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Discrete scale invariance and its logarithmic extension

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Abstract

It is known that discrete scale invariance leads to log-periodic corrections to scaling. We investigate the correlations of a system with discrete scale symmetry, discuss in detail possible extension of this symmetry such as translation and inversion, and find general forms for correlation functions.

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1. Introduction

Log-periodicity is a signature of *discrete* scale invariance (DSI) [1]. DSI is a symmetry weaker than (continuous) scale invariance. This latter symmetry manifests itself as the invariance of a correlator $\mathcal{O}(x)$ as a function of the *control* parameter x, under the scaling $x \to e^{\mu}x$ for arbitrary μ. This means, there exists a number $f(\mu)$ such that

$$\mathcal{O}(x) = f(\mu)\mathcal{O}\big(e^{\mu}x\big). \tag{1}$$

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The solution to (1) is simply a power law $\mathcal{O}(x) = x^\alpha$, with $\alpha = -(\log f)/\mu$, which can be directly verified. In a system having DSI, correlators obey scale invariance, only for *discrete* values of the magnification factor $\mu = n\mu_1$, where n is integer and μ_1 is fixed. μ_1 is the period of the resulting log-periodicity [1]. This property can be qualitatively seen to encode a *lacunarity* of the fractal structure. The most general solution to (1) with $\mu = n\mu_1$ is

$$\mathcal{O}(x) = x^\alpha P\left(\frac{\ln x}{\mu_1}\right), \tag{2}$$

where $P(y)$ is an arbitrary periodic function of period 1, hence the name log-periodicity. Expanding it in Fourier series $\sum_{n=-\infty}^{\infty} c_n \exp[2n\pi i(\ln x)/\mu_1]$, we see that $\mathcal{O}(x)$ becomes a sum of power laws with the exponents $\alpha_n = \alpha + (2n\pi i/\mu_1)$, where n is an arbitrary integer. So there is an infinite set of discrete complex exponents. Specifically, it has been established that for financial bubbles prior to large crashes, a first order representation of (2),

$$I(t) = A + B(t_c - t)^\beta + C(t_c - t)^\beta \cos\bigl(\omega \ln(t_c - t) - \phi\bigr), \tag{3}$$

captures well the behaviour of the market price $I(t)$ prior to a crash or large correction at a time $\approx t_c$ [2,3].

There are many mechanisms known to generate log-periodicity [1]. Let us stress that various dynamical mechanisms generate log-periodicity, without relying on a pre-existing discrete hierarchical structure. Thus, DSI may be produced dynamically (see in particular the recent nonlinear dynamical model introduced in [4]) and does not need to be predetermined by, e.g., a geometrical network. This is because there are many ways to break a symmetry, the subtlety here being to break it only partially. Thus, log-periodicity per se is not a signature of a critical point. Only within a well-defined model and mechanism can it be used as a potential signature.

Scale-invariance is a subgroup of a larger transformation group, the conformal group. In any conformal field theory, the system shows invariance under translation, rotation, dilatation, and special conformal transformation [5]. Conformal invariance is sufficient to determine the general form of simple correlation functions. It is known that generally the correlation functions in ordinary conformal field theories are in the form of scaling functions. This is the case, when the matrix of the weights is diagonalizable. There are, however, cases when this matrix is nondiagonalizable, and has a Jordanian form, that is the weight matrix is the sum of a diagonalizable matrix, and a nilpotent one (the latter commuting with the former). These theories are known as logarithmic conformal field theories (LCFTs). In these theories, there exist at least one partner for a primary field. The general form of one-, two-, and three-point functions in LCFTs have been obtained in [6–11].

Recently, people have become more interested in systems with discrete scale invariance. There does not exist, however, a complete study of general properties of such systems. In this paper we want to study the general form of correlation functions of systems with discrete scale invariance. In Section 2, subgroups of the conformal transformations are investigated, which contain discrete (*not full*) scale transformations. In Section 3, the general form of the correlators of systems possessing discrete scale invariance are obtained. In Sections 4 and 5, the same is done for the systems possessing discrete scale invariance plus translation- or special conformal transformation-invariance, respectively. Both ordinary and logarithmic cases are studied.

2. Subgroups of the conformal group

Consider the transformations

$$\begin{aligned} &T_\alpha : z \to z + \alpha, && \text{translation,} \\ &S_\mu : z \to e^\mu z, && \text{scaling-rotation,} \\ &C_a : z \to \frac{z}{1 - az}, && \text{special conformal transformation,} \end{aligned} \tag{4}$$

in the complex plane. The group of transformations constructed by the above transformations, is that subgroup of the conformal group, the central extension of which is trivial. It is well known that this group is isomorphic to $\mathrm{SL}_2(\mathbb{C})$. Let us call this group SCT. Throughout this section, we consider only those subgroups of SCT, which are *complete*, by which it is meant that if a convergent sequence of group-elements of SCT are in the subgroup, the limit of that sequence is also in the subgroup.

First consider subgroups of translations. Any subgroup of translations, consists of translations T_α, with

$$\alpha = x\beta + y\beta', \tag{5}$$

where β and β' are two fixed nonzero complex numbers, and β/β' is not real. x and y, each may take only real values, only integer values, or only zero.

A similar argument holds for the subgroups of special conformal transformations, as these transformations are in fact translations of $1/z$.

For the subgroups of scaling-rotations too, a similar argument holds. However, as a rotation by 2π (S_μ with $\mu = 2\pi i$) is equal to identity and hence should be in the subgroup, one concludes that any subgroup of scaling-rotations consists of the transformations S_μ, with

$$\mu = x\nu + y\nu', \tag{6}$$

where ν is pure imaginary, ν' is not pure imaginary, ν and ν' are fixed. The values x and y can take, are as in the subgroups of translations, except that x takes integer values only if ν is rational multiple of $2\pi i$. In this case, if $\nu = (2\pi i)n/m$, where n and m are integers prime with respect to each other and m is positive, then those values of x which result in distinct group-elements are integers from 0 up to $m - 1$. In this case, in fact one can use $\tilde{\nu} := 2\pi i/m$ instead of ν. This means that subgroups of scaling-rotations always contain subgroups of pure rotations, which may be trivial, discrete, or the full rotation group.

Next consider subgroups consisting of products of two of the above subgroups. These products are not direct products, since translations, scaling-rotations, and special conformal transformations do not commute with each other. Consider first a subgroup of SCT, consisting of transformations which are a scaling-rotation followed by a translation, where the translations are a subgroup of translations, and the scaling-rotations are a subgroup of scaling rotations:

$$z \to e^{\mu} z + \alpha, \tag{7}$$

where α takes value on a lattice, (5), and μ takes value on another lattice, (6). (Each of these lattices may be degenerate, that is, a collection of lines, the whole plane, just one point, etc.) This set of transformations is a subgroup, if and only if the lattice of the possible values of α is invariant under the action of scaling-rotations which are in the set. The only possible ways (apart from the trivial cases that the scaling-rotations or translations consist of only the identity element) are the following:

(i) a rectangular lattice of translations, with scaling-rotations only rotations by integer multiples of $\pi/2$;
(ii) a triangular lattice of translations, with scaling-rotations only rotations by integer multiples of $\pi/3$ or $2\pi/3$;
(iii) any lattice of translations, with scaling-rotations only rotations by integer multiples of π;
(iv) a degenerate lattice of translations, consisting of one continuous line, with scaling-rotations only a subgroup (any subgroup) of scalings, or the product of a subgroup of scalings and rotations by integers multiples of π;
(v) the full translations with any subgroup of scaling-rotations.

Again, a similar argument holds for subgroups of SCT, consisting of transformations which are a scaling-rotation followed by a special conformal transformation, where the special conformal translations are a subgroup of special conformal transformations, and the scaling-rotations are a subgroup of scaling rotations:

$$z^{-1} \to e^{-\mu} z^{-1} - a. \tag{8}$$

A set of transformations consisting of a translation followed by a special conformal transformation, where the special conformal translations are a subgroup of special conformal transformations, and the translations are a subgroup of translations, cannot be a subgroup, unless the only translation is identity, or the only special conformal transformation is the identity. To show this, one notices that the product $T_\alpha C_a$, can be equal to $C_{a'} T_{\alpha'}$ for some α' and a', only if $a\alpha = 0$ or $a\alpha = 2$. But this criterion is necessary for the above-mentioned set to be a subgroup (the multiplication be closed). If $a\alpha = 0$ for all translations and special conformal translations in the set, then at least on of these subgroups must contain no element apart from the identity. If $a\alpha = 2$ for some a and α, one notes that if the above-mentioned set contains T_α, it should also contain $T_{2\alpha}$, and obviously $2a\alpha \neq 2$.

Finally, let us consider a subgroup G of SCT, containing nonidentity translations, scaling-rotations, and special conformal transformation. First, assume also that G contains a scaling-rotation which is not a pure-rotation, that is, some S_μ, with μ not pure imaginary. The above arguments show that G must contain a subgroup of continuous translations at least in one direction, and a subgroup of continuous special conformal translations at least in one dimension. The generators of these two one-parameter transformations are

$$g_1 = e^{i\theta} L_{-1} + e^{-i\theta} \bar{L}_{-1}, \tag{9}$$

and

$$g_2 = e^{i\zeta} L_1 + e^{-i\zeta} \bar{L}_1, \tag{10}$$

for some θ and ζ. Using the commutation relation of the generators, it is seen that G contains another one-parameter continuous subgroup, with the generator

$$g_3 = e^{i(\theta+\zeta)} L_0 + e^{-i(\theta+\zeta)} \bar{L}_0. \tag{11}$$

So G contains a continuous subgroup of scaling-rotations. Combining this with the above arguments, it is seen that either

(i) G consists of transformations generated by g_1, g_2, and $L_0 + \bar{L}_0$, where $\alpha + \zeta$ is an integer multiple of π, or,
(ii) G is equal to SCT.

Second, assume that the only scaling-rotations contained in G are pure rotations. It can be shown that G cannot contain both nontrivial translations and nontrivial special conformal transformations. The argument is similar to that used to prove that a set containing translations followed by special conformal transformations cannot be a subgroup of SCT. The difference is that now, For any T_α and S_a in G, there should be some $T_{\alpha'}$ and $S_{a'}$ in G, so that $T_{-\alpha'} S_{-a'} T_\alpha S_a$ be a pure rotation.

The above arguments can be stated for the conformal group in one dimension as well. In one dimension, SCT is still constructed by the transformations T_α, S_μ, and C_a. But now the parameters α, μ, and a are real, the scaling-rotations are in fact only scalings, and SCT is isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Arguments similar to the above, then show that there are subgroups of SCT consisting of

- all translations, or discrete translations;
- all special conformal transformations, or discrete special conformal transformations;
- all scalings, or discrete scalings;
- all translations, and all scalings or discrete scalings;
- all special conformal translations, and all scalings or discrete scalings.

3. Systems with discrete scale invariance

Consider a one-dimensional (one real dimension) system with discrete scale invariance. A quasi-primary field transforms under the scaling as:

$$\phi(x) \to e^{\Delta\mu_1} \phi\big(e^{\mu_1} x\big). \tag{12}$$

Invariance under discrete scaling gives

$$\langle \phi(x) \rangle = e^{\Delta\mu_1} \big\langle \phi\big(e^{\mu_1} x\big)\big\rangle. \tag{13}$$

Defining

$$g(x) := x^{\Delta} \langle \phi(x) \rangle, \tag{14}$$

one arrives at

$$g\left(e^{\mu_1}x\right) = g(x). \tag{15}$$

This shows that g is periodic with respect to $\log x$, with the period μ_1. Hence it has a Fourier series:

$$g(x) = \sum_{n=-\infty}^{\infty} C_n \exp\left(\frac{2\pi ni \ln x}{\mu_1}\right), \tag{16}$$

from which one can obtain the one-point function $\langle\phi(x)\rangle$. The expectation value of the field ϕ is real, iff $C_n = \overline{C}_{-n}$. In this case, one arrives at

$$\langle\phi(x)\rangle = x^{-\Delta} \sum_{n=0}^{\infty} r_n \cos\left(\frac{2\pi n \ln x}{\mu_1} + \theta_n\right), \tag{17}$$

where r_n and θ_n are defined through $2C_n =: r_n \exp(i\theta_n)$.

The two-point function of two quasi-primary fields can be obtained similarly. Defining

$$g(x_1, x_2) := x_1^{\Delta_1} x_2^{\Delta_2} \langle\phi_1(x_1)\phi_2(x_2)\rangle, \tag{18}$$

and exploiting the discrete scale invariance, it is seen that

$$g\left(e^{\mu_1}x_1, e^{\mu_1}x_2\right) = g(x_1, x_2), \tag{19}$$

from which, using the new variables (x_2/x_1) and $\ln x_1$, one arrives at

$$g(x_1, x_2) = P\left(\frac{\ln x_1}{\mu_1}, \frac{x_2}{x_1}\right), \tag{20}$$

where P is a periodic function with period one, of its first variable, and arbitrary with respect to its second variable. One can then expand this function as a Fourier series, and obtain the two-point function as

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = x_1^{-\Delta_1} x_2^{-\Delta_2} \sum_{n=-\infty}^{\infty} C_n\left(\frac{x_2}{x_1}\right) \exp\left(\frac{2\pi ni \ln x_1}{\mu_1}\right), \tag{21}$$

where C_ns are arbitrary functions.

This argument can be extended to more-point functions of quasi-primary fields, and one arrives at

$$\langle\phi_1(x_1)\cdots\phi_k(x_k)\rangle = x_1^{-\Delta_1} \cdots x_k^{-\Delta_k} \sum_{n=-\infty}^{\infty} C_n\left(\frac{x_2}{x_1}, \ldots, \frac{x_k}{x_{k-1}}\right) \exp\left(\frac{2\pi ni \ln x_1}{\mu_1}\right). \tag{22}$$

Now let us assume a logarithmic partner ψ for the quasi-primary field ϕ. This means that under discrete scale transformation, ψ transforms like

$$\psi(x) \to e^{\Delta\mu_1}\left[\psi\left(e^{\mu_1}x\right) + \mu_1\phi\left(e^{\mu_1}x\right)\right]. \tag{23}$$

This is nothing by the formal derivative of (12) with respect to Δ, if one considers ψ as the formal derivative of ϕ with respect to Δ. So, in light of [8], it is not surprising that the expectation of ψ is just the formal derivative of that of ϕ:

$$\langle\psi(x)\rangle = x^{-\Delta}\sum_{n=-\infty}^{\infty}(C_n' - C_n \ln x)\exp\left(\frac{2\pi n i \ln x}{\mu_1}\right), \tag{24}$$

where C_n' is the formal derivative of C_n with respect to Δ. This can be readily extended to more-point functions: changing any quasi-primary field in the left-hand side of (22), into its logarithmic partner, one has to differentiate formally the right-hand side with respect to the weight of that field, treating arbitrary constants and functions as entities depending on that weight.

Finally, the above arguments for obtaining the correlators of a one-dimensional system with discrete scale invariance, can be easily extended to two-dimensional system with rotational invariance and discrete scale invariance. Here, a quasi-primary field ϕ transforms as

$$\begin{aligned}
&\phi(z) \to e^{i(\Delta-\bar{\Delta})\theta}\phi\left(e^{i\theta}z\right), \quad z \to e^{i\theta}z,\\
&\phi(z) \to e^{(\Delta+\bar{\Delta})\mu_1}\phi\left(e^{\mu_1}z\right), \quad z \to e^{\mu_1}z.
\end{aligned} \tag{25}$$

(It is not assumed that ϕ is holomorphic. Its dependence on z is a short-hand notation for its dependence on the real part and the imaginary part of z, or its dependence on z and $\bar{z}$.) Now, let us find the expectation of ϕ. Defining

$$g(z) := z^{\Delta}\bar{z}^{\bar{\Delta}}\langle\phi(z)\rangle, \tag{26}$$

it is seen that g depends on only $|z|$, and that g is periodic with respect to $\ln|z|$, with the period μ_1. So,

$$\langle\phi(z)\rangle = z^{-\Delta}\bar{z}^{-\bar{\Delta}}\sum_{n=-\infty}^{\infty}C_n\exp\left(\frac{2\pi n i \ln|z|}{\mu_1}\right). \tag{27}$$

A similar reasoning leads to

$$\begin{aligned}
\langle\phi_1(z_1)\cdots\phi_k(z_k)\rangle &= z_1^{-\Delta_1}\cdots z_k^{-\Delta_k}\bar{z}_1^{-\bar{\Delta}_1}\cdots\bar{z}_k^{-\bar{\Delta}_k}\\
&\quad\times\sum_{n=-\infty}^{\infty}C_n\left(\frac{z_2}{z_1},\ldots,\frac{z_k}{z_{k-1}}\right)\exp\left(\frac{2\pi n i \ln|z_1|}{\mu_1}\right),
\end{aligned} \tag{28}$$

for the k-point function of k quasi-primary fields. Differentiating this formally, with respect to appropriate weights, one arrives at the correlators containing logarithmic partners as well.

4. Discrete scale invariance plus translation invariance

First consider a one-dimensional system with discrete scale invariance as well as translation invariance. A quasi-primary field ϕ, is transformed under discrete scale transformation like (12), and is transformed under translation $x \to x+\alpha$ as

$$\phi(x) \to \phi(x+\alpha). \tag{29}$$

Translation invariance results in

$$\langle\phi(x+a)\rangle = \langle\phi(x)\rangle, \tag{30}$$

or

$$\langle\phi(x)\rangle = C, \tag{31}$$

where C is a constant. Discrete scale invariance, then gives

$$\left(e^{\Delta\mu_1} - 1\right)C = 0. \tag{32}$$

This means that the one-point function is nonzero, only if for fields with the weights

$$\Delta = \frac{2\pi k i}{\mu_1}, \tag{33}$$

where k is an integer. If there exists a logarithmic partner for ϕ, then translational invariance gives

$$\langle\phi(x)\rangle = C_1, \qquad \langle\psi(x)\rangle = C_2. \tag{34}$$

Here ψ is the logarithmic partner of ψ, being transformed under discrete scale invariance like (23), and behaving under translation like (29), with ϕ substituted with ψ. Imposing discrete scale invariance, it is seen that the one-point functions are zero, unless $e^{\Delta\mu_1} = 1$, and in this case C_1 is zero. The constraint on the weight is the same as (33). So,

$$\langle\phi(x)\rangle = 0, \qquad \langle\psi(x)\rangle = C_2, \qquad e^{\Delta\mu_1} = 1. \tag{35}$$

The more-point functions of quasi-primary fields, can be obtained similarly. Translation invariance makes the k-point function a function of $(k-1)$ independent differences of the k coordinates. Defining

$$\begin{aligned} g(x_2 - x_1, \ldots, x_k - x_{k-1}) :=\ & (x_1 - x_k)^{\Delta_1}(x_2 - x_1)^{\Delta_2}\cdots(x_k - x_{k-1})^{\Delta_k} \\ & \times \langle\phi(x_1)\cdots\phi_k(x_k)\rangle, \end{aligned} \tag{36}$$

it is seen that

$$g(x_2 - x_1, \ldots, x_k - x_{k-1}) = P\left[\frac{\ln(x_2 - x_1)}{\mu_1}, \frac{x_3 - x_2}{x_2 - x_1}, \ldots, \frac{x_k - x_{k_1}}{x_{k-1} - x_{k-2}}\right], \tag{37}$$

where P is periodic with the period one, in its first variable, and arbitrary in other variables. From this, the k-point function is obtained as

$$\begin{aligned} \langle\phi(x_1)\cdots\phi_k(x_k)\rangle =\ & (x_1 - x_k)^{-\Delta_1}(x_2 - x_1)^{-\Delta_2}\cdots(x_k - x_{k-1})^{-\Delta_k} \\ & \times \sum_{n=-\infty}^{\infty} C_n\left(\frac{x_3 - x_2}{x_2 - x_1}, \ldots, \frac{x_k - x_{k_1}}{x_{k-1} - x_{k-2}}\right) \\ & \times \exp\left[\frac{2\pi n i \ln(x_2 - x_1)}{\mu_1}\right]. \end{aligned} \tag{38}$$

Correlators containing logarithmic parts, can be obtained by formal differentiation with respect to appropriate weights. The one-point function is exceptional, as it is zero unless the weight is one of the members of a discrete set, Eq. (33). This case was discussed previously.

Finally, for a two-dimensional system, a quasi-primary field ϕ is transformed under rotation and discrete scaling like (25), and under translation like the obvious generalization of (29). Again, translation invariance makes the one-point function of the quasi-primary field ϕ independent of z:

$$\langle\phi(z)\rangle = C. \tag{39}$$

Rotation invariance and discrete scale invariance, then result in

$$e^{i(\Delta-\bar{\Delta})\theta}C = C, \qquad e^{(\Delta+\bar{\Delta})\mu_1}C = C, \tag{40}$$

respectively. From the first equation, C may be nonzero only if $\Delta = \bar{\Delta}$. Then the second equation shows that C may be nonzero only if

$$\Delta = \bar{\Delta} = \frac{\pi k i}{\mu_1}, \tag{41}$$

where k is an integer. If ϕ has a logarithmic partner ψ, then there may be nonzero one-point functions only as

$$\langle\phi(z)\rangle = 0, \qquad \langle\psi(z)\rangle = C_2, \qquad e^{(\Delta+\bar{\Delta})\mu_1} = 1. \tag{42}$$

k-point functions of k quasi-primary fields are also similarly obtained:

$$\begin{aligned}
&\langle\phi(z_1)\cdots\phi_k(z_k)\rangle \\
&\quad = (z_1 - z_k)^{-\Delta_1}(z_2 - z_1)^{-\Delta_2}\cdots(z_k - z_{k-1})^{-\Delta_k} \\
&\qquad \times (\bar{z}_1 - \bar{z}_k)^{-\bar{\Delta}_1}(\bar{z}_2 - \bar{z}_1)^{-\bar{\Delta}_2}\cdots(\bar{z}_k - \bar{z}_{k-1})^{-\bar{\Delta}_k} \\
&\qquad \times \sum_{n=-\infty}^{\infty} C_n\left(\frac{z_3 - z_2}{z_2 - z_1}, \ldots, \frac{z_k - z_{k_1}}{z_{k-1} - z_{k-2}}\right)\exp\left[\frac{2\pi n i \ln|z_2 - z_1|}{\mu_1}\right].
\end{aligned} \tag{43}$$

More-than-one-point functions containing logarithmic partners, are also obtained by formal differentiation of the above, with respect to appropriate weights.

5. Discrete scale invariance plus special conformal invariance

Now, let us consider the systems invariant under discrete scale transformation together with continuous special conformal transformation. First consider one-dimensional systems. A quasi-primary field ϕ, is transformed under special conformal transformation like

$$\phi(x) \to \frac{1}{(1-ax)^{2\Delta}}\phi\left(\frac{x}{1-ax}\right). \tag{44}$$

Defining a field $\tilde{\phi}$ through

$$\tilde{\phi}(x) := x^{-2\Delta}\phi(1/x), \tag{45}$$

it is seen that under C_a, this field is transformed like

$$\tilde{\phi}(x) \to \tilde{\phi}(x-a). \tag{46}$$

It is also seen that under discrete scaling, $\tilde{\phi}$ is transformed as

$$\tilde{\phi}(x) \to e^{-\Delta\mu_1}\tilde{\phi}\big(e^{-\mu_1}x\big). \tag{47}$$

These show that corresponding to the quasi-primary field ϕ, there exists another field $\tilde{\phi}$ such that the actions of special conformal transformation and discrete scaling on which are like translation and discrete scaling, respectively. So correlators containing these new fields are like those obtained in the previous section. From these, it is easy to obtain correlators containing the quasi-primary fields. If ϕ has a logarithmic partner ψ, then define the field $\tilde{\psi}$ through

$$\tilde{\psi}(x) := x^{-2\Delta}\big[\psi(1/x) - 2\Delta \ln x\,\phi(1/x)\big]. \tag{48}$$

This is in fact the formal derivative of $\tilde{\phi}$ with respect to Δ. Correlators containing $\tilde{\phi}$ and $\tilde{\psi}$ are of the general forms obtained in the previous section. From these, it is easy to obtain correlators containing ϕ and ψ, or several fields like them.

For two-dimensional systems, one defines $\tilde{\phi}$ corresponding to a quasi-primary field ϕ like

$$\tilde{\phi}(z) := z^{-2\Delta}\bar{z}^{-2\bar{\Delta}}\phi(1/z). \tag{49}$$

So correlators containing $\tilde{\phi}$ are of the general form obtained in the previous section. Correlators containing logarithmic partners can be obtained through differentiation with respect to appropriate weights.

6. Concluding remarks

The general form of correlation functions of systems with discrete scale invariance was obtained. This was done for systems in one real dimension, as well as systems in one complex dimension (two real dimensions). Logarithmic field theories with discrete scale invariance, were also introduced, and their corresponding correlators were obtained. It is still an open problem to find more physical systems with discrete scale invariance, and also measure more-than-one-point functions for systems showing log-periodicity.

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