

# The Exact N-Point Generating Function in Polyakov - Burgers Turbulence

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## Abstract

We find the exact N-point generating function in Polyakov's approach to Burgers turbulence.

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## 1- Introduction

A theoretical understanding of turbulence has eluded physicists for a long time. Recently Polyakov [1] has offered a field theoretic method for deriving of the probability distribution or density of states in (1+1)-dimensional turbulent systems. Polyakov formulates a new method for analyzing the inertial range correlation functions based on the two important ingredients in field theory and statistical physics namely the operator product expansion (OPE) and anomalies. Despite existence of many field theoretic approaches to turbulence [2,3,4], it appears that this new approach is more promising. Polyakov argues that in the limit of high Reynold's number because of existence of singularities at coinciding point, dissipation remains finite and all subleading terms give vanishing contributions in the inertial range. By develop the OPE one finds the leading singularities and can show that this approach is self-consistent. Here we consider Polyakov's approach [1] to the Burger turbulence when the gradient of pressure is negligible and solve the N-point master equation, calculating the N-point generating function. Our result also apply to the Kardar-Parisi-Zhang (KPZ) equation in (1+1)-dimensions, crystal growth [5], the nonlinear dynamics of a moving line [6], galaxy formations [7], dissipative transport [9], dynamics of a Sine-Gordan chain [10], behavior of magnetic flux line in superconductor [11], and spin glasses [12].

## 2- N-Point Generating Functions

The Burgers equation has following form

$$u_t + uu_x = \nu u_{xx} + f(x, t) \tag{1}$$

where  $u$  is the velocity field, and  $\nu$  is the viscosity and  $f(x, t)$  is the Gaussian random force

with the following correlation:

$$\langle f(x, t)f(x', t') \rangle = k(x - x')\delta(t - t') \quad (2)$$

The transformation,  $u(x, t) = -\lambda\partial_x h(x, t)$  maps eq.(1) to the well-known KPZ equation [5],

$$\partial_t h = \nu\partial_{xx}h + \frac{\lambda}{2}[\partial_x h]^2 + f(x, t) \quad (3)$$

It is noted that the parameter  $\lambda$  that appears in the above transformation is not renormalized under any renormalization procedure [8]. Following Polyakov [1] consider the following generating functional

$$Z_N(\lambda_1, \lambda_2, \dots, \lambda_N, x_1, \dots, x_N) = \langle \exp\left(\sum_{j=1}^N \lambda_j u(x_j, t)\right) \rangle \quad (4)$$

Noting that the random force  $f(x, t)$  has a Gaussian distribution  $Z_N$  satisfies a closed differential equation provided that the viscosity  $\nu$  tends to zero:

$$\dot{Z}_N + \sum \lambda_j \frac{\partial}{\partial \lambda_j} \left( \frac{1}{\lambda_j} \frac{\partial Z_N}{\partial x_j} \right) = \sum k(x_i - x_j) \lambda_i \lambda_j Z_N + D_N \quad (5)$$

where  $D_N$  is :

$$D_N = \nu \sum \lambda_j \langle u''(x_j, t) \exp \sum \lambda_k u(x_k, t) \rangle \quad (6)$$

To reach the inertial range we must, however, keep  $\nu$  infinitesimal but non-zero. Polyakov argues that the anomaly mechanism implies that infinitesimal viscosity produces a finite effect. To compute this effect Polakov makes the F-conjecture, which is the existence of an operator product expansion or the fusion rules. The fusion rule is the statement concerning the behaviour of correlation functions, when some subset of points are put close together.

Let us use the following notation;

$$Z(\lambda_1, \lambda_2, \dots, x_1, \dots, x_N) = \langle e_{\lambda_1}(x_1) \dots e_{\lambda_N}(x_N) \rangle \quad (7)$$

then Polyakov's F-conjecture is that in this case the OPE has the following form,

$$e_{\lambda_1}(x+y/2)e_{\lambda_2}(x-y/2) = A(\lambda_1, \lambda_2, y)e_{\lambda_1+\lambda_2}(x) + B(\lambda_1, \lambda_2, y)\frac{\partial}{\partial x}e_{\lambda_1+\lambda_2} + o(y^2) \quad (8)$$

This implies that  $Z_N$  fuses into functions  $Z_{N-1}$  as we fuse a couple of points together. The F-conjecture allows us to evaluate the following anomaly operator (i.e. the  $D_N$ -term in eq.(5)),

$$a_\lambda(x) = \lim_{\nu \rightarrow 0} \nu(\lambda u''(x) \exp(\lambda u(x))) \quad (9)$$

which can be written as:

$$a_\lambda(x) = \lim_{\xi, y, \nu \rightarrow 0} \lambda \nu \frac{\partial^3}{\partial \xi \partial y^2} e_\xi(x+y)e_\lambda(x) \quad (10)$$

As discussed in [1] the only possible Galilean invariant expression is:

$$a_\lambda(x) = a(\lambda)e_\lambda(x) + \tilde{\beta}(\lambda)\frac{\partial}{\partial x}e_\lambda(x) \quad (11)$$

Therefore in steady state the master equation takes the following form,

$$\begin{aligned} \sum \left( \frac{\partial}{\partial x_j} - \beta(\lambda_j) \right) \frac{\partial}{\partial x_j} Z_N - \sum k(x_i - x_j) \lambda_i \lambda_j Z_N &= \sum a(\lambda_j) Z_N \\ \beta(\lambda) &= \tilde{\beta}(\lambda) + \frac{1}{\lambda} \end{aligned} \quad (12)$$

Polyakov has found the explicit form of  $Z_2$  in the case that  $k(x_i - x_j) = K(0)(1 - (x_i - x_j)^2/l^2)$ , and the density of states as;

$$W(u, y) = \int_{c-i\infty}^{c+i\infty} \frac{d\mu}{2\pi i} e^{-\mu u} \phi(\mu y) \quad (13)$$

where

$$\phi(\mu y) = e^{2/3(\mu y)^{3/2}}$$

and

$$\mu = 2(\lambda_1 - \lambda_2), y = x_1 - x_2.$$

It can be easily shown that with the following definition of variables Polyakov's master equation with the scaling conjecture[1] is:

$$\left\{ \frac{\partial^2}{\partial \mu_2 \partial y_2} + \frac{\partial^2}{\partial \mu_3 \partial y_3} - (y_2 \mu_2 + y_3 \mu_3)^2 \right\} f_3 = 0 \quad (14)$$

where

$$f_3 = (\lambda_1 \lambda_2 \lambda_3)^{-b} Z_3$$

$$\begin{aligned} y_1 &= \frac{x_1 + x_2 + x_3}{3} & y_2 &= x_1 - \frac{x_2 + x_3}{2} & y_3 &= x_2 - x_3 \\ \mu_1 &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} & \mu_2 &= \frac{3}{2} \left( \lambda_1 - \frac{\lambda_2 + \lambda_3}{2} \right) & \mu_3 &= 2(\lambda_2 - \lambda_3) \end{aligned} \quad (15)$$

Now we set  $f$  as;

$$f_3 = \mu_2^{S_2} \mu_3^{S_3} g_3(\mu_2 y_2, \mu_3 y_3) \quad (16)$$

inserting in eq.(15) results in

$$g_3(\mu_2 y_2, \mu_3, y_3) = e^{2/3(\mu_2 y_2 + \mu_3 y_3)^{3/2}} \quad (17)$$

and

$$S_2 = S_3 = -5/4$$

Now if we use the following transformation;

$$\begin{aligned} y_1 &= \frac{x_1 + x_2 + x_3 + \dots x_N}{N} \\ y_2 &= x_1 - \frac{x_2 + x_3 + \dots x_N}{N-1} \\ y_3 &= x_2 - \frac{x_3 + x_4 + \dots x_N}{N-2} \end{aligned}$$

$$\text{and} \quad y_N = x_{N-1} - x_N \quad (18)$$

and

$$\begin{aligned} \mu_1 &= \frac{\lambda_1 + \lambda_2 + \dots + \lambda_N}{N} \\ \mu_2 &= \frac{N}{N-1} \left[ \lambda_1 - \frac{\lambda_2 + \lambda_3 + \dots + \lambda_N}{N-1} \right] \\ \mu_3 &= \frac{N-1}{N-2} \left[ \lambda_2 - \frac{\lambda_3 + \lambda_4 + \dots + \lambda_N}{N-2} \right] \\ \text{and} \quad \mu_N &= 2(\lambda_{N-1} - \lambda_N) \end{aligned} \quad (19)$$

we get the following partial differential equation for  $f_N$ ;

$$\left\{ \frac{\partial^2}{\partial y_2 \partial \mu_2} + \dots + \frac{\partial^2}{\partial y_N \partial \mu_N} \right\} f_N - (y_2 \mu_2 + \dots + y_N \mu_N)^2 f_N = 0 \quad (20)$$

which is solved by:

$$f_N = (\mu_2 \mu_3 \dots \mu_N)^{-[\frac{2N-1}{2(N-1)}]} \exp^{2/3(\mu_2 y_2 + \dots + \mu_N y_N)^{3/2}} \quad (21)$$

In principle the parameter  $b$  in eq.(15) can be evaluated by means of the exponents of  $(\mu_2 \dots \mu_N)$  term in eq.(21) which for  $Z_2$  it turns out that  $b = 3/4$ . Our exact result for  $Z_N$  allows us to determine the OPE coefficients in eq.(8), and calculate  $Z_N$  in the case where the random force is conservative, which is important for the KPZ-equation, work in this direction is on the way.

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