficulties which may arise in particular approaches towards the problem.

16) Organizing the process of coming to a solution

After a while when student worked out some details about the problem, it is time to formulate a solution plan. A solution plan is a strategy for attacking a problem and overcoming the obstacles and finally solving the problem. Make students to write down their solution plans for a particular problem which has many solutions and then compare their solution plans.

17) How to put down the proof

Make different students put down a particular proof and then compare their drafts. This makes them decide on how to put down a particular proof. They may decide that some lemmas should have easier proofs or decide where to put a lemma or the proof of the lemma or where to put the finishing argument in order to make the poof more readable. Ask them to write down such decisions and give them guidance.

Habits to find

Good habits are a result of supervised practice. Students should be provided with long term supervision and guidance in order to secure good habits of problem solving. One can teach how to become a good problem solver, but it takes time. Teachers should personally gain these habits and show off in their performance in front of the class. It is suggested that from time to time, the teacher should try to solve problems proposed by students by the board so that students have role models in problem solving.

18) Tasting the problem

For students to make comparison between relevant problems, they should find an understanding of the logic of problem, estimate the level of difficulty, and locate the center of complicacy. This is an intuitive process. The more students are experienced in problem solving, the better they are in tasting a problem.

19) Gaining personal view towards the problem

The process of understanding a problem is not independent of the mathematical personality of the students. Each student has personal ways to get in touch with the core of problem. Make the students translate problems into the language of personal preference and rewrite it in this new language.

20) Talking to oneself

Supervise students by asking them questions which could guide them through the process of solving problems. Then make them ask these questions themselves and make students conscious of the decisions they make in the process of thinking. They should tell themselves what to do and what not to do.

21) Considering all the cases

Solving a problem by case studies is a particular ability which is most effective in computational proofs. It happens often for students who are not experienced that they forget to consider some simple limiting or extreme cases which are naturally eliminated in the course of computations.

22) Checking special cases

Recognizing special cases which are easy enough to simplify the computations and are complicated enough to show the core of the problem, is an intuitive art which is gained by experience. Ask the students to suggest special cases of many problems and solve these particular cases and see if this can help solution of the problem in general.

23) Performing a few steps mentally

Students have different abilities in mental computations. Some of them are able to perform and relate quite a few steps mentally. This helps them to foresee the path each decision could lead to. This ability could be improved by supervised practice. Teachers should propose problems for students which are appropriate for their level of understanding to be solved mentally without the use of paper and pencil.

24) Thinking simple

Nature always chooses the simplest way. This could be imitated as a rule for efficient problem solving. There are many experienced problem solvers who are unable to think simple. All the complicated strategies and computations show off in their mind when thinking about a problem and influence how they react to the problem. This is why sometimes students of young age perform better in solving some problems.

Personality of good problem solvers

Good problem solvers have certain common characteristics which governs their success in this art. It is not possible to teach a particular person to have a certain personality. Some students naturally have these characteristics and some have hidden capacities which are developed by experience. The role of teachers is to face students with these personalities and motivate them to work hard for gaining them.

25) Patience

It happens often that a student has all the ideas needed to solve the problem but does not have enough patience to perform the computations or to relate different ideas to each other. When this happens students tries to solve the problem by forcing it into predetermined methods instead of trying to consider as an intellectual challenge. This impatience not only taken student’s mind away from the solution but also makes student nervous and angry. This is why many students dislike exams.

26) Divergent thinking

New ideas and scientific breakthroughs only come to the mind of researchers with divergent thinking. It has happened often that a student in a high class competition has proposed a much simpler solution to a problem that a group of experts has been working on it for hours. Divergent thinking is the main ingredient of a creative mind.

27) Criticizing conjectures

Many students jump into conclusions and waste time on trying to follow the path suggested by these conclusions. The most efficient students in problem solving are those whom are self critical. They try to find different independent reasons for following a solution plan. One can criticize conjectures by considering special cases, or by philosophical considerations.

28) Looking for equivalent formulations

Every student according to his or her mathematical personality chooses a particular language to think about the problem. Students are usually imprisoned in their choice of the language which is a very limiting characteristic for a problem solver. Good problem solvers are fluent in translating the set up to different languages and they do this on several occasions in the process of problem solving to think of the problem in a language most appropriate for computations.

29) Fluency in working with ideas and concepts

Good problem solvers are able to solve the problem in the language of concepts without any need for reference to particular symbols or mathematical formulations. This is a very abstract ability. Also translation of an idea to the language of mathematical symbols and performing computations using these algebraic symbols is an ability of good problem solvers.

30) Looking for simpler models

One shall constantly check if there is a possibility to use symbolizations which makes the problem look simpler than it is. This tendency towards using the simplest models could be recognized in many students occasionally, but only good problem solvers have developed this personality in a way that it is constantly in work.

Intuition

This is a mysterious behavior which has roots in external inspirations. Intuition does not work like personality. One should concentrate deep enough on a problem to become ready for receiving unconscious inspirations. There have been scientists who believed that thinking is nothing but getting prepared for inspirations. Only a healthy and patient mind can put itself in such a position and only by working hard. Intuitive thinking and logical thing are complements of each other and as skill they are independent. One can not have a good intuition and at the same time, be weak in arguments.

31) Geometric imagination

By geometric imagination we mean pictorial thinking which is against verbal or logical thinking. Good problem solvers always make pictures in their mind, even when doing discrete mathematics or algebra. The importance of pictorial intuition in mathematics is due to the geometric nature of the physical world. This does not mean that good problem solvers are not god logical thinkers.

32) Recognizing simple from difficult

A good problem solver after tasting a problem, estimates the level of difficulty of the problem, and then chooses solution plans accordingly. Doing particular cases of the problem introduces good indications about the level of difficulty of the problem. Only intuition is able to judge about simplicity of a problem before solving it. Similar problems could also give help for judging accurately

33) Decomposition and reduction to simpler problems

Reduction a problem to simpler problems usually can be done in several ways. Usually, it is completely unclear how one should decompose a problem to smaller ones. This is because some problems do not easily reveal their essence. This is when we need intuition to help us. Sometimes you have to translate the problem to a new language in order to make the reduction simpler.

34) Jumps of the mind

It is not at all the case that the process of solving a problem can always be entirely explained and understood. There are many instances that intuition makes the mind of the problem solver jump to new perspectives which can not be easily explained by the available data. Divergent thinkers are usually better jumpers. Convergent minds always do the trivial things.

35) Estimating how much progress has been made

Until the time the problem is solved, it is impossible to make sure how much progress is made. Intuition if supported by an accurate solution plan can reveal the level of progress. There are differences in opinions about recognizing parts of the solution to be more important than the other parts. Therefore this kind of estimation only provides personal information.

36) Finding the trivial propositions quickly

This quality is usually recognized as being smart or a fast thinker. But it has to do with intuition and fluency in arguments and embodying them in the form of propositions. The more experienced the problem solver is, the better are the guesses about what trivial propositions could be. One should think about the nature of such propositions.

37) Formulating good conjectures

Good problem solvers are often good in formalizing conjectures. A good conjecture should be tried against challenging cases and also formulated in a way that it introduces a better perspective. Intuition is the only tool our mind has possessed to introduce conjectures which are potential propositions. A good guesser chooses the best conjecture from a list of potential conjectures.

38) Being creative and directed in constructions

Introducing new constructions needs intuition. Because, one needs to foresee what one is trying to construct. By creativity in construction we mean combing different elements borrowed from different construction to introduce a new construction with new abilities. By being direct we mean going straight to the action and experiencing through the constructions.

39) Understanding an idea independent of the context

Students are always more comfortable with the language of mathematical symbols, rather than with the network of ideas. Only very abstract minds translate the problem to the language of concepts, and only some mathematicians are able to think through a problem only working with concepts related to the problem avoiding symbols. This is not a very concrete kind of understanding. It is rather intuitive.

40) Imagination comes before arguments and computations

Many people believe that proofs give insight. But it is never the case. Always insight comes before the arguments. This is what makes the mind of a problem solver an intuitive mind before being a logical mind. Giving a proof for a proposition is like finding an address which explains how to get to the facts mentioned in the proof. Therefore, it is by no means unique. One shall get rid of it as soon as one is familiar with the surrounding area around the facts mentioned in the proposition. A problem solver who is good in arguments is good in imagination.

Final word

It is not recommended that one shall follow good problem solvers in detail. But one should keep in touch with such people in order to absorb their skills by watching them use their skills. The main aim of students should be finding their own abilities and making the blossoms of their skills bloom so that they can see in themselves what could become of them as a problem solver.