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Generalizations of the Liouville theorem. (English summary)
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The author obtains a Liouville-type theorem for a C^2-function $w\colon N \to \mathbb{R}$ on a complete non-compact Riemannian manifold N with Ricci curvature bounded from below by a constant b. Namely, if $\Delta w \geq 1$, then

$$\limsup_{r_N \to \infty} w(x)/r_N(x) > 0,$$

where $r_N(x) = d_N(x,p)$ is the distance function in N to a fixed point p. In particular, w is unbounded.

The proof is based on the fact that $\Delta\eta(x_0) \leq 0$ for a continuous function $\eta\colon N \to \mathbb{R}$ in the sense of support functions (or the barrier sense) given in [J. Cheeger, in *Geometric topology: recent developments (Montecatini Terme, 1990)*, 1–38, Lecture Notes in Math., 1504, Springer, Berlin, 1991; MR1168042 (94a:53075)] and in [J.-H. Eschenburg, "Comparison theorems in Riemannian geometry", lecture notes, Univ. Trento, Trento, 1994; available at http://www.math.uni-augsburg.de/~eschenbu/comparison.pdf], and a comparison theorem for $\Delta_N(\Psi \circ r_N)$ (valid on all N, even at points where r_N is not differentiable) relating to $\Delta_b(\Psi \circ r_b)$, where r_b is the distance function of a space form of constant sectional curvature b, and $\Psi\colon [0,\infty) \to \mathbb{R}$ is a C^2-function such that $\Psi' \geq 0$. Continuous functions η on any connected complete Riemannian N satisfy the generalized Hopf-Calabi maximum principle: if $\Delta\eta \geq 0$ in the barrier sense, and η attains a local maximum, then η is constant.

By using this comparison result and choosing a convenient Ψ, and the fact that

$$\Delta(\Psi \circ r_b) \leq \Delta_N(Kw)$$

for a suitable constant $K > 0$, then an application of the maximum principle leads to the proof of the Liouville theorem.

If N is a surface with non-negative curvature, the author proves an improved version of the Liouville theorem for $\Delta w \geq 0$, w non-constant, obtaining the conclusion that

$$\limsup_{r_N(x) \to \infty} w(x)/\log(r_N(x)) > 0.$$

An application of this Liouville theorem to an isometric immersion

$$f\colon M^n \to \overline{M}^{n+k}$$

between complete Riemannian manifolds gives, under certain conditions, a generalization of a result of L. P. M. Jorge and F. V. Xavier [Math. Z. **178** (1981), no. 1, 77–82; MR0627095 (82k:53080)].

{Reviewer's remark: This paper is very nice, concise and deep.}

Reviewed by *Isabel M. C. Salavessa*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.