# Universal Constant of American Option Pricing

Dr. Hassan Nojumi

# Continuous Compounding

$A(t)$ = **investment after time** $t$ **with interest added** *continuously*.

$$A(t) = e^{rt} A_0$$

**Corollary.**

**The asset with value $K$ at time $T$ has value equal to $e^{-r(T-t)}K$ at time $t$.**

**Corollary.**

**The asset with value $K$ at time $T$ has value equal to $e^{-r(T-t)}K$ at time $t$.**

**Quantity $e^{-r(T-t)}$ is the** ***discount factor*****.**

**In the real world, all quantities vary with time in random, unexpected ways.**

**In the real world, all quantities vary with time in random, unexpected ways.**

**Hence an accurate modeling of financial markets requires methods of stochastic analysis.**

# Call Option

**A contract between two parties by which one party, called** *option holder* **or** *party in the long position*, **has the** *right*, **but not the obligation, to buy a specified asset, called** *the underlying asset*, **at a specified amount** $K$, **called** *exercise value*, **at a specified time** $T$, **called** *expiration date*, **from the other party.**

# Call Option

**If the holder decides to** *exercise* **his right, then the other party, called** *option writer* **or** *party in short position* **has the** *obligation* **to sell the asset.**

# Call Option

**If the holder decides to** *exercise* **his right, then the other party, called** *option writer* **or** *party in short position* **has the** *obligation* **to sell the asset.**

**With this contract, the holder** *insures* **himself against the large potential loss, but still has a large potential gain.**

# Put Option

**A contract between two parties by which one party, called** *option holder* **or** *party in the long position*, **has the** *right*, **but not the obligation, to sell a specified asset, called** *the underlying asset*, **at a specified amount** $K$**, called** *exercise value*, **at a specified time** $T$, **called** *expiration date*, **from the other party.**

# Put Option

**If the holder decides to** *exercise* **his right, then the other party, called** *option writer* **or** *party in short position* **has the** *obligation* **to buy the asset.**

# Put Option

**If the holder decides to** *exercise* **his right, then the other party, called** *option writer* **or** *party in short position* **has the** *obligation* **to buy the asset.**

**With this contract, the holder** *insures* **himself against the large potential loss, but still has a large potential gain.**

# Various types of Options include :

## Various types of Options include :

- European Option

## Various types of Options include :

- European Option
- American Option

## Various types of Options include :

- European Option
- American Option
- Bermudan Option

## Various types of Options include :

- European Option
- American Option
- Bermudan Option
- Asian Option

## Various types of Options include :

- European Option
- American Option
- Bermudan Option
- Asian Option
- Russian Option

## Various types of Options include :

- European Option
- American Option
- Bermudan Option
- Asian Option
- Russian Option
- Barrier Options

## Various types of Options include :

- European Option
- American Option
- Bermudan Option
- Asian Option
- Russian Option
- Barrier Options
- Exotic Options

## Various types of Options include :

- European Option
- **American Option**
- Bermudan Option
- Asian Option
- Russian Option
- Barrier Options
- Exotic Options

**Why would anyone want to be the writer of an option?**

**Why would anyone want to be the writer of an option?**

**The** *right* **possessed by the holder has some value. The holder should pay this value, called** *option premium*, **to enter a option contract.**

**Why would anyone want to be the writer of an option?**

**Call option is not exercised if $S_t < K$.**

**Why would anyone want to be the writer of an option?**

**Call option is not exercised if $S_t < K$.**

**Put option is not exercised if $S_t > K$.**

**Why would anyone want to be the writer of an option?**

**Call option is not exercised if $S_t < K$.**

**Put option is not exercised if $S_t > K$.**

**In case the option is not exercised, the writer will have gained the premium.**

# Dynamics of asset price

$$dS_t = a(S_t, t)\ dt$$

$a(S_t, t)\ dt$ **drift term**

# Dynamics of asset price

$$dS_t = a(S_t, t)\ dt + b(S_t, t)\ dW_t$$

$a(S_t, t)\ dt$ **drift term**

$b(S_t, t)\ dW_t$ **diffusion term**

# Standard Wiener Process

**A stochastic process $\{W_t : t \geq 0\}$ with the properties:**

- $W_0 = 0$
- $t \mapsto W_t$ **is a continuous function of** $t$
- **independent increments property**
- **stationary increments property**
- **For every** $t \geq 0$**:**

$$W_t \sim \text{Normal}(0, t), \quad \text{Var } W_t = t$$

# Standard Wiener Process

**Very important property:** $dW_t^2 = dt$

**an abbreviation for** $\int_0^T dW_t^2 = T = \int_0^T dt$
**which means**

$$\lim_{n\to\infty} E\left( \left[ \sum_0^{n-1} \left(W_{t_{i+1}} - W_{t_i}\right)^2 - T \right]^2 \right) = 0$$

**The problem of valuation of call option is determining the option value process $C_t$, and thereon the option premium $C_0$.**

$$C_t = C(S_t, t)$$
$$= \text{Price of a Call Option at time } t.$$

# Black-Scholes Formulation

$$\mathcal{L}_{BS} = \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} + r S_t \frac{\partial C}{\partial S_t} - rC = 0$$

# Black-Scholes Formulation

$$\begin{cases} \mathcal{L}_{BS} = \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 C}{\partial S_t^2} + r S_t \dfrac{\partial C}{\partial S_t} - rC = 0 \\ \ldots \; final \;\; condition \; \ldots \\ \ldots \; boundary \;\; condition \; \ldots \\ \ldots \; boundary \;\; condition \; \ldots \end{cases}$$

# Black-Scholes Formulation

$$\begin{cases} \mathcal{L}_{BS} = \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 C}{\partial S_t^2} + r S_t \dfrac{\partial C}{\partial S_t} - rC = 0 \\ C(S_T, T) = \max(S_T - K, 0) \\ \ldots\ boundary\ condition\ \ldots \\ \ldots\ boundary\ condition\ \ldots \end{cases}$$

# Black-Scholes Formulation

$$\begin{cases} \mathcal{L}_{BS} = \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 C}{\partial S_t^2} + r S_t \dfrac{\partial C}{\partial S_t} - rC = 0 \\ C(S_T, T) = \max(S_T - K, 0) \\ C(0, t) = 0 \\ \ldots \textit{boundary condition} \ldots \end{cases}$$

# Black-Scholes Formulation

$$\begin{cases} \mathcal{L}_{BS} = \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 C}{\partial S_t^2} + r S_t \dfrac{\partial C}{\partial S_t} - rC = 0 \\ C(S_T, T) = \max(S_T - K, 0) \\ C(0, t) = 0 \\ C(S, t) \sim S \text{ as } S \to \infty \end{cases}$$

# BS Price of European Call

$$C(S_t, t) = \qquad S_t\, N(d_1) - K\, e^{-r(T-t)}\, N(d_2)$$

$$d_1 := \frac{1}{\sigma\ \sqrt{T-t}} \left[\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right]$$

$$d_2 := \frac{1}{\sigma\ \sqrt{T-t}} \left[\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]$$

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-s^2/2} ds$$

# BS Price of European Call

$$C(S_t, t) = e^{-q(T-t)} S_t \, N(d_1) - K \, e^{-r(T-t)} \, N(d_2)$$

$$d_1 := \frac{1}{\sigma \, \sqrt{T-t}} \left[ \log\left(\frac{S_t}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t) \right]$$

$$d_2 := \frac{1}{\sigma \, \sqrt{T-t}} \left[ \log\left(\frac{S_t}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t) \right]$$

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-s^2/2} ds$$

# Valuation of American Call

$C_t = C(S_t, t)$

$=$ **Price of American call at time** $t$

# Valuation of American Call

$C_t = C(S_t, t)$

$=$ **Price of American call at time** $t$

$\geq$ **payoff of European call with expiry** $t$

# Valuation of American Call

$C_t = C(S_t, t)$

$=$ **Price of American call at time** $t$

$\geq \max\ (S_t - K, 0)$

# Valuation of American Call

**And:**

**return from delta-hedged portfolio**

$\leq$ **return from risk free bank account**

# Valuation of American Call

**For European Call:** $d\Pi_t \;=\; r\Pi_t \; dt$

# Valuation of American Call

**For European Call:** $d\Pi_t \;=\; r\Pi_t\, dt$

**For American Call:** $d\Pi_t \;\leq\; r\Pi_t\, dt$

# Valuation of American Call

**For European Call:** $\mathcal{L}_{BS} C_t = 0$

# Valuation of American Call

**For European Call:** $\mathcal{L}_{BS}C_t \; = \; 0$

**For American Call:** $\mathcal{L}_{BS}C_t \; \leq \; 0$

# Valuation of American Call

$$C_t \geq \max\,(S_t - K, 0)$$

$$\mathcal{L}_{BS} C_t \leq 0$$

# Valuation of American Call

**For every** $0 \leq t \leq T$ **there exists a** *free boundary* $S_f(t)$ **such that:**

**in case** $S_t < S_f(t)$ **we have:**

$$C_t \;\geq\; \max\,(S_t - K, 0) \qquad \mathcal{L}_{BS} C_t \;=\; 0$$

# Valuation of American Call

**For every** $0 \leq t \leq T$ **there exists a** *free boundary* $S_f(t)$ **such that:**

**in case** $S_t < S_f(t)$ **we have:**

$$C_t \geq \max(S_t - K, 0) \qquad \mathcal{L}_{BS} C_t = 0$$

**In this case, the optimal policy at time** $t$ **is to keep the option.**

# Valuation of American Call

**For every** $0 \leq t \leq T$ **there exists a** *free boundary* $S_f(t)$ **such that:**

**in case** $S_t > S_f(t)$ **we have:**

$$C_t = \max(S_t - K, 0) \qquad \mathcal{L}_{BS} C_t \leq 0$$

# Valuation of American Call

**For every** $0 \leq t \leq T$ **there exists a** *free boundary* $S_f(t)$ **such that:**

**in case** $S_t > S_f(t)$ **we have:**

$$C_t = \max(S_t - K, 0) \qquad \mathcal{L}_{BS} C_t \leq 0$$

**In this case, the optimal policy at time** $t$ **is to exercise the option.**

# Valuation of American Call

**Function** $t \mapsto S_f(t)$ **is the** *free boundary* **function for this problem.**

# Valuation of American Call

**Function** $t \mapsto S_f(t)$ **is the** *free boundary* **function for this problem.**

**As a corollary the above arguments, it has the important property:**

$$\frac{\partial C_t}{\partial S_t}\left( S_f(t) \ , \ t \right) = 1$$

# Valuation of American Call

**Unknowns:**

. . .

. . .

# Valuation of American Call

**Unknowns:**

**value of the option** $C_t(S_t, t)$

. . .

# Valuation of American Call

**Unknowns:**

**value of the option** $C_t(S_t, t)$

**free boundary function** $S_f(t)$

# Valuation of American Call

**So for every** $0 \le t \le T$**:**

$$[\ C_t - \max(S_t - K, 0)\ ]\ [\ \mathcal{L}_{BS}\ C_t\ ] = 0$$

# Valuation of American Call

**So for every** $0 \leq t \leq T$**:**

$$[\ C_t - \max(S_t - K, 0)\ ]\ \ [\ \mathcal{L}_{BS}\ C_t\ ] = 0$$

**Linear Complementarity Formulation**

**Advantage: ?**

# Valuation of American Call

**So for every** $0 \leq t \leq T$**:**

$$[\ C_t - \max(S_t - K, 0)\ ]\ \ [\ \mathcal{L}_{BS}\ C_t\ ] = 0$$

**Linear Complementarity Formulation**

**Advantage: No explicit free boundary**

# Valuation of American Call

**Black-Scholes equation:**

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} + (r-q) S_t \frac{\partial C}{\partial S_t} - rC = 0$$

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}}\ \tau$

$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$

**Black-Scholes equation:**

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} + (r-q)S_t \frac{\partial C}{\partial S_t} - rC = 0$$

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}} \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (k' - 1)\frac{\partial c}{\partial x} - kc + f(x)$$

$$k := \frac{r}{\frac{1}{2}\sigma^2}, \; k' := \frac{r-q}{\frac{1}{2}\sigma^2}, \; f(x) := -(k - k')e^x + k$$

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}} \tau$

$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$

**Free boundary in the new coordinates is** $x_f(\tau)$ **related to** $S_f(t)$ **via:**

$$S_f\left(T - \frac{1}{\frac{1}{\sigma^2}} \tau\right) = K \, \exp\left(x_f(\tau)\right)$$

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}}\ \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$C_t(S_t, t) \ \geq \ \text{max}\ (S_t - K, 0)$$

. . . . . . . . . . . . . . . .

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}} \tau$

$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$

$$C_t(S_t, t) \;\geq\; \max\ (S_t - K, 0)$$

$$c(x,\tau) \;\geq\; \max\ (e^x - 1, 0)$$

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}}\ \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$C\left(S_f(t), t\right) = S_f(t) - K$$

. . . . . . . . .

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}}\ \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$C\left(S_f(t), t\right) = S_f(t) - K$$

$$c(\ x_f(\tau), \tau\ ) = 0 \quad \Longrightarrow \quad \ldots\ldots$$

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}} \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$C\left(S_f(t), t\right) = S_f(t) - K$$

$$c(\ x_f(\tau), \tau\ ) = 0 \quad \Longrightarrow \quad \frac{\partial c}{\partial \tau}(\ x_f(\tau), \tau\ ) = 0$$

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}}\ \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$\frac{\partial C}{\partial S_t}\Big(S_f(t), t\Big) = 1$$

. . . . . . . . .

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}}\,\tau$

$$S = Ke^{x}, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$\frac{\partial C}{\partial S_t}\Big(S_f(t), t\Big) = 1$$

$$\frac{\partial c}{\partial x}\Big(x_f(\tau), \tau\Big) = 0$$

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}} \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (k' - 1)\frac{\partial c}{\partial x} - kc + f(x)$$

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}} \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

**The only value of $x_f(0^+)$ satisfying $c(\ x_f(0^+), 0^+) = 0$ is the root of $f$.**

# Valuation of American Call

**Change of variables:** $t = T - \frac{1}{\frac{1}{\sigma^2}} \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$f(x) = -(k - k')e^x + k$$

# Valuation of American Call

**Change of variables:** $t = T - \dfrac{1}{\frac{1}{\sigma^2}}\ \tau$

$$S = Ke^x, \quad C(S,t) = S - K + Kc(x,\tau)$$

$$f(x) = -(k-k')e^x + k$$

$$x_0 \ = \ x_f(0^+) \ = \ \log \ \frac{k}{k-k'}$$

**For $x$ near $x_0$,**

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

**For $x$ near $x_0$,**

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = -(k - k')e^x + k$$

**For $x$ near $x_0$,**

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = -(k - k')e^x + k$$

$$f'(x) = -(k - k')e^x$$

## For $x$ near $x_0$,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = -(k - k')e^x + k$$

$$f'(x) = -(k - k')e^x$$

$$x_0 = \log \frac{k}{k - k'} \;\Rightarrow\; f'(x_0) = -k$$

**For $x$ near $x_0$,**

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = -(k - k')e^x + k$$

$$f'(x) = -(k - k')e^x$$

$$x_0 = \log \frac{k}{k - k'} \;\Rightarrow\; f'(x_0) = -k$$

$$f(x) \approx -k(x - x_0)$$

**For $x$ near $x_0$, changes in function $c(x,\tau)$ are rapid, so**

$$\frac{\partial^2 c}{\partial x^2} \gg \frac{\partial c}{\partial x} \qquad \frac{\partial^2 c}{\partial x^2} \gg c$$

**For $x$ near $x_0$, changes in function $c(x,\tau)$ are rapid, so**

$$\frac{\partial^2 c}{\partial x^2} \gg \frac{\partial c}{\partial x} \qquad\qquad \frac{\partial^2 c}{\partial x^2} \gg c$$

**Hence, for $x$ near $x_0$, function $\tilde{c}(x,\tau)$, the** *local* **solution for $x$ near $x_0$, satisfies the equation**

$$\frac{\partial \tilde{c}}{\partial \tau} = \frac{\partial^2 \tilde{c}}{\partial x^2} - k(x - x_0)$$

**Equation**

$$\frac{\partial \tilde{c}}{\partial \tau} = \frac{\partial^2 \tilde{c}}{\partial x^2} - k(x - x_0)$$

**has a** *similarity solution* $c^*$ **in terms of a variable of the form**

$$\xi = \frac{x - x_0}{\tau^\beta}$$

**related to** $c$ **via**

$$c(x, \tau) = \tau^\alpha \; c^*(\xi)$$

**Free boundary function $x_f(\tau)$ takes the form**

$$x_f(\tau) = x_0 + \xi_0 \ \tau^\beta$$

**with $\xi_0$ a constant to be determined.**

# Similarity Solution

$$c(x, \tau) = \tau^{\alpha} \; c^*(\xi)$$

# Similarity Solution

$$c(x,\tau) = \tau^{\alpha}\ c^{*}(\xi)$$

$$\frac{\partial \tilde{c}}{\partial \tau} = \alpha \tau^{\alpha-1} c^{*} + \tau^{\alpha}\ \frac{\partial \xi}{\partial \tau}\ \frac{dc^{*}}{d\xi}$$

# Similarity Solution

$$c(x,\tau) = \tau^{\alpha}\ c^{*}(\xi)$$

$$\frac{\partial \tilde{c}}{\partial \tau} = \alpha \tau^{\alpha-1} c^{*} + \tau^{\alpha}\ \frac{\partial \xi}{\partial \tau}\ \frac{dc^{*}}{d\xi}$$

$$\xi = \frac{x - x_0}{\tau^{\beta}} \ \Rightarrow\ \frac{\partial \xi}{\partial \tau} = -\frac{\beta \xi}{\tau}$$

# Similarity Solution

$$c(x,\tau) = \tau^{\alpha}\ c^{*}(\xi)$$

$$\frac{\partial \tilde{c}}{\partial \tau} = \alpha\tau^{\alpha-1}c^{*} + \tau^{\alpha}\ \frac{\partial \xi}{\partial \tau}\ \frac{dc^{*}}{d\xi}$$

$$\xi = \frac{x - x_0}{\tau^{\beta}} \quad \Rightarrow \quad \frac{\partial \xi}{\partial \tau} = -\frac{\beta\xi}{\tau}$$

$$\frac{\partial \tilde{c}}{\partial \tau} = \alpha\tau^{\alpha-1}c^{*} - \beta\xi\tau^{\alpha-1}\ \frac{dc^{*}}{d\xi}$$

# Similarity Solution

$$c(x,\tau) = \tau^{\alpha}\ c^{*}(\xi)$$

# Similarity Solution

$$c(x,\tau) = \tau^{\alpha} \; c^*(\xi)$$

$$\frac{\partial \tilde{c}}{\partial x} = \tau^{\alpha} \; \frac{\partial \xi}{\partial x} \; \frac{dc^*}{d\xi}$$

# Similarity Solution

$$c(x,\tau) = \tau^{\alpha}\ c^{*}(\xi)$$

$$\frac{\partial \tilde{c}}{\partial x} = \tau^{\alpha}\ \frac{\partial \xi}{\partial x}\ \frac{dc^{*}}{d\xi}$$

$$\xi = \frac{x - x_0}{\tau^{\beta}} \quad \Rightarrow \quad \frac{\partial \xi}{\partial x} = \frac{1}{\tau^{\beta}}$$

# Similarity Solution

$$c(x,\tau) = \tau^{\alpha}\ c^{*}(\xi)$$

$$\frac{\partial \tilde{c}}{\partial x} = \tau^{\alpha}\ \frac{\partial \xi}{\partial x}\ \frac{dc^{*}}{d\xi}$$

$$\xi = \frac{x - x_0}{\tau^{\beta}} \ \Rightarrow\ \frac{\partial \xi}{\partial x} = \frac{1}{\tau^{\beta}}$$

$$\frac{\partial \tilde{c}}{\partial x} = \tau^{\alpha-\beta}\ \frac{dc^{*}}{d\xi}$$

# Similarity Solution

$$c(x, \tau) = \tau^{\alpha} \; c^*(\xi)$$

# Similarity Solution

$$c(x,\tau) = \tau^{\alpha}\ c^{*}(\xi)$$

$$\frac{\partial \tilde{c}}{\partial x} = \tau^{\alpha-\beta}\ \frac{dc^{*}}{d\xi}$$

## Similarity Solution

$$c(x,\tau) = \tau^{\alpha}\ c^*(\xi)$$

$$\frac{\partial \tilde{c}}{\partial x} = \tau^{\alpha-\beta}\ \frac{dc^*}{d\xi}$$

$$\frac{\partial^2 \tilde{c}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \tilde{c}}{\partial x}\right) = \frac{\partial \xi}{\partial x}\ \frac{\partial}{\partial \xi}\left(\frac{\partial \tilde{c}}{\partial x}\right)$$

# Similarity Solution

$$c(x,\tau) = \tau^{\alpha}\ c^*(\xi)$$

$$\frac{\partial \tilde{c}}{\partial x} = \tau^{\alpha-\beta}\ \frac{dc^*}{d\xi}$$

$$\frac{\partial^2 \tilde{c}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \tilde{c}}{\partial x}\right) = \frac{\partial \xi}{\partial x}\ \frac{\partial}{\partial \xi}\left(\frac{\partial \tilde{c}}{\partial x}\right)$$

$$\frac{\partial^2 \tilde{c}}{\partial x^2} = \tau^{\alpha-2\beta}\ \frac{d^2 c^*}{d\xi^2}$$

**Equation for $c^*$:**

$$\tau^{\alpha-3\beta}\ \frac{d^2c^*}{d\xi^2} + \beta\xi\tau^{\alpha-1-\beta}\ \frac{dc^*}{d\xi} - \alpha\tau^{\alpha-1-\beta}c^* = k\xi$$

**Equation for $c^*$:**

$$\tau^{\alpha-3\beta}\frac{d^2c^*}{d\xi^2}+\beta\xi\tau^{\alpha-1-\beta}\frac{dc^*}{d\xi}-\alpha\tau^{\alpha-1-\beta}c^*=k\xi$$

**Values of $\alpha$ and $\beta$ for eliminating $\tau$:**

$$\begin{cases}\alpha-3\beta=0\\ \alpha-1-\beta=0\end{cases}\Rightarrow\begin{cases}\alpha=\dfrac{3}{2}\\ \beta=\dfrac{1}{2}\end{cases}$$

**Result.** **Changes of variable and function**

$$\xi = \frac{x - x_0}{\sqrt{\tau}} \qquad c(x,\tau) = \tau^{3/2}\ c^*(\xi)$$

**transforms the PDE problem for** $c(x,\tau)$

**into the following ODE problem for $c^*$:**

$$\begin{cases} \dfrac{d^2c^*}{d\xi^2} + \dfrac{1}{2}\,\xi\dfrac{dc^*}{d\xi} - \dfrac{3}{2}\,c^* = k\xi \\[2ex] c^*(\xi_0) = 0 \\[2ex] \dfrac{dc^*}{d\xi}(\xi_0) = 0 \\[2ex] c^*(\xi) \sim -k\xi \quad \text{as } \xi \to -\infty \end{cases}$$

**The unknowns are the function $c^*(\xi)$ and the constant $\xi_0$.**

**The unknowns are the function $c^*(\xi)$ and the constant $\xi_0$.**

**The free boundary function is of the form:**

$$x_f(\tau) = x_0 + \xi_0 \sqrt{\tau}$$

**Two linearly independent solutions of the corresponding homogeneous equation**

$$\frac{d^2c^*}{d\xi^2} + \frac{1}{2}\ \xi\frac{dc^*}{d\xi} - \frac{3}{2}\ c^* = 0$$

**are**

$$c^*_{1h}(\xi) = \xi^3 + 6\xi$$

**Two linearly independent solutions of the corresponding homogeneous equation**

$$\frac{d^2c^*}{d\xi^2} + \frac{1}{2}\ \xi\frac{dc^*}{d\xi} - \frac{3}{2}\ c^* = 0$$

**are**

$$c^*_{2h}(\xi) = \frac{1}{2}\left(\xi^3 + 6\xi\right)\int_{-\infty}^{\xi} e^{-s^2/4}ds + \left(\xi^2 + 4\right)e^{-\xi^2/4}$$

**A particular solution of the equation**

$$\frac{d^2c^*}{d\xi^2} + \frac{1}{2}\ \xi\frac{dc^*}{d\xi} - \frac{3}{2}\ c^* = k\xi$$

**is**

$$c_p^*(\xi) = -k\xi$$

**A particular solution of the equation**

$$\frac{d^2c^*}{d\xi^2} + \frac{1}{2}\ \xi \frac{dc^*}{d\xi} - \frac{3}{2}\ c^* = k\xi$$

**is**

$$c_p^*(\xi) = -k\xi$$

**General solution:**

$$c^*(\xi) = Ac_{1h}^*(\xi) + Bc_{2h}^*(\xi) - k\xi$$

**General solution:**

$$c^*(\xi) = Ac^*_{1h}(\xi) + Bc^*_{2h}(\xi) - k\xi$$

**General solution:**

$$c^*(\xi) = Ac^*_{1h}(\xi) + Bc^*_{2h}(\xi) - k\xi$$

$c^*(\xi) \sim -k\xi$ as $\xi \to -\infty$

**General solution:**

$$c^*(\xi) = Ac^*_{1h}(\xi) + Bc^*_{2h}(\xi) - k\xi$$

$c^*(\xi) \sim -k\xi$ as $\xi \to -\infty$

$c^*_{h1}(\xi) \to -\infty$ as $\xi \to -\infty$

**General solution:**

$$c^*(\xi) = Ac^*_{1h}(\xi) + Bc^*_{2h}(\xi) - k\xi$$

$c^*(\xi) \sim -k\xi$ as $\xi \to -\infty$

$c^*_{h1}(\xi) \to -\infty$ as $\xi \to -\infty$

$c^*_{h2}(\xi) \to 0$ as $\xi \to -\infty$

**General solution:**

$$c^*(\xi) = Ac^*_{1h}(\xi) + Bc^*_{2h}(\xi) - k\xi$$

$$\left.\begin{array}{l} c^*(\xi) \sim -k\xi \quad \text{as } \xi \to -\infty \\ \\ c^*_{h1}(\xi) \to -\infty \quad \text{as } \xi \to -\infty \\ \\ c^*_{h2}(\xi) \to 0 \quad \text{as } \xi \to -\infty \end{array}\right\} \Rightarrow \quad A = 0$$

**Solution:** $c^*(\xi) = Bc^*_{2h}(\xi) - k\xi$

**Solution:** $c^*(\xi) = Bc^*_{2h}(\xi) - k\xi$

$$c^*(\xi_0) = 0 \;\Rightarrow\; Bc^*_{2h}(\xi_0) = k\xi_0$$

**Solution:** $c^*(\xi) = Bc^*_{2h}(\xi) - k\xi$

$$c^*(\xi_0) = 0 \;\Rightarrow\; Bc^*_{2h}(\xi_0) = k\xi_0$$

$$\frac{\partial c^*}{\partial \xi}(\xi_0) = 0 \;\Rightarrow\; \frac{dBc^*_{2h}}{\partial \xi}(\xi_0) = k$$

**Solution:** $c^*(\xi) = Bc^*_{2h}(\xi) - k\xi$

$$c^*(\xi_0) = 0 \;\Rightarrow\; Bc^*_{2h}(\xi_0) = k\xi_0$$

$$\frac{\partial c^*}{\partial \xi}(\xi_0) = 0 \;\Rightarrow\; \frac{dBc^*_{2h}}{\partial \xi}(\xi_0) = k$$

$$c^*_{2h}(\xi_0) = \xi_0 \, \frac{dBc^*_{2h}}{\partial \xi}(\xi_0)$$

$$
\begin{aligned}
c_{2h}^{*}(\xi) \; = \; & \frac{1}{2}\left(\xi^{3}+6\xi\right)\int_{-\infty}^{\xi} e^{-s^{2}/4} ds \\
& + \left(\xi^{2}+4\right) e^{-\xi^{2}/4}
\end{aligned}
$$

$$
\begin{aligned}
c^*_{2h}(\xi) \;=\; & \frac{1}{2}\left(\xi^3 + 6\xi\right)\int_{-\infty}^{\xi} e^{-s^2/4} ds \\
& + \left(\xi^2 + 4\right) e^{-\xi^2/4}
\end{aligned}
$$

$$
c^*_{2h}(\xi_0) = \xi_0 \; \frac{dBc^*_{2h}}{\partial \xi}(\xi_0)
$$

$$c_{2h}^{*}(\xi) = \frac{1}{2}\left(\xi^{3}+6\xi\right)\int_{-\infty}^{\xi} e^{-s^{2}/4}ds + \left(\xi^{2}+4\right)e^{-\xi^{2}/4}$$

$$c_{2h}^{*}(\xi_0) = \xi_0 \, \frac{dBc_{2h}^{*}}{\partial\xi}(\xi_0)$$

$$\xi_0^{3} \, e^{\xi_0^{2}/4}\int_{-\infty}^{\xi_0} e^{-s^{2}/4}ds = 2\left(2-\xi_0^{2}\right)$$

**It can be shown that the transcendental equation**

$$\xi_0^3 \; e^{\xi_0^2/4} \int_{-\infty}^{\xi_0} e^{-s^2/4} ds = 2\left(2 - \xi_0^2\right)$$

**has a unique solution $\xi_0$; the Universal Constant of American Option Pricing.**

**It can be shown that the transcendental equation**

$$\xi_0^3 \; e^{\xi_0^2/4} \int_{-\infty}^{\xi_0} e^{-s^2/4} ds = 2\left(2 - \xi_0^2\right)$$

**has a unique solution** $\xi_0$**; the Universal Constant of American Option Pricing.**

**Numerical solution gives** $\xi_0 \approx 0.9034$ **.**

**The free boundary function:**

$$x_f(\tau) \; = \; x_0 \; + \; \xi_0\sqrt{\tau}$$

**The free boundary function:**

$$x_f(\tau) = x_0 + \xi_0\sqrt{\tau}$$

$$x_f(\tau) \approx \log\left(\frac{r}{q}\right) + 0.9034\,\sqrt{\tau}$$

## The free boundary function:

$$x_f(\tau) = x_0 + \xi_0\sqrt{\tau}$$

$$x_f(\tau) \approx \log\left(\frac{r}{q}\right) + 0.9034\sqrt{\tau}$$

$$S_f(t) = K \exp\left( x_f\left( \frac{T-t}{\sigma^2} \right) \right)$$

**The free boundary function:**

$$
\begin{aligned}
x_f(\tau) &= x_0 + \xi_0\sqrt{\tau} \\
x_f(\tau) &\approx \log\left(\frac{r}{q}\right) + 0.9034\,\sqrt{\tau} \\
S_f(t) &= K\,\exp\left( x_f\left( \frac{T-t}{\sigma^2} \right) \right) \\
S_f(t) &= \frac{r}{q}\,K\,\exp\left( 0.9034\,\frac{\sqrt{T-t}}{\sigma} \right)
\end{aligned}
$$

# Various types of Options include :

- European Option
- American Option
- Bermudan Option
- Asian Option
- Russian Option
- Barrier Options
- Exotic Options

## Various types of Options include :

- European Option
- American Option
- **Bermudan Option**
- Asian Option
- Russian Option
- Barrier Options
- Exotic Options

# What is the relation between $\xi_0$ and value of American put option?

. . .

. . .

**What is the relation between $\xi_0$ and value of American put option?**

**What are differences in pricing American calls and American puts?**

**. . .**

**What is the relation between $\xi_0$ and value of American put option?**

**What are differences in pricing American calls and American puts?**

**What is the value of Bermudan option, discretized version of American option?**