Stochastic Processes	Student ID:
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Spring 2007	

In The Name of God, the Compassionate, the Merciful

Final Exam

Take Home: Due Friday 15th of Tir at 12 Midnight

Send by email to: rabiee@sharif.edu or my office at Sharif phone number: 6600-6399

1. [20 points] Basic Concepts

a. (10 points) Suppose you have a real random variable, X, that is Gaussian with zero mean and a variance of σ^2. No take two independent, identically distributed Bernoulli random variables, Y_1 and Y_2 with

$$p = 0.5(p(Y_i = 1) = 0.5,\ p(Y_i = -1) = 0.5)$$

Define two new random variables: $W_1 = Y_1 X$ and $W_2 = Y_2 X$.

(1) Find the probability density functions of W_1 and W_2.

(2) Find the $E[W_1 W_2]$.

(3) Are W_1 and W_2 uncorrelated?

(4) Find the probability $W_1 = 0$ given $W_2 = 1$. Find the probability $W_1 = 1$ given $W_2 = 1$. Are the two random variables W_1 and W_2 independent? Why?

b. (5 points) Show that if $\mathbf{x}(t)$ is a stochastic process with zero mean and autocorrelation $f(t_1)f(t_2)w(t_1-t_2)$, then the process $\mathbf{y}(t) = \mathbf{x}(t)/f(t)$ is WSS with autocorrelation $w(\tau)$. If $\mathbf{x}(t)$ is white noise with autocorrelation $q(t_1)\sigma(t_1-t_2)$, then the process $z(t) = x(t)/\sqrt{q(t)}$ is WSS white noise with autocorrelation $\sigma(t)$.

c. (5 points) Show that if in an LTI system the output-input relation is given by $\mathbf{y}(t) = \mathbf{x}(t+a) - \mathbf{x}(t-a)$, then

$R_y(\tau) = 2R_x(\tau) - R_x(\tau + 2a) - R_x(\tau - 2a)$ and $S_y(\omega) = 4S_x(\omega)\sin^2 a\omega$.

2. [20 points] Linear Systems & Stochastic Processes

Consider an experiment in which a point with coordinates (w_1 , w_2) is drawn at random from the unit square:

$\Omega = \{(w_1, w_2) : 0 \le w_1, w_2 \le 1\}$.

A continuous-Parameter random process (field) is defined on the same square according to

$\underline{f}_{w_1,w_2}(x,y) = \mathrm{sgn}[(x-w_1)(y-w_2)],\ 0 \le x\ ,\ y \le 1.$

a) Draw a typical sample function of this process.

b) What is its probability?

c) Calculate E{ $\underline{f}(x,y)$ }.

d) Calculate the second moment E{ $\underline{f}^2(x,y)$ }. and the variance $s^2_{\underline{f}}(x,y)$ of this random field.

e) Calculate the autocorrelation $R_{\underline{ff}}(x_1,y_1,x_2,y_2) = E\{\ \underline{f}(x_1,y_1)\underline{f}(x_2,y_2)\ \}$

3. [20 points] Maximum Likelihood (ML) & Fisher Information Matrix (I)

Consider the random vector

$$y = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 5 \\ 4 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Where ***x*** is a normal Gaussian random vector with mean $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and covariance $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Given ***y*** find the ML estimator of $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$.

(b) Find the covariance matrix of the ML estimate of q_1 and q_2.

(c) Find the Fisher information matrix for estimating q_1 and q_2.

(d) Find the Fisher information matrix for estimating q_1 given you know q_2.

(e) Find the ML estimate of q_1 given you know q_2.

(f) Find the variance of q_1 given you know q_2.

(g) Compare the variances and the Cramer-Rao bound for the case where q_2 is known and the case where q_2 is not known.

4. [20 points] Bayesian Estimation

Given an m-dimensional Gaussian random vector q and an n-dimensional vector **y** such that

$$\mathbf{y} = \mathbf{H}q + \mathbf{n}$$

where **n** is **N**(0,**I**) and q is **N**(a,**I**), **n** and q are independent and **H** is a n ***x*** m matrix, and $n \geq m$. Find

(a) The joint density of q and **y**.

(b) Find the Bayes estimator for minimizing MSE of q.

(c) Is the above Bayes estimator unbiased?

(d) Find the Bayes Risk (probability of error) of the estimator.

(e) Find the MVUB estimator of q.

(f) Find the probability of error of the MVUB estimator.

5. [20 points] Misc.

Let $\underline{b}_{k\ell}\, k = 0,\ldots,M-1,\quad \ell = 0,\ldots,N-1$ be the basis images associated with the M×N unitary transform: $\underline{F} = \underline{P}\,\underline{f}\,\underline{Q}$, and assume that $\underline{f}(m,n)$ is a zero mean random field with autocorrelation function:

$$R_{\underline{ff}}(m,n;r,s) = E\{\ \underline{f}(m,n)\underline{f}(r,s)\ \}$$

Show that if the basis images satisfy the equation

$$\sum_{r=0}^{M-1}\ \sum_{s=0}^{N-1} R_{\underline{ff}}(m,n;r,s)\left[\underline{b}_{k\ell}\right]_{rs} = g_{k\ell}\left[\underline{b}_{k\ell}\right]_{mn}$$

For a set of constants $g_{k\ell}$, then the transform coefficients are uncorrelated, i.e.

$$\mathrm{E}\left\{ \underline{F}(k,\ell)\underline{\mathrm{F}}(\mathrm{r,s}) \right\} = s^{2}_{\underline{F}(k,\ell)} d(k-r,\ell-s).$$

6. [20 points] Markov Chains

Consider a discrete-time Markov chain {Xn : n>=0} with values in the positive integers. Assume that the transition probabilities are all positive, i.e., Pi,j>0 for all i and j. Let the Markov chain start off with initial probability vector a, i.e., $a_j \equiv P(X_0 = j)$ for j>=1.

Consider the following two statements:

A. There exist a probability vector $p \equiv \{p_j : j \geq 1\}$ such that

$$p_i P_{i,j} = p_j P_{j,i} \quad \text{for all i and j.}$$

B. (i) there exist a probability vector p such that

$$p_j = \sum_{i=1}^{\infty} p_i P_{i,j} \quad \text{for all j}$$

And (ii) for all I, j and k,

$$P_{i,j}.P_{j,k}.P_{k,i} = P_{i,k}.P_{k,j}.P_{j,i}$$

Prove or disprove:

(i) A implies B.

(ii) B implies A.

Appendix of the Take Home Midterm Exam

- **The Score Function**

Given $f_X(x|q)$ of a random vector **X** and its log-likelihood function **L**(q|**X**), the score function **U** is defined to be the gradient of **L**:

$$\mathbf{U}(q) = d\mathbf{L}/dq$$

- **Fisher Information Matrix**

In general, the Fisher information measures how much "information" is known about a parameter q.

Given $f_X(x|q)$ of a random vector **X**, the Fisher information matrix, **I**, is the variance of the score function **U**. Therefore,

$$\mathbf{I} = \mathrm{var}\,(\mathbf{U})$$

If there is only one parameter involved, then **I** is simply called the Fisher information or information of $f_X(x|q)$.

Remarks

- If $f_X(x|q)$ belongs to a exponential family, $\mathbf{I} = E(\mathbf{U}^T\mathbf{U})$. Furthermore, with some regularity conditions imposed, $\mathbf{I} = -E(d\mathbf{U}/dq)$.
- As an example, the normal distribution, $N(\mu, \sigma^2)$, belongs to the exponential family and its log-likelihood function is **L**(q|**X**),

$$-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2},$$

Where q = (μ, σ^2). Then the score function **U**(q) is given by

$$(\frac{x-\mu}{\sigma^2}, \frac{(x-\mu)^2}{2\sigma^4} - \frac{1}{2\sigma^2}).$$

Taking the derivative with respect to q, we have

$$\partial U/\partial \boldsymbol{\theta} = \begin{pmatrix} -1/\sigma^2 & -(x-\mu)/\sigma^4 \\ -(x-\mu)(\sigma^4) & 1/(2\sigma^4) - (x-\mu)^2/(4\sigma^6) \end{pmatrix}.$$

Therefore, the Fisher information matrix **I** is

$$-\mathrm{E}(\partial U/\partial \boldsymbol{\theta}) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{pmatrix}.$$

- **Cramer-Rao Lower Bound**

If **T** is an unbiased estimator of q, it can be shown that

$$\mathrm{Var}[T(X)] \geq \frac{1}{I(\theta)}$$

This is known as the Cramer-Rao inequality, and the number 1/**I**(q) is known as the Cramer-Rao lower bound. The samller the variance of the estimate of q, the more information we have on q. If there is more than one parameter, the above can be generalized by saying that

$$\mathrm{Var}[T(X)] - I(\boldsymbol{\theta})^{-1}$$

is positive semi-definite, where **I** is the Fisher information matrix.