2 ANALYSIS OF LINEAR CONTROL SYSTEMS

2.1 INTRODUCTION

In this introduction we give a brief description of control problems and of the contents of this chapter.

A control system is a dynamic system which as time evolves behaves in a certain prescribed way, generally without human interference. Control theory deals with the analysis and synthesis of control systems.

The essential components of a control system (Fig. 2.1) are: (1) the *plant*, which is the system to be controlled; (2) one or more *sensors*, which give information about the plant; and (3) the *controller*, the "heart" of the control system, which compares the measured values to their desired values and adjusts the input variables to the plant.

An example of a control system is a self-regulating home heating system, which maintains at all times a fairly constant temperature inside the home even though the outside temperature may vary considerably. The system operates without human intervention, except that the desired temperature must be set. In this control system the *plant* is the home and the heating equipment. The *sensor* generally consists of a temperature transducer inside the home, sometimes complemented by an outside temperature transducer. The *controller* is usually combined with the inside temperature sensor in the thermostat, which switches the heating equipment on and off as necessary.

Another example of a control system is a tracking antenna, which without human aid points at all times at a moving object, for example, a satellite. Here the plant is the antenna and the motor that drives it. The sensor consists of a potentiometer or other transducer which measures the antenna displacement, possibly augmented with a tachometer for measuring the angular velocity of the antenna shaft. The controller consists of electronic equipment which supplies the appropriate input voltage to the driving motor.

Although at first glance these two control problems seem different, upon further study they have much in common. First, in both cases the plant and the controller are described by differential equations. Consequently, the mathematical tool needed to analyze the behavior of the control system in

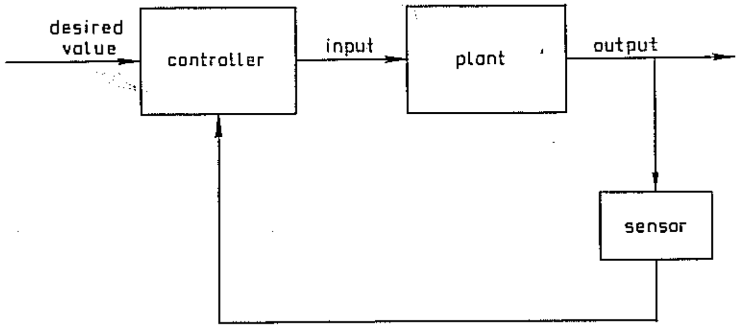

Fig. 2.1. Schematic representation of a control system.

both cases consists of the collection of methods usually referred to as system theory. Second, both control systems exhibit the feature of *feedback*, that is, the actual operation of the control system is compared to the desired operation and the input to the plant is adjusted on the basis of this comparison.

Feedback has several attractive properties. Since the actual operation is continuously compared to the desired operation, feedback control systems are able to operate satisfactorily despite adverse conditions, such as *disturbances* that act upon the system, or *variations in plant properties*. In a home heating system, disturbances are caused by fluctuations in the outside temperature and wind speed, and variations in plant properties may occur because the heating equipment in parts of the home may be connected or disconnected. In a tracking antenna disturbances in the form of wind gusts act upon the system, and plant variations occur because of different friction coefficients at different temperatures.

In this chapter we introduce control problems, describe possible solutions to these problems, analyze those solutions, and present basic design objectives. In the chapters that follow, we formulate control problems as mathematical optimization problems and use the results to synthesize control systems.

The basic design objectives discussed are stated mainly for time-invariant linear control systems. Usually, they are developed in terms of frequency domain characteristics, since in this domain the most acute insight can be gained. We also extensively discuss the time domain description of control systems via state equations, however, since numerical computations are often more conveniently performed in the time domain.

This chapter is organized as follows. In Section 2.2 a general description is given of tracking problems, regulator problems, and terminal control problems. In Section 2.3 closed-loop controllers are introduced. In the remaining sections various properties of control systems are discussed, such as stability, steady-state tracking properties, transient tracking properties, effects of disturbances and observation noise, and the influence of plant variations. Both single-input single-output and multivariable control systems are considered.

2.2 THE FORMULATION OF CONTROL PROBLEMS

2.2.1 Introduction

In this section the following two types of control problems are introduced: (1) *tracking problems* and, as special cases, *regulator problems;* and (2) *terminal control problems.*

In later sections we give detailed descriptions of possible control schemes and discuss at length how to analyze these schemes. In particular, the following topics are emphasized: root mean square (rms) tracking error, rms input, stability, transmission, transient behavior, disturbance suppression, observation noise suppression, and plant parameter variation compensation.

2.2.2 The Formulation of Tracking and Regulator Problems

We now describe in general terms an important class of control problems—*tracking problems.* Given is a system, usually called the *plant*, which cannot be altered by the designer, with the following variables associated with it (see Fig. 2.2).

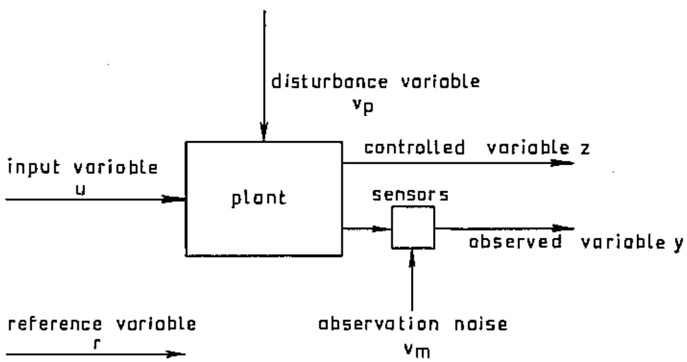

Fig. 2.2. The plant.

1. An *input variable* $u(t)$ which influences the plant and which can be manipulated;

2. A *disturbance variable* $v_p(t)$ which influences the plant but which cannot be manipulated;

3. An *observed variable* $y(t)$ which is measured by means of *sensors* and which is used to obtain information about the state of the plant; this observed variable is usually contaminated with *observation noise* $v_m(t)$;

4. A *controlled variable* $z(t)$ which is the variable we wish to control;

5. A *reference variable* $r(t)$ which represents the prescribed value of the controlled variable $z(t)$.

The *tracking problem* roughly is the following. For a given reference variable, find an appropriate input so that the controlled variable tracks the reference variable, that is,

$$z(t) \simeq r(t), \qquad t \geq t_0, \qquad \textbf{2-1}$$

where t_0 is the time at which control starts. Typically, the reference variable is not known in advance. A practical constraint is that the range of values over which the input $u(t)$ is allowed to vary is limited. Increasing this range usually involves replacement of the plant by a larger and thus more expensive one. As will be seen, this constraint is of major importance and prevents us from obtaining systems that track perfectly.

In designing tracking systems so as to satisfy the basic requirement **2-1**, the following aspects must be taken into account.

1. The disturbance influences the plant in an unpredictable way.
2. The plant parameters may not be known precisely and may vary.
3. The initial state of the plant may not be known.
4. The observed variable may not directly give information about the state of the plant and moreover may be contaminated with observation noise.

The input to the plant is to be generated by a piece of equipment that will be called the *controller*. We distinguish between two types of controllers: *open-loop* and *closed-loop* controllers. Open-loop controllers generate $u(t)$ on the basis of past and present values of the reference variable only (see Fig. 2.3), that is,

$$u(t) = f_{OL}[r(\tau), t_0 \leq \tau \leq t], \qquad t \geq t_0. \qquad \textbf{2-2}$$

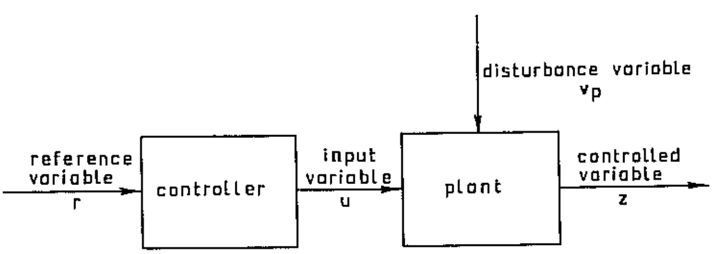

Fig. 2.3. An open-loop control system.

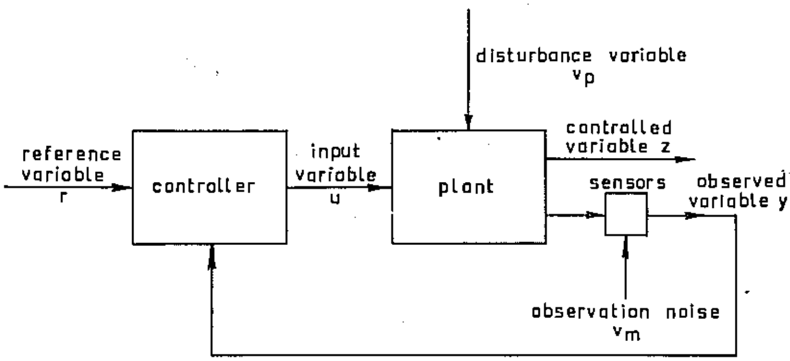

Fig. 2.4. A closed-loop control system.

Closed-loop controllers take advantage of the information about the plant that comes with the observed variable; this operation can be represented by (see Fig. 2.4)

$$u(t) = f_{CL}[r(\tau),\ t_0 \leq \tau \leq t;\ y(\tau),\ t_0 \leq \tau \leq t], \qquad t \geq t_0. \qquad \textbf{2-3}$$

Note that neither in **2-2** nor in **2-3** are future values of the reference variable or the observed variable used in generating the input variable since they are unknown. The plant and the controller will be referred to as the *control system*.

Already at this stage we note that closed-loop controllers are much more powerful than open-loop controllers. Closed-loop controllers can accumulate information about the plant during operation and thus are able to collect information about the initial state of the plant, reduce the effects of the disturbance, and compensate for plant parameter uncertainty and variations. Open-loop controllers obviously have no access to any information about the plant except for what is available before control starts. The fact that open-loop controllers are not afflicted by observation noise since they do not use the observed variable does not make up for this.

An important class of tracking problems consists of those problems where the reference variable is constant over long periods of time. In such cases it is customary to refer to the reference variable as the *set point* of the system and to speak of *regulator problems*. Here the main problem usually is to maintain the controlled variable at the set point in spite of disturbances that act upon the system. In this chapter tracking and regulator problems are dealt with simultaneously.

This section is concluded with two examples.

Example 2.1. *A position servo system*

In this example we describe a control problem that is analyzed extensively later. Imagine an object moving in a plane. At the origin of the plane is a rotating antenna which is supposed to point in the direction of the object at all times. The antenna is driven by an electric motor. The control problem is to command the motor such that

$$\theta(t) \simeq \theta_r(t), \qquad t \geq t_0, \tag{2-4}$$

where $\theta(t)$ denotes the angular position of the antenna and $\theta_r(t)$ the angular position of the object. We assume that $\theta_r(t)$ is made available as a mechanical angle by manually pointing binoculars in the direction of the object.

The *plant* consists of the antenna and the motor. The *disturbance* is the torque exerted by wind on the antenna. The *observed variable* is the output of a potentiometer or other transducer mounted on the shaft of the antenna, given by

$$\eta(t) = \theta(t) + \nu(t), \tag{2-5}$$

where $\nu(t)$ is the measurement noise. In this example the angle $\theta(t)$ is to be controlled and therefore is the *controlled variable*. The *reference variable* is the direction of the object $\theta_r(t)$. The *input* to the plant is the input voltage to the motor μ.

A possible method of forcing the antenna to point toward the object is as follows. Both the angle of the antenna $\theta(t)$ and the angle of the object $\theta_r(t)$ are converted to electrical variables using potentiometers or other transducers mounted on the shafts of the antenna and the binoculars. Then $\theta(t)$ is subtracted from $\theta_r(t)$; the difference is amplified and serves as the input voltage to the motor. As a result, when $\theta_r(t) - \theta(t)$ is positive, a positive input voltage is produced that makes the antenna rotate in a positive direction so that the difference between $\theta_r(t)$ and $\theta(t)$ is reduced. Figure 2.5 gives a representation of this control scheme.

This scheme obviously represents a closed-loop controller. An open-loop controller would generate the driving voltage $\mu(t)$ on the basis of the reference angle $\theta_r(t)$ alone. Intuitively, we immediately see that such a controller has no way to compensate for external disturbances such as wind torques, or plant parameter variations such as different friction coefficients at different temperatures. As we shall see, the closed-loop controller does offer protection against such phenomena.

This problem is a typical *tracking* problem.

Example 2.2. *A stirred tank regulator system*

The preceding example is relatively simple since the plant has only a single input and a single controlled variable. *Multivariable* control problems, where the plant has several inputs and several controlled variables, are usually

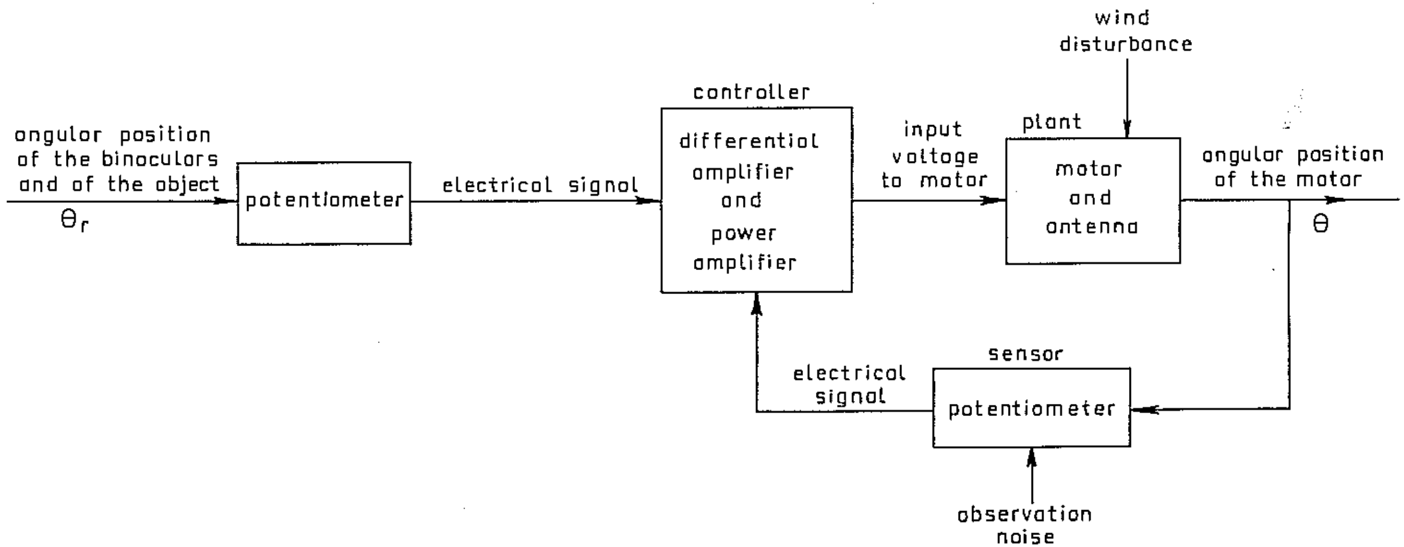

Fig. 2.5. A position servo system.

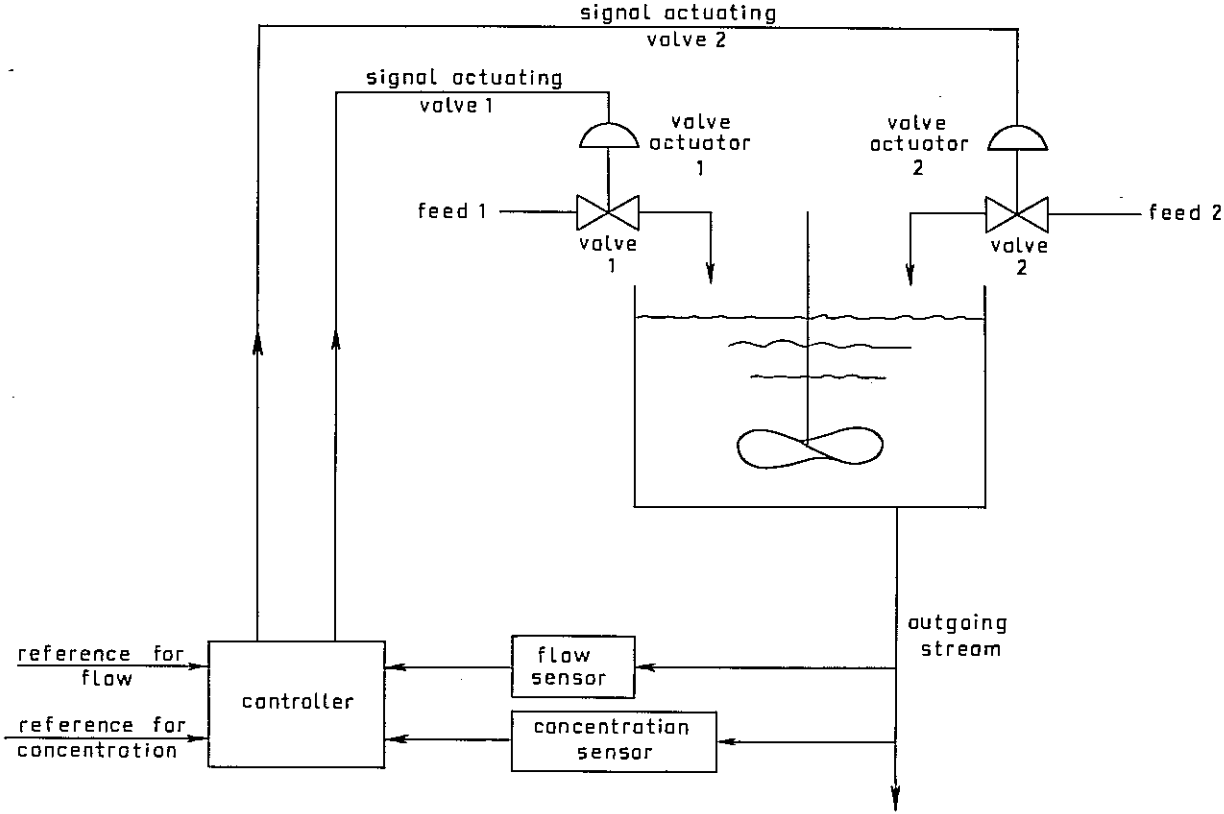

Fig. 2.6. The stirred-tank control system.

much more difficult to deal with. As an example of a multivariable problem, we consider the stirred tank of Example 1.2 (Section 1.2.3). The tank has two feeds; their flows can be adjusted by valves. The concentration of the material dissolved in each of the feeds is fixed and cannot be manipulated. The tank has one outlet and the control problem is to design equipment that automatically adjusts the feed valves so as to maintain both the outgoing flow and the concentration of the outgoing stream constant at given reference values (see Fig. 2.6).

This is a typical *regulator* problem. The components of the input variable are the flows of the incoming feeds. The components of the controlled variable are the outgoing flow and the concentration of the outgoing stream. The set point also has two components: the desired outgoing flow and the desired outgoing concentration. The following disturbances may occur: fluctuations in the incoming concentrations, fluctuations in the incoming flows resulting from pressure fluctuations before the valves, loss of fluid because of leaks and evaporation, and so on. In order to control the system well, both the outgoing flow and concentration should be measured; these then are the components of the observed variable. A closed-loop controller uses these measurements as well as the set points to produce a pneumatic or electric signal which adjusts the valves.

2.2.3 The Formulation of Terminal Control Problems

The framework of terminal control problems is similar to that of tracking and regulator problems, but a somewhat different goal is set. Given is a plant with input variable u, disturbance variable v_p, observed variable y, and controlled variable z, as in the preceding section. Then a typical terminal control problem is roughly the following. Find $u(t)$, $t_0 \leq t \leq t_1$, so that $z(t_1) \simeq r$, where r is a given vector and where the terminal time t_1 may or may not be specified. A practical restriction is that the range of possible input amplitudes is limited. The input is to be produced by a controller, which again can be of the closed-loop or the open-loop type.

In this book we do not elaborate on these problems, and we confine ourselves to giving the following example.

Example 2.3. *Position control as a terminal control problem*

Consider the antenna positioning problem of Example 2.1. Suppose that at a certain time t_0 the antenna is at rest at an angle θ_0. Then the problem of repositioning the antenna at an angle θ_1, where it is to be at rest, in as short a time as possible without overloading the motor is an example of a terminal control problem.

2.3 CLOSED-LOOP CONTROLLERS; THE BASIC DESIGN OBJECTIVE

In this section we present detailed descriptions of the plant and of closed-loop controllers. These descriptions constitute the framework for the discussion of the remainder of this chapter. Furthermore, we define the mean square tracking error and the mean square input and show how these quantities can be computed.

Throughout this chapter and, indeed, throughout most of this book, it is assumed that the plant can be described as a linear differential system with some of its inputs stochastic processes. The state differential equation of the system is

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + v_p(t), \qquad x(t_0) = x_0. \tag{2-6}$$

Here $x(t)$ is the *state* of the plant and $u(t)$ the *input variable*. The *initial state* x_0 is a stochastic variable, and the *disturbance variable* $v_p(t)$ is assumed to be a stochastic process. The *observed variable* $y(t)$ is given by

$$y(t) = C(t)x(t) + v_m(t), \tag{2-7}$$

where the *observation noise* $v_m(t)$ is also assumed to be a stochastic process. The *controlled variable* is

$$z(t) = D(t)x(t). \tag{2-8}$$

Finally, the *reference variable* $r(t)$ is assumed to be a stochastic process of the same dimension as the controlled variable $z(t)$.

The general *closed-loop controller* will also be taken to be a linear differential system, with the reference variable $r(t)$ and the observed variable $y(t)$ as inputs, and the plant input $u(t)$ as output. The state differential equation of the closed-loop controller will have the form

$$\dot{q}(t) = L(t)q(t) + K_r(t)r(t) - K_f(t)y(t), \qquad q(t_0) = q_0, \tag{2-9}$$

while the output equation of the controller is of the form

$$u(t) = F(t)q(t) + H_r(t)r(t) - H_f(t)y(t). \tag{2-10}$$

Here the index r refers to the reference variable and the index f to feedback. The quantity $q(t)$ is the state of the controller. The initial state q_0 is either a given vector or a stochastic variable. Figure 2.7 clarifies the interconnection of plant and controller, which is referred to as the *control system*. If $K_f(t) \equiv 0$ and $H_f(t) \equiv 0$, the closed-loop controller reduces to an *open-loop controller* (see Fig. 2.8). We refer to a control system with a closed-loop

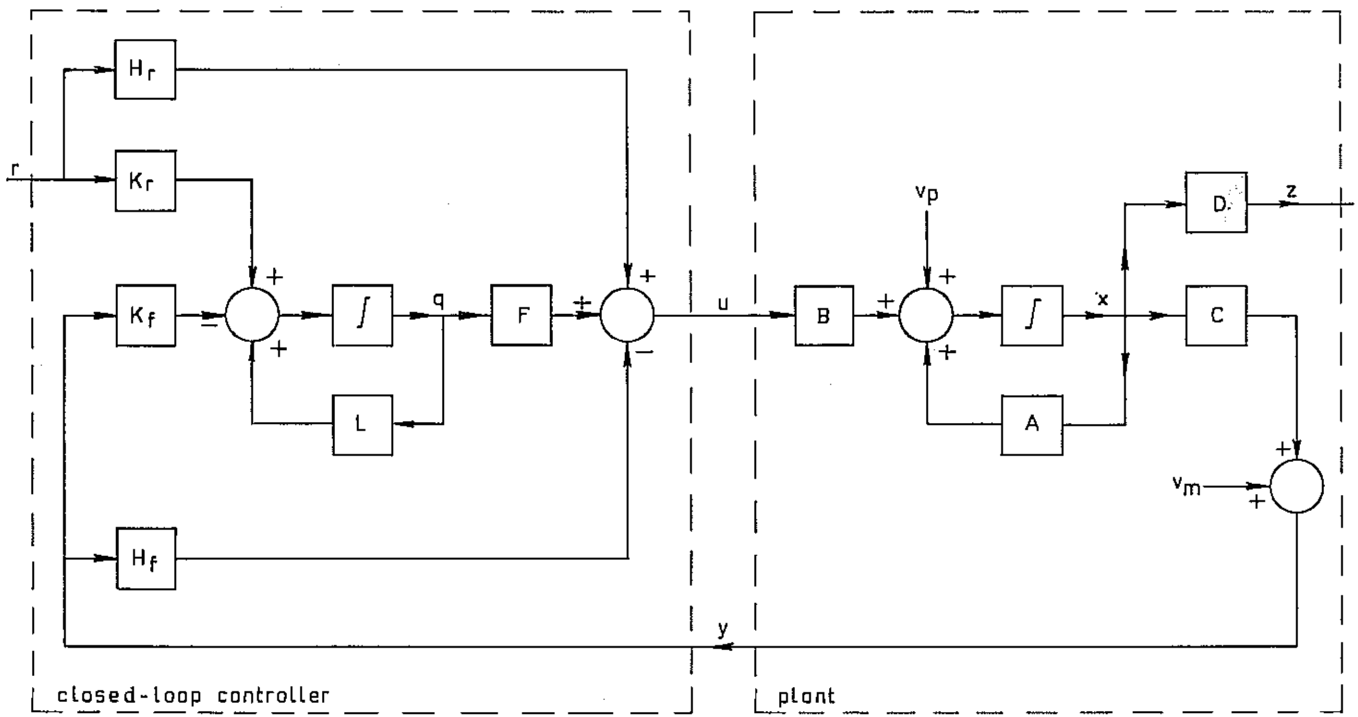

Fig. 2.7. A closed-loop control system.

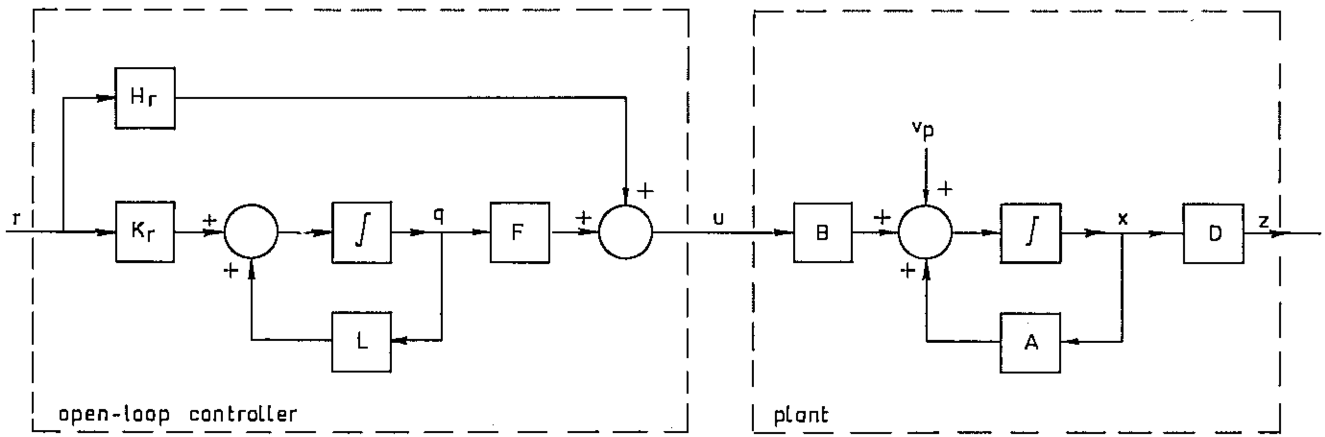

Fig. 2.8. An open-loop control system.

controller as a *closed-loop control system*, and to a control system with an open-loop controller as an *open-loop control system*.

We now define two measures of control system performance that will serve as our main tools in evaluating how well a control system performs its task:

Definition 2.1. *The **mean square tracking error** $C_e(t)$ and the **mean square input** $C_u(t)$ are defined as:*

$$C_e(t) = E\{e^T(t)W_e(t)e(t)\}, \qquad t \geq t_0,$$
$$C_u(t) = E\{u^T(t)W_u(t)u(t)\}, \qquad t \geq t_0. \qquad \textbf{2-11}$$

*Here the **tracking error** $e(t)$ is given by*

$$e(t) = z(t) - r(t), \qquad t \geq t_0, \qquad \textbf{2-12}$$

and $W_e(t)$ and $W_u(t)$, $t \geq t_0$, are given nonnegative-definite symmetric weighting matrices.

When $W_e(t)$ is diagonal, as it usually is, $C_e(t)$ is the weighted sum of the mean square tracking errors of each of the components of the controlled variable. When the error $e(t)$ is a scalar variable, and $W_e = 1$, then $\sqrt{C_e(t)}$ is the *rms tracking error*. Similarly, when the input $u(t)$ is scalar, and $W_u = 1$, then $\sqrt{C_u(t)}$ is the *rms input*.

Our aim in designing a control system is to reduce the mean square tracking error $C_e(t)$ as much as possible. Decreasing $C_e(t)$ usually implies increasing the mean square input $C_u(t)$. Since the maximally permissible value of the mean square input is determined by the capacity of the plant, a compromise must be found between the requirement of a small mean square tracking error and the need to keep the mean square input down to a reasonable level. We are thus led to the following statement.

Basic Design Objective. *In the design of control systems, the lowest possible mean square tracking error should be achieved without letting the mean square input exceed its maximally permissible value.*

In later sections we derive from the basic design objective more specific design rules, in particular for time-invariant control systems.

We now describe how the mean square tracking error $C_e(t)$ and the mean square input $C_u(t)$ can be computed. First, we use the state augmentation technique of Section 1.5.4 to obtain the state differential equation of the control system. Combining the various state and output equations we find

$$\begin{pmatrix} \dot{x}(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} A(t) - B(t)H_f(t)C(t) & B(t)F(t) \\ -K_f(t)C(t) & L(t) \end{pmatrix} \begin{pmatrix} x(t) \\ q(t) \end{pmatrix} + \begin{pmatrix} B(t)H_r(t) \\ K_r(t) \end{pmatrix} r(t) + \begin{pmatrix} I & -B(t)H_f(t) \\ 0 & -K_f(t) \end{pmatrix} \begin{pmatrix} v_p(t) \\ v_m(t) \end{pmatrix}. \qquad \textbf{2-13}$$

For the tracking error and the input we write

$$\begin{aligned} e(t) &= [D(t), 0]\begin{pmatrix} x(t) \\ q(t) \end{pmatrix} - r(t), \\ u(t) &= [-H_f(t)C(t), F(t)]\begin{pmatrix} x(t) \\ q(t) \end{pmatrix} + H_r(t)r(t) - H_f(t)v_m(t). \end{aligned} \qquad \textbf{2-14}$$

The computation of $C_e(t)$ and $C_u(t)$ is performed in two stages. First, we determine the *mean* or *deterministic part* of $e(t)$ and $u(t)$, denoted by

$$\bar{e}(t) = E\{e(t)\}, \qquad \bar{u}(t) = E\{u(t)\}, \qquad t \geq t_0. \qquad \textbf{2-15}$$

These means are computed by using the augmented state equation **2-13** and the output relations **2-14** where the stochastic processes $r(t)$, $v_p(t)$, and $v_m(t)$ are replaced with their means, and the initial state is taken as the mean of col $[x(t_0), q(t_0)]$.

Next we denote by $\tilde{x}(t)$, $\tilde{q}(t)$, and so on, the variables $x(t)$, $q(t)$, and so on, with their means $\bar{x}(t)$, $\bar{q}(t)$, and so on, subtracted:

$$\tilde{x}(t) = x(t) - \bar{x}(t), \qquad \tilde{q}(t) = q(t) - \bar{q}(t), \text{ and so on}, \qquad t \geq t_0. \qquad \textbf{2-16}$$

With this notation we write for the mean square tracking error and the mean square input

$$\begin{aligned} C_e(t) &= E\{e^T(t)W_e(t)e(t)\} = \bar{e}^T(t)W_e(t)\bar{e}(t) + E\{\tilde{e}^T(t)W_e(t)\tilde{e}(t)\}, \\ C_u(t) &= E\{u^T(t)W_u(t)u(t)\} = \bar{u}^T(t)W_u(t)\bar{u}(t) + E\{\tilde{u}^T(t)W_u(t)\tilde{u}(t)\}. \end{aligned} \qquad \textbf{2-17}$$

The terms $E\{\tilde{e}^T(t)W_e(t)\tilde{e}(t)\}$ and $E\{\tilde{u}^T(t)W_u(t)\tilde{u}(t)\}$ can easily be found when the variance matrix of col $[\tilde{x}(t), \tilde{q}(t)]$ is known. In order to determine this variance matrix, we must model the zero mean parts of $r(t)$, $v_p(t)$, and $v_m(t)$ as output variables of linear differential systems driven by white noise (see Section 1.11.4). Then col $[\tilde{x}(t), \tilde{q}(t)]$ is augmented with the state of the models generating the various stochastic processes, and the variance matrix of the resulting augmented state can be computed using the differential equation for the variance matrix of Section 1.11.2. The entire procedure is illustrated in the examples.

Example 2.4. *The position servo with three different controllers*

We continue Example 2.1 (Section 2.2.2). The motion of the antenna can be described by the differential equation

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = \tau(t) + \tau_d(t). \qquad \textbf{2-18}$$

Here J is the moment of inertia of all the rotating parts, including the antenna. Furthermore, B is the coefficient of viscous friction, $\tau(t)$ is the torque applied by the motor, and $\tau_d(t)$ is the disturbing torque caused by the wind. The motor torque is assumed to be proportional to $\mu(t)$, the input voltage to the motor, so that

$$\tau(t) = k\mu(t).$$

Defining the state variables $\xi_1(t) = \theta(t)$ and $\xi_2(t) = \dot{\theta}(t)$, the state differential equation of the system is

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_d(t), \qquad \textbf{2-19}$$

where

$$x(t) = \text{col}\,[\xi_1(t), \xi_2(t)], \qquad \alpha = \frac{B}{J}, \qquad \kappa = \frac{k}{J}, \qquad \gamma = \frac{1}{J}. \qquad \textbf{2-20}$$

The controlled variable $\zeta(t)$ is the angular position of the antenna:

$$\zeta(t) = (1, 0)x(t). \qquad \textbf{2-21}$$

When appropriate, the following numerical values are used:

$$\begin{aligned} \alpha &= 4.6\ \text{s}^{-1}, \\ \kappa &= 0.787\ \text{rad}/(\text{V s}^2), \\ J &= 10\ \text{kg m}^2. \end{aligned} \qquad \textbf{2-22}$$

Design I. *Position feedback via a zero-order controller*

In a first attempt to design a control system, we consider the control scheme outlined in Example 2.1. The only variable measured is the angular position $\theta(t)$, so that we write for the observed variable

$$\eta(t) = (1, 0)x(t) + \nu(t), \qquad \textbf{2-23}$$

where $\nu(t)$ is the measurement noise. The controller proposed can be described by the relation

$$\mu(t) = \lambda[\theta_r(t) - \eta(t)], \qquad \textbf{2-24}$$

where $\theta_r(t)$ is the reference angle and λ a gain constant. Figure 2.9 gives a simplified block diagram of the control scheme. Here it is seen how an input voltage to the motor is generated that is proportional to the difference between the reference angle $\theta_r(t)$ and the observed angular position $\eta(t)$.

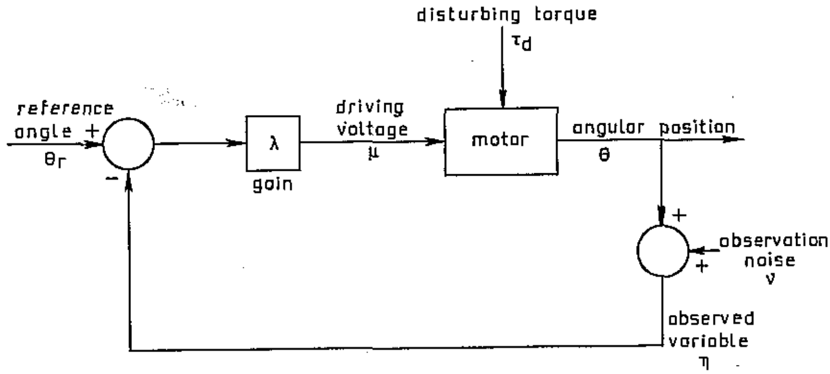

Fig. 2.9. Simplified block diagram of a position feedback control system via a zero-order controller.

The signs are so chosen that a positive value of $\theta_r(t) - \eta(t)$ results in a positive torque upon the shaft of the antenna. The question what to choose for λ is left open for the time being; we return to it in the examples of later sections.

The state differential equation of the closed-loop system is obtained from **2-19**, **2-23**, and **2-24**:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -\kappa\lambda & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa\lambda \end{pmatrix} \theta_r(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_d(t) + \begin{pmatrix} 0 \\ -\kappa\lambda \end{pmatrix} \nu(t). \qquad \textbf{2-25}$$

We note that the controller **2-24** does not increase the dimension of the closed-loop system as compared to the plant, since it does not contain any dynamics. We refer to controllers of this type as *zero-order controllers*.

In later examples it is seen how the mean square tracking error and the mean square input can be computed when specific models are assumed for the stochastic processes $\theta_r(t)$, $\tau_d(t)$, and $\nu(t)$ entering into the closed-loop system equation.

Design II. *Position and velocity feedback via a zero-order controller*

As we shall see in considerable detail in later chapters, the more information the control system has about the state of the system the better it can be made to perform. Let us therefore introduce, in addition to the potentiometer that measures the angular position, a tachometer, mounted on the shaft of the antenna, which measures the angular velocity. Thus we observe the complete state, although contaminated with observation noise, of course. We write for the observed variable

$$y(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + v(t), \qquad \textbf{2-26}$$

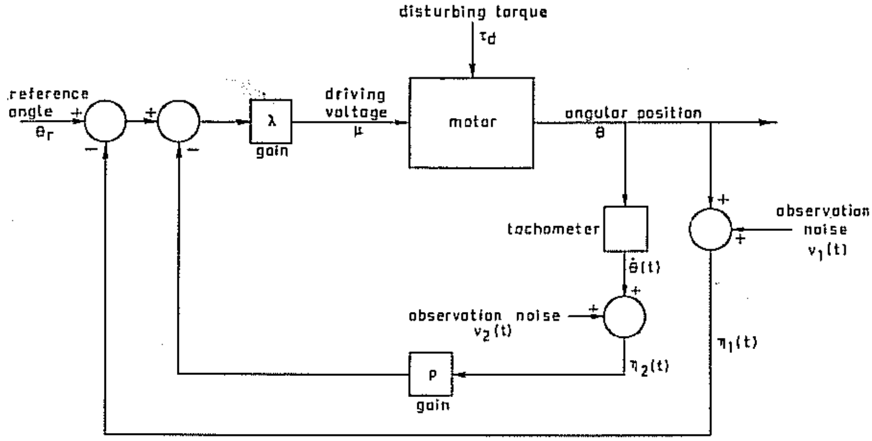

Fig. 2.10. Simplified block diagram of a position and velocity feedback control system via a zero-order controller.

where $y(t) = \text{col}\,[\eta_1(t), \eta_2(t)]$ and where $\nu(t) = \text{col}\,[\nu_1(t), \nu_2(t)]$ is the observation noise.

We now suggest the following simple control scheme (see Fig. 2.10):

$$\mu(t) = \lambda[\theta_r(t) - \eta_1(t)] - \lambda\rho\eta_2(t). \qquad \textbf{2-27}$$

This time the motor receives as input a voltage that is not only proportional to the tracking error $\theta_r(t) - \theta(t)$ but which also contains a contribution proportional to the angular velocity $\dot{\theta}(t)$. This serves the following purpose. Let us assume that at a given instant $\theta_r(t) - \theta(t)$ is positive, and that $\dot{\theta}(t)$ is positive and large. This means that the antenna moves in the right direction but with great speed. Therefore it is probably advisable not to continue driving the antenna but to start decelerating and thus avoid "overshooting" the desired position. When ρ is correctly chosen, the scheme **2-27** can accomplish this, in contrast to the scheme **2-24**. We see later that the present scheme can achieve much better performance than that of Design I.

Design III. *Position feedback via a first-order controller*

In this design approach it is assumed, as in Design I, that only the angular position $\theta(t)$ is measured. If the observation did not contain any noise, we could use a differentiator to obtain $\dot{\theta}(t)$ from $\theta(t)$ and continue as in Design II. Since observation noise is always present, however, we cannot differentiate since this greatly increases the noise level. We therefore attempt to use an approximate differentiator (see Fig. 2.11), which has the property of "filtering" the noise to some extent. Such an approximate differentiator can be realized as a system with transfer function

$$\frac{s}{T_d s + 1}, \qquad \textbf{2-28}$$

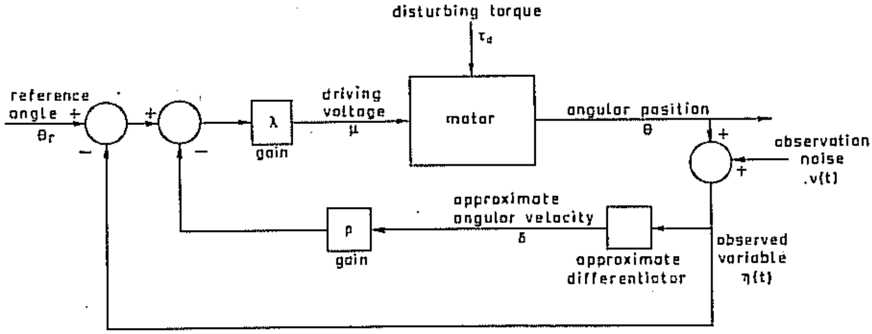

Fig. 2.11. Simplified block diagram of a position feedback control system using a first-order controller.

where T_d is a (small) positive time constant. The larger T_d is the less accurate the differentiator is, but the less the noise is amplified.

The input to the plant can now be represented as

$$\mu(t) = \lambda[\theta_r(t) - \eta(t)] - \lambda\rho\delta(t), \qquad \textbf{2-29}$$

where $\eta(t)$ is the observed angular position as in **2-23** and where $\delta(t)$ is the "approximate derivative," that is, $\delta(t)$ satisfies the differential equation

$$T_d\dot{\delta}(t) + \delta(t) = \dot{\eta}(t). \qquad \textbf{2-30}$$

This time the controller is dynamic, of order one. Again, we defer to later sections the detailed analysis of the performance of this control system; this leads to a proper choice of the time constant T_d and the gains λ and ρ. As we shall see, the performance of this design is in between those of Design I and Design II; better performance can be achieved than with Design I, although not as good as with Design II.

2.4 THE STABILITY OF CONTROL SYSTEMS

In the preceding section we introduced the control system performance measures $C_e(t)$ and $C_u(t)$. Since generally we expect that the control system will operate over long periods of time, the least we require is that both $C_e(t)$ and $C_u(t)$ remain bounded as t increases. This leads us directly to an investigation of the stability of the control system.

If the control system is not stable, sooner or later some variables will start to grow indefinitely, which is of course unacceptable in any control system that operates for some length of time (i.e., during a period larger than the time constant of the growing exponential). If the control system is unstable, usually $C_e(t)$ or $C_u(t)$, or both, will also grow indefinitely. We thus arrive at the following design objective.

Design Objective 2.1. *The control system should be asymptotically stable.*

Under the assumption that the control system is time-invariant, Design Objective 2.1 is equivalent to the requirement that all characteristic values of the augmented system **2-13**, that is, the characteristic values of the matrix

$$\begin{pmatrix} A - BH_fC & BF \\ -K_fC & L \end{pmatrix}, \tag{2-31}$$

have strictly negative real parts. By referring back to Section 1.5.4, Theorem 1.21, the characteristic polynomial of **2-31** can be written as

$$\det (sI - A) \det (sI - L) \det [I + H(s)G(s)], \tag{2-32}$$

where we have denoted by

$$H(s) = C(sI - A)^{-1}B \tag{2-33}$$

the transfer matrix of the plant from the input u to be the observed variable y, and by

$$G(s) = F(sI - L)^{-1}K_f + H_f \tag{2-34}$$

the transfer matrix of the controller from y to $-u$.

One of the functions of the controller is to move the poles of the plant to better locations in the left-hand complex plane so as to achieve an improved system performance. If the plant by itself is unstable, *stabilizing* the system by moving the closed-loop poles to proper locations in the left-half complex plane is the *main* function of the controller (see Example 2.6).

Example 2.5. *Position servo*

Let us analyze the stability of the zero-order position feedback control system proposed for the antenna drive system of Example 2.4, Design I. The plant transfer function (the transfer function from the driving voltage to the antenna position) is given by

$$H(s) = \frac{\kappa}{s(s + \alpha)}. \tag{2-35}$$

The controller transfer function is

$$G(s) = \lambda. \tag{2-36}$$

Thus by **2-32** the closed-loop poles are the roots of

$$s(s + \alpha)\left[1 + \frac{\kappa\lambda}{s(s + \alpha)}\right] = s^2 + \alpha s + \kappa\lambda. \tag{2-37}$$

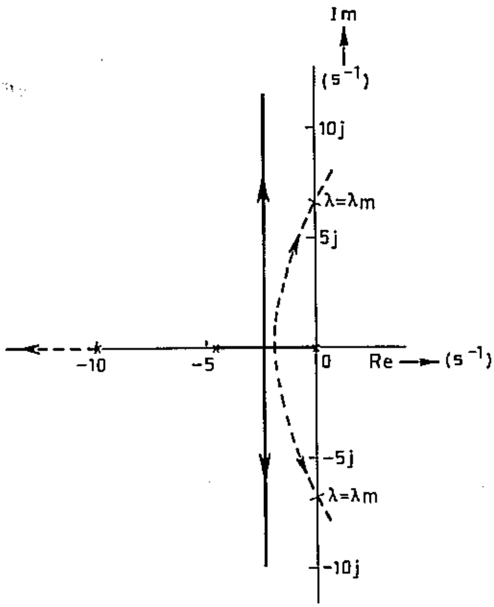

Fig. 2.12. Root loci for position servo. Solid lines, loci for second-order system; dashed lines, modifications of loci due to the presence of the pole at $-10\ s^{-1}$.

Figure 2.12 shows the loci of the closed-loop poles with λ as a parameter for the numerical values **2-22.**

It is seen that ideally the control system is stable for all positive values of λ. In practice, however, the system becomes unstable for large λ. The reason is that, among other things, we have neglected the electrical time constant T_e of the motor. Taking this into account, the transfer function of motor plus antenna is

$$H(s) = \frac{\kappa}{s(s + \alpha)(sT_e + 1)}. \qquad \textbf{2-38}$$

As a result, the closed-loop characteristic polynomial is

$$s(s + \alpha)\left(s + \frac{1}{T_e}\right) + \frac{\kappa\lambda}{T_e}. \qquad \textbf{2-39}$$

Figure 2.12 shows the modification of the root loci that results for

$$T_e = 0.1\ \text{s}. \qquad \textbf{2-40}$$

For $\lambda \geq \lambda_m$, where

$$\lambda_m = \frac{\alpha}{\kappa}\left(\alpha + \frac{1}{T_e}\right), \qquad \textbf{2-41}$$

the closed-loop system is unstable. In the present case $\lambda_m = 85.3$ V/rad.

Example 2.6. *The stabilization of the inverted pendulum*

As an example of an unstable plant, we consider the inverted pendulum of Example 1.1 (Section 1.2.3). In Example 1.16 (Section 1.5.4), we saw that by feeding back the angle $\phi(t)$ via a zero-order controller of the form

$$\mu(t) = \lambda\phi(t) \tag{2-42}$$

it is not possible to stabilize the system for any value of the gain λ. It is possible, however, to stabilize the system by feeding back the complete state $x(t)$ as follows

$$\mu(t) = -kx(t). \tag{2-43}$$

Here k is a constant row vector to be determined. We note that implementation of this controller requires measurement of all four state variables.

In Example 1.1 we gave the linearized state differential equation of the system, which is of the form

$$\dot{x}(t) = Ax(t) + b\mu(t), \tag{2-44}$$

where b is a column vector. Substitution of **2-43** yields

$$\dot{x}(t) = Ax(t) - bkx(t), \tag{2-45}$$

or

$$\dot{x}(t) = (A - bk)x(t). \tag{2-46}$$

The stability of this system is determined by the characteristic values of the matrix $A - bk$. In Chapter 3 we discuss methods for determining *optimal* controllers of the form **2-43** that stabilize the system. By using those methods, and using the numerical values of Example 1.1, it can be found, for example, that

$$k = (86.81,\ 12.21,\ -118.4,\ -33.44) \tag{2-47}$$

stabilizes the linearized system. With this value for k, the closed-loop characteristic values are $-4.706 \pm j1.382$ and $-1.902 \pm j3.420$.

To determine the stability of the actual (nonlinear) closed-loop system, we consider the nonlinear state differential equation

$$\begin{aligned}
\dot{\xi}_1(t) &= \xi_2(t),\\
\dot{\xi}_2(t) &= \frac{1}{M}\mu(t) - \frac{F}{M}\xi_2(t),\\
\dot{\xi}_3(t) &= \xi_4(t),\\
\dot{\xi}_4(t) &= g\sin\left[\frac{\xi_3(t) - \xi_1(t)}{L'}\right]\\
&\qquad + \left[\frac{\mu(t) - F\xi_2(t)}{M}\right]\left[1 - \cos\frac{\xi_3(t) - \xi_1(t)}{L'}\right],
\end{aligned} \tag{2-48}$$

where the definitions of the components ξ_1, ξ_2, ξ_3, and ξ_4 are the same as for the linearized equations. Substitution of the expression **2-43** for $\mu(t)$ into **2-48** yields the closed-loop state differential equation. Figure 2.13 gives the closed-loop response of the angle $\phi(t)$ for different initial values $\phi(0)$ while all other initial conditions are zero. For $\phi(0) = 10°$ the motion is indistinguishable from the motion that would be found for the linearized system. For $\phi(0) = 20°$ some deviations occur, while for $\phi(0) = 30°$ the system is no longer stabilized by **2-47**.

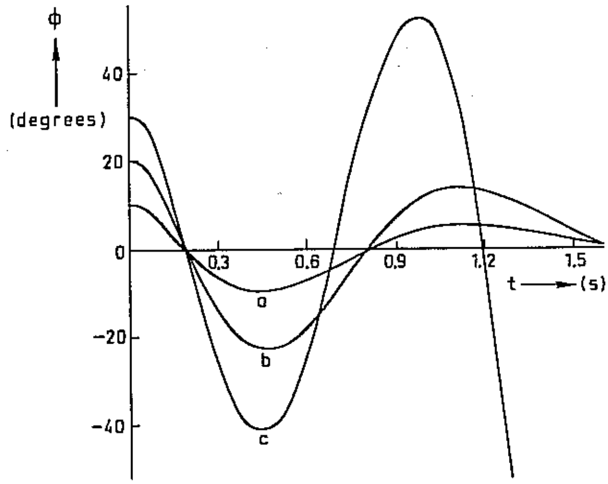

Fig. 2.13. The behavior of the angle $\phi(t)$ for the stabilized inverted pendulum: (*a*) $\phi(0) = 10°$; (*b*) $\phi(0) = 20°$; (*c*) $\phi(0) = 30°$.

This example also illustrates Theorem 1.16 (Section 1.4.4), where it is stated that when a linearized system is asymptotically stable the nonlinear system from which it is derived is also asymptotically stable. We see that in the present case the range over which linearization gives useful results is quite large.

2.5 THE STEADY-STATE ANALYSIS OF THE TRACKING PROPERTIES

2.5.1 The Steady-State Mean Square Tracking Error and Input

In Section 2.3 we introduced the mean square tracking error C_e and the mean square input C_u. From the control system equations **2-13** and **2-14**, it can be seen that all three processes $r(t)$, $v_p(t)$, and $v_m(t)$, that is, the reference variable, the disturbance variable, and the observation noise, have an effect on C_e and C_u. From now until the end of the chapter, we assume that $r(t)$, $v_p(t)$, and $v_m(t)$ are *statistically uncorrelated* stochastic processes so that their contributions to C_e and C_u can be investigated separately. In the present and the following section, we consider the contribution of the reference variable $r(t)$ to $C_e(t)$ and $C_u(t)$ alone. The effect of the disturbance and the observation noise are investigated in later sections.

We divide the duration of a control process into two periods: the *transient* and the *steady-state* period. These two periods can be characterized as follows. The transient period starts at the beginning of the process and terminates when the quantities we are interested in (usually the mean square tracking error and input) approximately reach their steady-state values. From that time on we say that the process is in its steady-state period. We assume, of course, that the quantities of interest converge to a certain limit as time increases. The duration of the transient period will be referred to as the *settling time*.

In the design of control systems, we must take into account the performance of the system during both the transient period and the steady-state period. The present section is devoted to the analysis of the steady-state properties of tracking systems. In the next section the transient analysis is discussed. In this section and the next, the following assumptions are made.

1. *Design Objective* 2.1 *is satisfied, that is, the control system is asymptotically stable;*
2. *The control system is time-invariant and the weighting matrices* W_e *and* W_u *are constant;*
3. *The disturbance* $v_p(t)$ *and the observation noise* $v_m(t)$ *are identical to zero;*
4. *The reference variable* $r(t)$ *can be represented as*

$$r(t) = r_0 + r_v(t), \tag{2-49}$$

where r_0 *is a stochastic vector and* $r_v(t)$ *is a zero-mean wide-sense stationary vector stochastic process, uncorrelated with* r_0.

Here the stochastic vector r_0 is the *constant part* of the reference variable and is in fact the *set point* for the controlled variable. The zero-mean process $r_v(t)$ is the *variable part* of the reference variable. We assume that the second-order moment matrix of r_0 is given by

$$E\{r_0r_0^T\} = R_0, \tag{2-50}$$

while the variable part $r_v(t)$ will be assumed to have the power spectral density matrix $\Sigma_r(\omega)$.

Under the assumptions stated the mean square tracking error and the mean square input converge to constant values as t increases. We thus define the *steady-state mean square tracking error*

$$C_{e\infty} = \lim_{t\to\infty} C_e(t), \qquad \text{2-51}$$

and the *steady-state mean square input*

$$C_{u\infty} = \lim_{t\to\infty} C_u(t). \qquad \text{2-52}$$

In order to compute $C_{e\infty}$ and $C_{u\infty}$, let us denote by $T(s)$ the *transmission* of the closed-loop control system, that is, the transfer matrix from the reference variable r to the controlled variable z. We furthermore denote by $N(s)$ the transfer matrix of the closed-loop system from the reference variable r to the input variable u.

In order to derive expressions for the steady-state mean square tracking error and input, we consider the contributions of the constant part r_0 and the variable part $r_v(t)$ of the reference variable separately. The constant part of the reference variable yields a steady-state response of the controlled variable and input as follows

$$\lim_{t\to\infty} z(t) = T(0)r_0 \qquad \text{2-53}$$

and

$$\lim_{t\to\infty} u(t) = N(0)r_0, \qquad \text{2-54}$$

respectively. The corresponding contributions to the steady-state square tracking error and input are

$$[T(0)r_0 - r_0]^T W_e[T(0)r_0 - r_0] = \text{tr}\,\{r_0 r_0^T[T(0) - I]^T W_e[T(0) - I]\} \qquad \text{2-55}$$

and

$$[N(0)r_0]^T W_u[N(0)r_0] = \text{tr}\,[r_0 r_0^T N^T(0) W_u N(0)]. \qquad \text{2-56}$$

It follows that the contributions of the *constant* part of the reference variable to the steady-state mean square tracking error and input, respectively, are

$$\text{tr}\,\{R_0[T(0) - I]^T W_e[T(0) - I] \qquad \text{and} \qquad \text{tr}\,[R_0 N^T(0) W_u N(0)]. \qquad \text{2-57}$$

The contributions of the *variable* part of the reference variable to the steady-state mean square tracking error and input are easily found by using the results of Section 1.10.4 and Section 1.10.3. The steady-state mean square tracking error turns out to be

$$C_{e\infty} = \text{tr}\,\Big\{R_0[T(0) - I]^T W_e[T(0) - I] + \int_{-\infty}^{\infty} \Sigma_r(\omega)[T(-j\omega) - I]^T W_e[T(j\omega) - I]\, df\Big\}, \qquad \text{2-58}$$

while the steady-state mean square input is

$$C_{u\infty} = \text{tr}\left\{R_0 N^T(0) W_u N(0) + \int_{-\infty}^{\infty} \Sigma_r(\omega) N^T(-j\omega) W_u N(j\omega)\, df\right\}. \qquad \textbf{2-59}$$

These formulas are the starting point for deriving specific design objectives. In the next subsection we confine ourselves to the single-input single-output case, where both the input u and the controlled variable z are scalar and where the interpretation of the formulas **2-58** and **2-59** is straightforward. In Section 2.5.3 we turn to the more general multiinput multioutput case.

In conclusion we obtain expressions for $T(s)$ and $N(s)$ in terms of the various transfer matrices of the plant and the controller. Let us denote the transfer matrix of the plant **2-6–2-8** (now assumed to be time-invariant) from the input u to the controlled variable z by $K(s)$ and that from the input u to the observed variable y by $H(s)$. Also, let us denote the transfer matrix of the controller **2-9**, **2-10** (also time-invariant) from the reference variable r to u by $P(s)$, and from the plant observed variable y to $-u$ by $G(s)$. Thus we have:

$$\begin{aligned} K(s) &= D(sI - A)^{-1}B, & H(s) &= C(sI - A)^{-1}B, \\ P(s) &= F(sI - L)^{-1}K_r + H_r, & G(s) &= F(sI - L)^{-1}K_f + H_f. \end{aligned} \qquad \textbf{2-60}$$

The block diagram of Fig. 2.14 gives the relations between the several system variables in terms of transfer matrices. From this diagram we see that, if

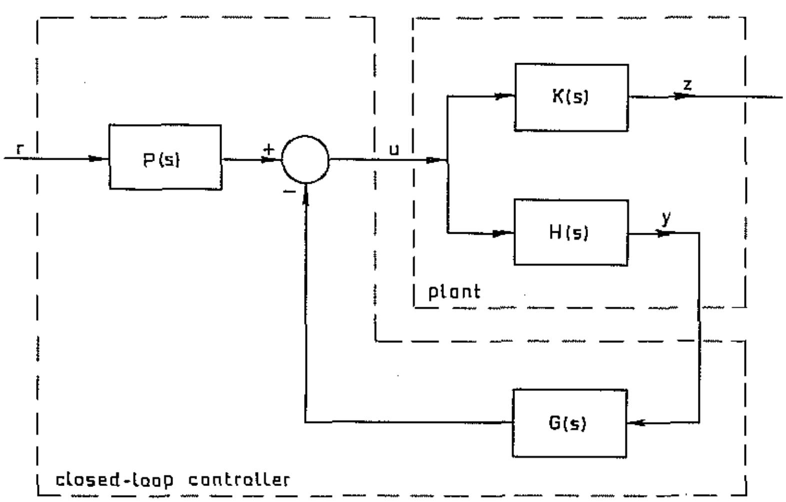

Fig. 2.14. The transfer matrix block diagram of a linear time-invariant closed-loop control system.

$r(t)$ has a Laplace transform $\mathbf{R}(s)$, in terms of Laplace transforms the several variables are related by

$$\begin{aligned} \mathbf{U}(s) &= P(s)\mathbf{R}(s) - G(s)\mathbf{Y}(s), \\ \mathbf{Y}(s) &= H(s)\mathbf{U}(s), \\ \mathbf{Z}(s) &= K(s)\mathbf{U}(s). \end{aligned} \tag{2-61}$$

Elimination of the appropriate variables yields

$$\begin{aligned} \mathbf{Z}(s) &= T(s)\mathbf{R}(s), \\ \mathbf{U}(s) &= N(s)\mathbf{R}(s), \end{aligned} \tag{2-62}$$

where

$$\begin{aligned} T(s) &= K(s)[I + G(s)H(s)]^{-1}P(s), \\ N(s) &= [I + G(s)H(s)]^{-1}P(s). \end{aligned} \tag{2-63}$$

$T(s)$ and $N(s)$ are of course related by

$$T(s) = K(s)N(s). \tag{2-64}$$

2.5.2 The Single-Input Single-Output Case

In this section it is assumed that both the input u and the controlled variable z, and therefore also the reference variable r, are scalar variables. Without loss of generality we take both $W_e = 1$ and $W_u = 1$. As a result, the steady-state mean square tracking error and the steady-state mean square input can be expressed as

$$C_{e\infty} = R_0\,|T(0) - 1|^2 + \int_{-\infty}^{\infty} \Sigma_r(\omega)\,|T(j\omega) - 1|^2\,df, \tag{2-65a}$$

$$C_{u\infty} = R_0\,|N(0)|^2 + \int_{-\infty}^{\infty} \Sigma_r(\omega)\,|N(j\omega)|^2\,df. \tag{2-65b}$$

From the first of these expressions, we see that since we wish to design tracking systems with a small steady-state mean square tracking error the following advice must be given.

Design Objective 2.2. *In order to obtain a small steady-state mean square tracking error, the transmission $T(s)$ of a time-invariant linear control system should be designed such that*

$$\Sigma_r(\omega)\,|T(j\omega) - 1|^2 \tag{2-66}$$

is small for all real ω. In particular, when nonzero set points are likely to occur, $T(0)$ should be made close to 1.

The remark about $T(0)$ can be clarified as follows. In certain applications it is important that the set point of the control system be maintained very accurately. In particular, this is the case in regulator problems, where the variable part of the reference variable is altogether absent. In such a case it may be necessary that $T(0)$ very precisely equal 1.

We now examine the contributions to the integral in **2-65a** from various frequency regions. Typically, as ω increases, $\Sigma_r(\omega)$ decreases to zero. It thus follows from **2-65a** that it is sufficient to make $|T(j\omega) - 1|$ small for those frequencies where $\Sigma_r(\omega)$ assumes significant values.

In order to emphasize these remarks, we introduce two notions: the *frequency band of the control system* and the *frequency band of the reference variable.* The frequency band of the control system is roughly the range of frequencies over which $T(j\omega)$ is "close" to 1:

Definition 2.2. *Let $T(s)$ be the scalar transmission of an asymptotically stable time-invariant linear control system with scalar input and scalar controlled variable. Then the* ***frequency band*** *of the control system is defined as the set of frequencies ω, $\omega \geq 0$, for which*

$$|T(j\omega) - 1| \leq \varepsilon, \qquad \textbf{2-67}$$

where ε is a given number that is small with respect to 1. If the frequency band is an interval $[\omega_1, \omega_2]$, we call $\omega_2 - \omega_1$ the ***bandwidth*** *of the control system. If the frequency band is an interval $[0, \omega_c]$, we refer to ω_c as the* ***cutoff frequency*** *of the system.*

Figure 2.15 illustrates the notions of frequency band, bandwidth, and cutoff frequency.

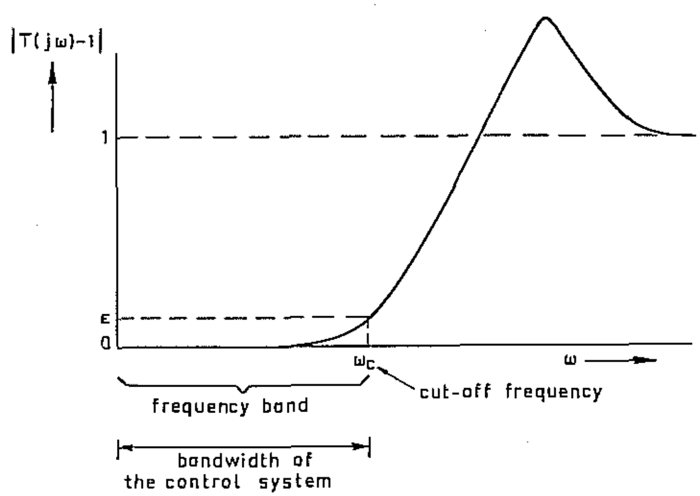

Fig. 2.15. Illustration of the definition of the frequency band, bandwidth, and cutoff frequency of a single-input single-output time-invariant control system. It is assumed that $T(j\omega) \to 0$ as $\omega \to \infty$.

In this book we usually deal with *low-pass* transmissions where the frequency band is the interval from the zero frequency to the cutoff frequency ω_c. The precise value of the cutoff frequency is of course very much dependent upon the number ε. When $\varepsilon = 0.01$, we refer to ω_c as the 1% *cutoff frequency*. We use a similar terminology for different values of ε. Frequently, however, we find it convenient to speak of the *break frequency* of the control system, which we define as that corner frequency where the asymptotic Bode plot of $|T(j\omega)|$ breaks away from unity. Thus the break frequency of the first-order transmission

$$T(s) = \frac{\alpha}{s + \alpha} \tag{2-68}$$

is α, while the break frequency of the second-order transmission

$$T(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{2-69}$$

is ω_0. Note, however, that in both cases the cutoff frequency is considerably *smaller* than the break frequency, dependent upon ε, and, in the second-order case, dependent upon the relative damping ζ. Table 2.1 lists the 1% and 10% cut-off frequencies for various cases.

Table 2.1 Relation between Break Frequency and Cutoff Frequency for First- and Second-Order Scalar Transmissions

	First-order system with break frequency α	Second-order system with break frequency ω_0		
		$\zeta = 0.4$	$\zeta = 0.707$	$\zeta = 1.5$
1% cutoff freq.	0.01α	$0.012\omega_0$	$0.0071\omega_0$	$0.0033\omega_0$
10% cutoff freq.	0.1α	$0.12\omega_0$	$0.071\omega_0$	$0.033\omega_0$

Next we define the frequency band of the reference variable, which is the range of frequencies over which $\Sigma_r(\omega)$ is significantly different from zero:

Definition 2.3. *Let r be a scalar wide-sense stationary stochastic process with power spectral density function $\Sigma_r(\omega)$. The* ***frequency band*** *Ω of $r(t)$ is defined as the set of frequencies ω, $\omega \geq 0$, for which*

$$\Sigma_r(\omega) \geq \alpha. \tag{2-70}$$

Here α is so chosen that the frequency band contains a given fraction 1 − ε where ε is small with respect to 1, of half of the power of the process, that is

$$\int_{\omega \in \Omega} \Sigma_r(\omega)\, df = (1 - \varepsilon) \int_{\omega > 0} \Sigma_r(\omega)\, df. \qquad \text{2-71}$$

If the frequency band is an interval $[\omega_1, \omega_2]$, we define $\omega_2 - \omega_1$ as the ***bandwidth*** *of the process. If the frequency band is an interval $[0, \omega_c]$, we refer to ω_c as the* ***cutoff frequency*** *of the process.*

Figure 2.16 illustrates the notions of frequency band, bandwidth, and cutoff frequency of a stochastic process.

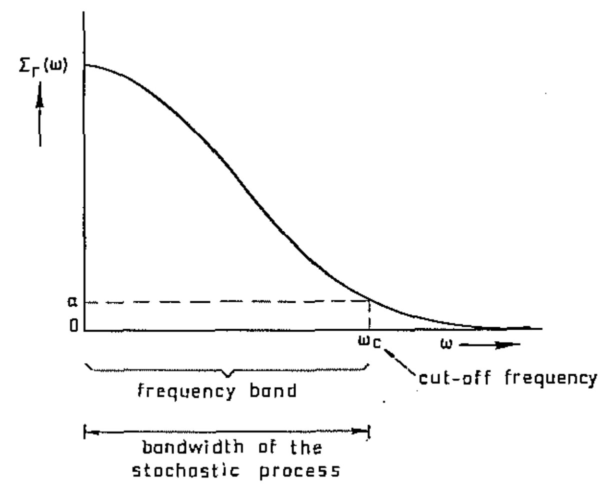

Fig. 2.16. Illustration of the definition of the frequency band, bandwidth, and cutoff frequency of a scalar stochastic process *r*.

Usually we deal with low-pass-type stochastic processes that have an interval of the form $[0, \omega_c]$ as a frequency band. The precise value of the cutoff frequency is of course very much dependent upon the value of ε. When $\varepsilon = 0.01$, we speak of the 1% *cutoff frequency*, which means that the interval $[0, \omega_c]$ contains 99% of half the power of the process. A similar terminology is used for other values of ε. Often, however, we find it convenient to speak of the *break frequency* of the process, which we define as the corner frequency where the asymptotic Bode plot of $\Sigma_r(\omega)$ breaks away from its low-frequency asymptote, that is, from $\Sigma_r(0)$. Let us take as an example exponentially correlated noise with rms value σ and time constant θ. This process has the power spectral density function

$$\frac{2\sigma^2\theta}{1 + \omega^2\theta^2}, \tag{2-72}$$

so that its break frequency is $1/\theta$. Since this power spectral density function decreases very slowly with ω, the 1 and 10% cutoff frequencies are much *larger* than $1/\theta$; in fact, they are $63.66/\theta$ and $6.314/\theta$, respectively.

Let us now reconsider the integral in **2-65a**. Using the notions just introduced, we see that the main contribution to this integral comes from those frequencies which are in the frequency band of the reference variable but *not* in the frequency band of the system (see Fig. 2.17). We thus rephrase Design Objective 2.2 as follows.

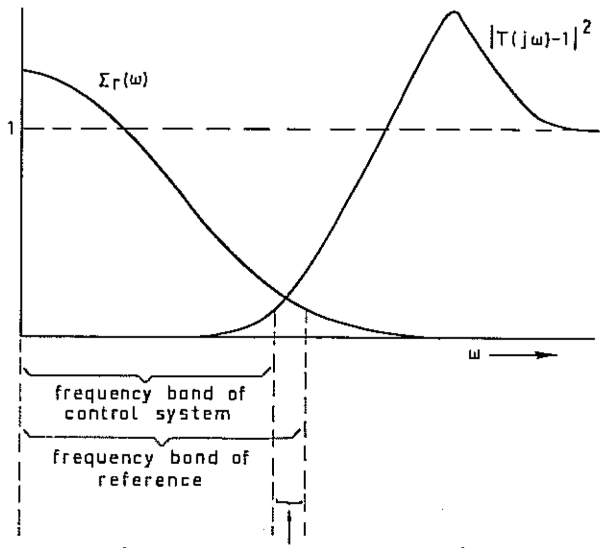

Fig. 2.17. Illustration of Design Objective 2.2.A.

Design Objective 2.2A. *In order to obtain a small steady-state mean square tracking error, the frequency band of the control system should contain as much as possible of the frequency band of the variable part of the reference variable. If nonzero set paints are likely to occur,* $T(0)$ *should be made close to* 1.

An important aspect of this design rule is that it is also useful when very little is known about the reference variable except for a rough idea of its frequency band.

Let us now consider the second aspect of the design—the steady-state mean square input. A consideration of **2-65b** leads us to formulate our next design objective.

Design Objective 2.3. *In order to obtain a small steady-state mean square input in an asymptotically stable single-input single-output time-invariant linear control system,*

$$\Sigma_r(\omega)\,|N(j\omega)|^2 \tag{2-73}$$

should be made small for all real ω. This can be achieved by making $|N(j\omega)|$ sufficiently small over the frequency band of the reference variable.

It should be noted that this objective does not contain the advice to keep $N(0)$ small, such as would follow from considering the first term of **2-65b**. This term represents the contribution of the constant part of the reference variable, that is, the set point, to the input. The set point determines the desired level of the controlled variable and therefore also that of the input. It must be assumed that the plant is so designed that it is capable of sustaining this level. The second term in **2-65b** is important for the dynamic range of the input, that is, the variations in the input about the set point that are permissible. Since this dynamic range is restricted, the magnitude of the second term in **2-65b** must be limited.

It is not difficult to design a control system so that *one* of the Design Objectives 2.2A or 2.3 is completely satisfied. Since $T(s)$ and $N(s)$ are related by

$$T(s) = K(s)N(s), \tag{2-74}$$

however, the design of $T(s)$ affects $N(s)$, and vice-versa. We elaborate a little on this point and show how Objectives 2.2 and 2.3 may conflict. The plant frequency response function $|K(j\omega)|$ usually decreases beyond a certain frequency, say ω_p. If $|T(j\omega)|$ is to stay close to 1 beyond this frequency, it is seen from **2-74** that $|N(j\omega)|$ must *increase* beyond ω_p. The fact that $|T(j\omega)|$ is not allowed to decrease beyond ω_p implies that the reference variable frequency band extends beyond ω_p. As a result, $|N(j\omega)|$ will be large over a frequency range where $\Sigma_r(\omega)$ is not small, which may mean an important contribution to the mean square input. If this results in overloading the plant, either the bandwidth of the control system must be reduced (at the expense of a larger tracking error), or the plant must be replaced by a more powerful one.

The designer must find a technically sound compromise between the requirements of a small mean square tracking error and a mean square input that matches the dynamic range of the plant. This compromise should be based on the specifications of the control system such as the maximal allowable rms tracking error or the maximal power of the plant. In later chapters, where we are concerned with the synthesis problem, *optimal* compromises to this dilemma are found.

At this point a brief comment on computational aspects is in order. In Section 2.3 we outlined how time domain methods can be used to calculate the mean square tracking error and mean square input. In the time-invariant case, the integral expressions **2-65a** and **2-65b** offer an alternative computational approach. Explicit solutions of the resulting integrals have been tabulated for low-order cases (see, e.g., Newton, Gould, and Kaiser (1957), Appendix E; Seifert and Steeg (1960), Appendix). For numerical computations we usually prefer the time-domain approach, however, since this is better suited for digital computation. Nevertheless, the frequency domain expressions as given are extremely important since they allow us to formulate design objectives that cannot be easily seen, if at all, from the time domain approach.

Example 2.7. *The tracking properties of the position servo*

Let us consider the position servo problem of Examples 2.1 (Section 2.2.2) and 2.4 (Section 2.3), and let us assume that the reference variable is adequately represented as zero-mean exponentially correlated noise with rms value σ and time constant T_r. We use the numerical values

$$\begin{aligned} \sigma &= 1 \text{ rad}, \\ T_r &= 10 \text{ s}. \end{aligned} \qquad \textbf{2-75}$$

It follows from the value of the time constant and from **2-72** that the reference variable break frequency is 0.1 rad/s, its 10% cutoff frequency 0.63 rad/s, and its 1% cutoff frequency 6.4 rad/s.

Design I. Let us first consider Design I of Example 2.4, where zero-order feedback of the position has been assumed. It is easily found that the transmission $T(s)$ and the transfer function $N(s)$ are given by

$$\begin{aligned} T(s) &= \frac{\kappa\lambda}{s^2 + \alpha s + \kappa\lambda}, \\ N(s) &= \frac{\lambda s(s + \alpha)}{s^2 + \alpha s + \kappa\lambda}. \end{aligned} \qquad \textbf{2-76}$$

We rewrite the transmission as

$$T(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \qquad \textbf{2-77}$$

where

$$\omega_0 = \sqrt{\kappa\lambda} \qquad \textbf{2-78}$$

is the undamped natural frequency, and

$$\zeta = \frac{\alpha}{2\sqrt{\kappa\lambda}} \tag{2-79}$$

the relative damping. In Fig. 2.18 we plot $|T(j\omega)|$ as a function of ω for various values of the gain λ. Following Design Objective 2.2A the gain λ should probably not be chosen less than about 15 V/rad, since otherwise the cutoff frequency of the control system would be too small as compared to the 1% cutoff frequency of the reference variable of 6.4 rad/s. However, the cutoff

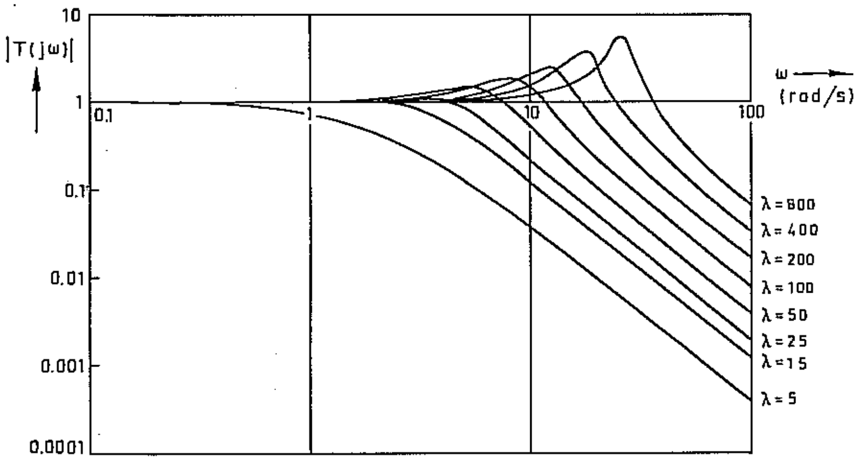

Fig. 2.18. Bode plots of the transmission of the position control system, Design I, for various values of the gain λ.

frequency does not seem to increase further with the gain, due to the peaking effect which becomes more and more pronounced. The value of 15 V/rad for the gain corresponds to the case where the relative damping ζ is about 0.7.

It remains to be seen whether or not this gain leads to acceptable values of the rms tracking error and the rms input voltage. To this end we compute both. The reference variable can be modeled as follows

$$\dot{\theta}_r(t) = -\frac{1}{T_r}\theta_r(t) + w(t), \tag{2-80}$$

where $w(t)$ is white noise with intensity $2\sigma^2/T_r$. The combined state equations of the control system and the reference variable are from **2-19**, **2-24**, and **2-80**:

$$\begin{pmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \dot{\theta}_r(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa\lambda & -\alpha & \kappa\lambda \\ 0 & 0 & -\dfrac{1}{T_r} \end{pmatrix} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \theta_r(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w(t). \qquad \textbf{2-81}$$

With this equation as a starting point, it is easy to set up and solve the Lyapunov equation for the steady-state variance matrix $\bar{Q}$ of the augmented state col $[\xi_1(t), \xi_2(t), \theta_r(t)]$ (Theorem 1.53, Section 1.11.3). The result is

$$\bar{Q} = \begin{pmatrix} \dfrac{\kappa\lambda\left(\dfrac{1}{\alpha} + T_r\right)}{\alpha + \dfrac{1}{T_r} + \kappa\lambda T_r}\sigma^2 & 0 & \dfrac{\kappa\lambda T_r}{\alpha + \dfrac{1}{T_r} + \kappa\lambda T_r}\sigma^2 \\ 0 & \dfrac{\dfrac{(\kappa\lambda)^2}{\alpha}}{\alpha + \dfrac{1}{T_r} + \kappa\lambda T_r}\sigma^2 & \dfrac{\kappa\lambda}{\alpha + \dfrac{1}{T_r} + \kappa\lambda T_r}\sigma^2 \\ \dfrac{\kappa\lambda T_r}{\alpha + \dfrac{1}{T_r} + \kappa\lambda T_r}\sigma^2 & \dfrac{\kappa\lambda}{\alpha + \dfrac{1}{T_r} + \kappa\lambda T_r}\sigma^2 & \sigma^2 \end{pmatrix}. \qquad \textbf{2-82}$$

As a result, we obtain for the steady-state mean square tracking error:

$$C_{e\infty} = \lim_{t\to\infty} E\{[\theta(t) - \theta_r(t)]^2\} = \bar{q}_{11} - 2\bar{q}_{13} + \bar{q}_{33}$$

$$= \frac{\alpha + \dfrac{1}{T_r} + \dfrac{\kappa\lambda}{\alpha}}{\alpha + \dfrac{1}{T_r} + \kappa\lambda T_r}\sigma^2, \qquad \textbf{2-83}$$

where the $\bar{q}_{ij}$ are the entries of $\bar{Q}$. A plot of the steady-state rms tracking error is given in Fig. 2.19. We note that increasing λ beyond 15–25 V/rad decreases the rms tracking error only very little. The fact that $C_{e\infty}$ does not decrease to zero as $\lambda \to \infty$ is attributable to the peaking effect in the transmission which becomes more and more pronounced as λ becomes larger.

The steady-state rms input voltage can be found to be given by

$$C_{u\infty} = E\{\mu^2(t)\} = E\{\lambda^2[\theta(t) - \theta_r(t)]^2\} = \lambda^2 C_{e\infty}. \qquad \textbf{2-84}$$

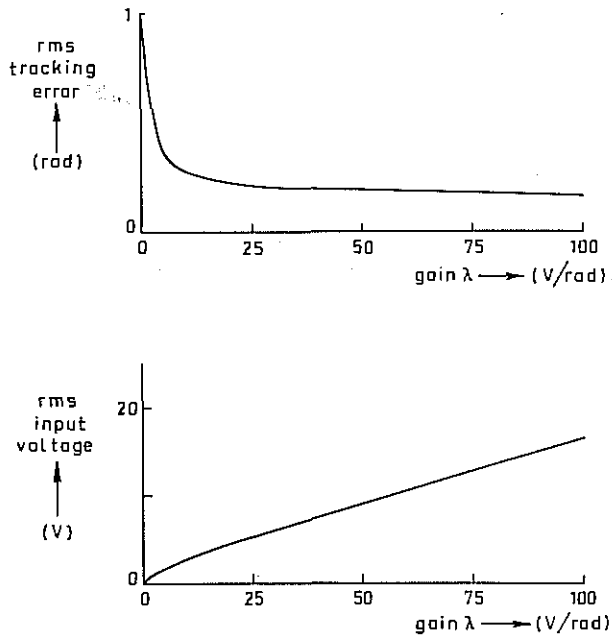

Fig. 2.19. Rms tracking error and rms input voltage as functions of the gain λ for the position servo, Design I.

Figure 2.19 shows that, according to what one would intuitively feel, the rms input keeps increasing with the gain λ. Comparing the behavior of the rms tracking error and the rms input voltage confirms the opinion that there is very little point in increasing the gain beyond 15–25 V/rad, since the increase in rms input voltage does not result in any appreciable reduction in the rms tracking error. We observe, however, that the resulting design is not very good, since the rms tracking error achieved is about 0.2 rad, which is not very small as compared to the rms value of the reference variable of 1 rad.

Design II. The second design suggested in Example 2.4 gives better results, since in this case the tachometer feedback gain factor ρ can be so chosen that the closed-loop system is well-damped for each desired bandwidth, which eliminates the peaking effect. In this design we find for the transmission

$$T(s) = \frac{\kappa\lambda}{s^2 + (\alpha + \kappa\lambda\rho)s + \kappa\lambda}, \qquad \textbf{2-85}$$

which is similar to **2-76** except that α is replaced with $\alpha + \kappa\lambda\rho$. As a result, the undamped natural frequency of the system is

$$\omega_0 = \sqrt{\kappa\lambda} \tag{2-86}$$

and the relative damping

$$\zeta = \frac{\alpha + \kappa\lambda\rho}{2\sqrt{\kappa\lambda}}. \tag{2-87}$$

The break frequency of the system is ω_0, which can be made arbitrarily large by choosing λ large enough. By choosing ρ such that the relative damping is in the neighborhood of 0.7, the cutoff frequency of the control system can be made correspondingly large. The steady-state rms tracking error is

$$C_{e\infty} = \frac{(\alpha + \kappa\lambda\rho)^2 + (\alpha + \kappa\lambda\rho)\dfrac{1}{T_r} + \kappa\lambda}{(\alpha + \kappa\lambda\rho)\left(\alpha + \kappa\lambda\rho + \dfrac{1}{T_r} + \kappa\lambda T_r\right)}\sigma^2, \tag{2-88}$$

while the steady-state mean square input voltage is given by

$$C_{\mu\infty} = \lambda^2 \frac{\alpha^2 + \dfrac{\alpha}{T_r} + \dfrac{\kappa\lambda\rho}{T_r} + \kappa\lambda}{\left(\alpha + \kappa\lambda\rho + \dfrac{1}{T_r} + \kappa\lambda T_r\right)(\alpha + \kappa\lambda\rho)}\sigma^2. \tag{2-89}$$

$C_{e\infty}$ can be made arbitrarily small by choosing λ and ρ large enough. For a given rms input voltage, it is possible to achieve an rms tracking error that is less than for Design I. The problem of how to choose the gains λ and ρ such that for a given rms input a minimal rms tracking error is obtained is a mathematical optimization problem.

In Chapter 3 we see how this optimization problem can be solved. At present we confine ourselves to an intuitive argument as follows. Let us suppose that for each value of λ the tachometer gain ρ is so chosen that the relative damping ζ is 0.7. Let us furthermore suppose that it is given that the steady-state rms input voltage should not exceed 30 V. Then by trial and error it can be found, using the formulas **2-88** and **2-89**, that for

$$\lambda = 500 \text{ V/rad}, \qquad \rho = 0.06 \text{ s}, \tag{2-90}$$

the steady-state rms tracking error is 0.1031 rad, while the steady-state rms input voltage is 30.64 V. These values of the gain yield a near-minimal rms tracking error for the given rms input. We observe that this design is better than Design I, where we achieved an rms tracking error of about 0.2 rad. Still Design II is not very good, since the rms tracking error of 0.1 rad is not very small as compared to the rms value of the reference variable of 1 rad.

This situation can be remedied by either replacing the motor by a more powerful one, or by lowering the bandwidth of the reference variable. The 10% cutoff frequency of the present closed-loop design is $0.071\omega_0 = 0.071\sqrt{\kappa\lambda} \simeq 1.41$ rad/s, where ω_0 is the break frequency of the system (see Table 2.1). This cutoff frequency is not large enough compared to the 1% cutoff frequency of 6.4 rad/s of the reference variable.

Design III. The third design proposed in Example 2.4 is an intermediate design: for $T_d = 0$ it reduces to Design II and for $T_d = \infty$ to Design I. For a given value of T_d, we expect its performance to lie in between that of the two other designs, which means that for a given rms input voltage an rms tracking error may be achieved that is less than that for Design I but larger than that for Design II.

From the point of view of tracking performance, T_d should of course be chosen as small as possible. A too small value of T_d, however, will unduly enhance the effect of the observation noise. In Example 2.11 (Section 2.8), which concludes the section on the effect of observation noise in the control system, we determine the most suitable value of T_d.

2.5.3 The Multiinput Multioutput Case

In this section we return to the case where the plant input, the controlled variable, and the reference variable are multidimensional variables, for which we rephrase the design objectives of Section 2.5.2.

When we first consider the steady-state mean square tracking error as given by **2-58**, we see that Design Objective 2.2 should be modified in the sense that

$$\text{tr}\,\{\Sigma_r(\omega)[T(-j\omega) - I]^T W_e[T(j\omega) - I]\} \qquad \textbf{2-91}$$

is to be made small for all real $\omega \geq 0$, and that when nonzero set points are likely to occur,

$$\text{tr}\,\{R_0[T(0) - I]^T W_e[T(0) - I]\} \qquad \textbf{2-92}$$

must be made small. Obviously, this objective is achieved when $T(j\omega)$ equals the unit matrix for all frequencies. It clearly is *sufficient*, however, that $T(j\omega)$ be close to the unit matrix for all frequencies for which $\Sigma_r(\omega)$ is significantly different from zero. In order to make this statement more precise, the following assumptions are made.

1. *The variable part of the reference variable is a stochastic process with uncorrelated components, so that its power spectral density matrix can be expressed as*

$$\Sigma_r(\omega) = \text{diag}\,[\Sigma_{r,1}(\omega), \Sigma_{r,2}(\omega), \cdots, \Sigma_{r,m}(\omega)]. \qquad \textbf{2-93}$$

2. *The constant part of the reference variable is a stochastic variable with uncorrelated components, so that its second-order moment matrix can be expressed as*

$$R_0 = \text{diag}\,(R_{0,1}, R_{0,2}, \cdots, R_{0,m}). \tag{2-94}$$

From a practical point of view, these assumptions are not very restrictive. By using **2-93** and **2-94**, it is easily found that the steady-state mean square tracking error can be expressed as

$$C_{e\infty} = \sum_{i=1}^{m} R_{0,i}\{[T(0) - I]^T W_e[T(0) - I]\}_{ii} + \sum_{i=1}^{m} \int_{-\infty}^{\infty} \Sigma_{r,i}(\omega)\{[T(-j\omega) - I]^T W_e[T(j\omega) - I]\}_{ii}\, df, \tag{2-95}$$

where

$$\{[T(-j\omega) - I]^T W_e[T(j\omega) - I]\}_{ii} \tag{2-96}$$

denotes the i-th diagonal element of the matrix $[T(-j\omega) - I]^T W_e[T(j\omega) - I]$.

Let us now consider one of the terms on the right-hand side of **2-95**:

$$\int_{-\infty}^{\infty} \Sigma_{r,i}(\omega)\{[T(-j\omega) - I]^T W_e[T(j\omega) - I]\}_{ii}\, df. \tag{2-97}$$

This expression describes the contribution of the i-th component of the reference variable to the tracking error as transmitted through the system. It is therefore appropriate to introduce the following notion.

Definition 2.4. *Let $T(s)$ be the $m \times m$ transmission of an asymptotically stable time-invariant linear control system. Then we define the* ***frequency band of the i-th link*** *of the control system as the set of frequencies ω, $\omega \geq 0$, for which*

$$\{[T(-j\omega) - I]^T W_e[T(j\omega) - I]\}_{ii} \leq \varepsilon^2 W_{e,ii}. \tag{2-98}$$

Here ε is a given number which is small with respect to 1, W_e is the weighting matrix for the mean square tracking error, and $W_{e,ii}$ denotes the i-th diagonal element of W_e.

Once the frequency band of the i-th link is established, we can of course define the *bandwidth* and the *cutoff frequency* of the i-th link, if they exist, as in Definition 2.2. It is noted that Definition 2.4 also holds for nondiagonal weighting matrices W_e. The reason that the magnitude of

$$\{[T(-j\omega) - I]^T W_e[T(j\omega) - I]\}_{ii}$$

is compared to $W_{e,ii}$ is that it is reasonable to compare the contribution **2-97** of the i-th component of the reference variable to the mean square tracking error to its contribution when no control is present, that is, when

$T(s) = 0$. This latter contribution is given by

$$\int_{-\infty}^{\infty} \Sigma_{r,i}(\omega) W_{e,ii} \, df. \tag{2-99}$$

We refer to the normalized function $\{[T(-j\omega) - I]^T W_e [T(j\omega) - I]\}_{ii} / W_{e,ii}$ as the *difference function* of the i-th link. In the single-input single-output case, this function is $|T(j\omega) - 1|^2$.

We are now in a position to extend Design Objective 2.2A as follows.

Design Objective 2.2B. *Let $T(s)$ be the $m \times m$ transmission of an asymptotically stable time-invariant linear control system for which both the constant part and the variable part of the reference variable have uncorrelated components. Then in order to obtain a small steady-state mean square tracking error, the frequency band of each of the m links should contain as much as possible of the frequency band of the corresponding component of the reference variable. If the i-th component, $i = 1, 2, \cdots, m$, of the reference variable is likely to have a nonzero set point, $\{[T(0) - I]^T W_e [T(0) - I]\}_{ii}$ should be made small as compared to $W_{e,ii}$.*

As an amendment to this rule, we observe that if the contribution to $C_{e\infty}$ of one particular term in the expression **2-95** is much larger than those of the remaining terms, then the advice of the objective should be applied more severely to the corresponding link than to the other links.

In view of the assumptions 1 and 2, it is not unreasonable to suppose that the weighting matrix W_e is diagonal, that is,

$$W_e = \text{diag}\,(W_{e,11}, W_{e,22}, \cdots, W_{e,mm}). \tag{2-100}$$

Then we can write

$$\{[T(-j\omega) - I]^T W_e [T(j\omega) - I]\}_{ii} = \sum_{l=1}^{m} |\{T(j\omega) - I\}_{li}|^2 \, W_{e,ll}, \qquad i = 1, 2, \cdots, m, \tag{2-101}$$

where $\{T(j\omega) - I\}_{li}$ denotes the (l, i)-th element of $T(j\omega) - I$. This shows that the frequency band of the i-th link is determined by the i-th column of the transmission $T(s)$.

It is easy to see, especially in the case where W_e is diagonal, that the design objective forces the diagonal elements of the transmission $T(j\omega)$ to be close to 1 over suitable frequency bands, while the off-diagonal elements are to be small in an appropriate sense. If all off-diagonal elements of $T(j\omega)$ are zero, that is, $T(j\omega)$ is diagonal, we say that the control system is completely *decoupled*. A control system that is not completely decoupled is said to exhibit *interaction*. A well-designed control system shows little interaction. A control system for which $T(0)$ is diagonal will be called *statically decoupled*.

We consider finally the steady-state mean square input. If the components of the reference variable are uncorrelated (assumptions 1 and 2), we can write

$$C_{u\infty} = \sum_{i=1}^{m} R_{0,i}\{N^T(0)W_uN(0)\}_{ii} + \sum_{i=1}^{m}\int_{-\infty}^{\infty}\Sigma_{r,i}(\omega)\{N^T(-j\omega)W_uN(j\omega)\}_{ii}\,df, \tag{2-102}$$

where $\{N^T(-j\omega)W_uN(j\omega)\}_{ii}$ is the i-th diagonal element of $N^T(-j\omega)\cdot W_uN(j\omega)$. This immediately leads to the following design objective.

Design Objective 2.3A. *In order to obtain a small steady-state mean square input in an asymptotically stable time-invariant linear control system with an m-dimensional reference variable with uncorrelated components,*

$$\{N^T(-j\omega)W_uN(j\omega)\}_{ii} \tag{2-103}$$

should be made small over the frequency band of the i-th component of the reference variable, for $i = 1, 2, \cdots, m$.

Again, as in Objective 2.3, we impose no special restrictions on $\{N^T(0)W_uN(0)\}_{ii}$ even if the i-th component of the reference variable is likely to have a nonzero set point, since only the fluctuations *about* the set point of the input need be restricted.

Example 2.8. *The control of a stirred tank*

Let us take up the problem of controlling a stirred tank, as described in Example 2.2 (Section 2.2.2). The linearized state differential equation is given in Example 1.2 (Section 1.2.3); it is

$$\dot{x}(t) = \begin{pmatrix} -0.01 & 0 \\ 0 & -0.02 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ -0.25 & 0.75 \end{pmatrix} u(t). \tag{2-104}$$

As the components of the controlled variable $z(t)$ we choose the outgoing flow and the outgoing concentration so that we write

$$z(t) = \begin{pmatrix} \zeta_1(t) \\ \zeta_2(t) \end{pmatrix} = \begin{pmatrix} 0.01 & 0 \\ 0 & 1 \end{pmatrix} x(t). \tag{2-105}$$

The reference variable $r(t)$ thus has as its components $\rho_1(t)$ and $\rho_2(t)$, the desired outgoing flow and the desired outgoing concentration, respectively.

We now propose the following simple controller. If the outgoing flow is too small, we adjust the flow of feed 1 proportionally to the difference between the actual flow and the desired flow; thus we let

$$\mu_1(t) = k_1[\rho_1(t) - \zeta_1(t)]. \tag{2-106}$$

However, if the outgoing concentration differs from the desired value, the flow of feed 2 is adjusted as follows:

$$\mu_2(t) = k_2[\rho_2(t) - \zeta_2(t)]. \tag{2-107}$$

Figure 2.20 gives a block diagram of this control scheme. The reason that

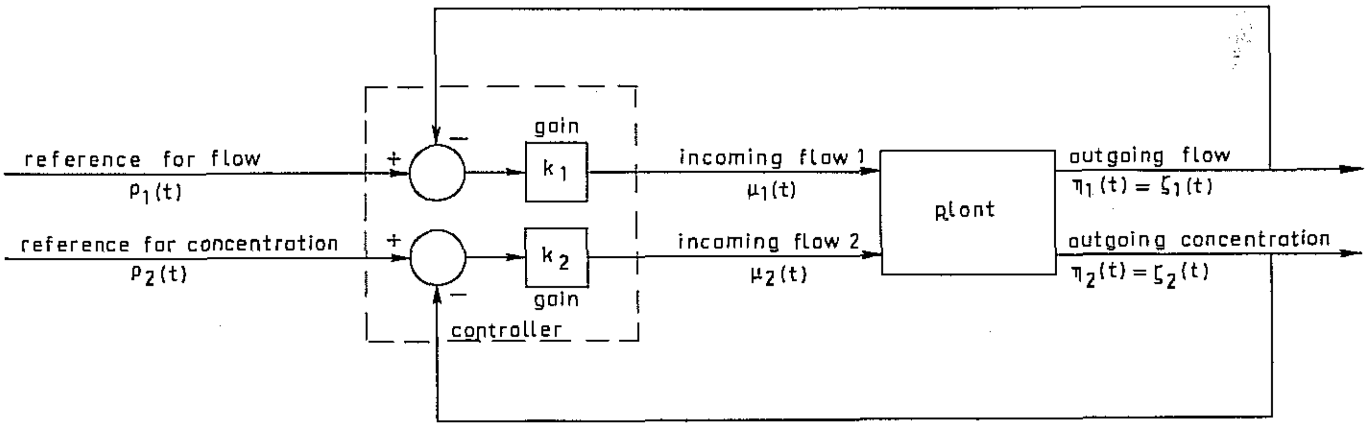

Fig. 2.20. A closed-loop control scheme for the stirred tank.

this simple scheme is expected to work is that feed 2 has a higher concentration than feed 1; thus the concentration is more sensitive to adjustments of the second flow. As a result, the first flow is more suitable for regulating the outgoing flow. However, since the second flow also affects the outgoing flow, and the first flow the concentration, a certain amount of interaction seems unavoidable in this scheme.

For this control system the various transfer matrices occurring in Fig. 2.14 can be expressed as follows:

$$K(s) = H(s) = \begin{pmatrix} \dfrac{0.01}{s + 0.01} & \dfrac{0.01}{s + 0.01} \\ \dfrac{-0.25}{s + 0.02} & \dfrac{0.75}{s + 0.02} \end{pmatrix}, \qquad \textbf{2-108}$$

$$P(s) = G(s) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

In Example 1.17 (Section 1.5.4), we found that the characteristic polynomial of the closed-loop system is given by

$$\phi_c(s) = s^2 + s(0.01k_1 + 0.75k_2 + 0.03) + (0.0002k_1 + 0.0075k_2 + 0.01k_1k_2 + 0.0002), \qquad \textbf{2-109}$$

from which we see that the closed-loop system is asymptotically stable for all positive values of the gains k_1 and k_2.

It can be found that the transmission of the system is given by

$$T(s) = K(s)[I + G(s)H(s)]^{-1}P(s) = \frac{1}{\phi_c(s)} \begin{pmatrix} 0.01k_1(s + k_2 + 0.02) & 0.01k_2(s + 0.02) \\ -0.25k_1(s + 0.01) & k_2(0.75s + 0.01k_1 + 0.0075) \end{pmatrix}. \qquad \textbf{2-110}$$

As a result, we find that

$$T(s) - I = \frac{1}{\phi_c(s)} \begin{pmatrix} -[s^2 + s(0.75k_2 + 0.03) + 0.0075k_2 + 0.0002] & 0.01k_2(s + 0.02) \\ -0.25k_1(s + 0.01) & -[s^2 + s(0.01k_1 + 0.03) + 0.0002k_1 + 0.0002] \end{pmatrix}. \qquad \textbf{2-111}$$

It is easy to see that if k_1 and k_2 simultaneously approach infinity then $[T(s) - I] \to 0$ so that perfect tracking is obtained.

The transfer matrix $N(s)$ can be found to be

$$N(s) = [I + G(s)H(s)]^{-1}P(s)$$
$$= \frac{1}{\phi_c(s)} \begin{pmatrix} k_1[s^2 + s(0.75k_2 + 0.03) + 0.0075k_2 + 0.0002] & -0.01k_1k_2(s + 0.02) \\ 0.25k_1k_2(s + 0.01) & k_2[s^2 + s(0.01k_1 + 0.03) + 0.0002k_1 + 0.0002] \end{pmatrix}. \quad \textbf{2-112}$$

When k_1 and k_2 simultaneously approach infinity,

$$N(s) \to \begin{pmatrix} 75(s + 0.01) & -(s + 0.02) \\ 25(s + 0.01) & s + 0.02 \end{pmatrix}, \quad \textbf{2-113}$$

which means that the steady-state mean square input $C_{u\infty}$ will be infinite unless the entries of $\Sigma_r(\omega)$ decrease fast enough with ω.

In order to find suitable values for the gains k_1 and k_2, we now apply Design Objective 2.2B and determine k_1 and k_2 so that the frequency bands of the two links of the system contain the frequency bands of the components of the reference variable. This is a complicated problem, however, and therefore we prefer to use a trial-and-error approach that is quite typical of the way multivariable control problems are commonly solved. This approach is as follows. To determine k_1 we assume that the second feedback link has not yet been connected. Similarly, in order to determine k_2, we assume that the first feedback link is disconnected. Thus we obtain two single-input single-output problems which are much easier to solve. Finally, the control system with both feedback links connected is analyzed and if necessary the design is revised.

When the second feedback link is disconnected, the transfer function from the first input to the first controlled variable is

$$H_{11}(s) = \frac{0.01}{s + 0.01}. \quad \textbf{2-114}$$

Proportional feedback according to **2-106** results in the following closed-loop transfer function from $\rho_1(t)$ to $\zeta_1(t)$:

$$\frac{0.01k_1}{s + 0.01k_1 + 0.01}. \quad \textbf{2-115}$$

We immediately observe that the zero-frequency transmission is different from 1; this can be remedied by inserting an extra gain f_1 into the connection from the first component of the reference variable as follows:

$$\mu_1(t) = k_1[f_1\rho_1(t) - \zeta_1(t)]. \quad \textbf{2-116}$$

With this **2-115** is modified to

$$\frac{0.01k_1f_1}{s + 0.01k_1 + 0.01}. \qquad \textbf{2-117}$$

For each value of k_1, it is possible to choose f_1 so that the zero-frequency transmission is 1. Now the value of k_1 depends upon the cutoff frequency desired. For $k_1 = 10$ the 10% cutoff frequency is 0.011 rad/s (see Table 2.1). Let us assume that this is sufficient for the purpose of the control system. The corresponding value that should be chosen for f_1 is 1.1.

When studying the second link in a similar manner, it can be found that the feedback scheme

$$\mu_2(t) = k_2[f_2\rho_2(t) - \zeta_2(t)] \qquad \textbf{2-118}$$

results in the following closed-loop transfer function from $\rho_2(t)$ to $\zeta_2(t)$ (assuming that the first feedback link is disconnected):

$$\frac{0.75k_2f_2}{s + 0.75k_2 + 0.02}. \qquad \textbf{2-119}$$

For $k_2 = 0.1$ and $f_2 = 1.267$, the zero-frequency transmission is 1 and the 10% cutoff frequency 0.0095 rad/s.

Let us now investigate how the multivariable control system with

$$G(s) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 0.1 \end{pmatrix} \qquad \textbf{2-120}$$

and

$$P(s) = \begin{pmatrix} k_1f_1 & 0 \\ 0 & k_2f_2 \end{pmatrix} = \begin{pmatrix} 11 & 0 \\ 0 & 0.1267 \end{pmatrix} \qquad \textbf{2-121}$$

performs. It can be found that the control system transmission is given by

$$T(s) = \frac{1}{s^2 + 0.205s + 0.01295} \begin{pmatrix} 0.11s + 0.0132 & 0.001267s + 0.00002534 \\ -2.75s - 0.0275 & 0.09502s + 0.01362 \end{pmatrix}, \qquad \textbf{2-122}$$

hence that

$$T(s) - I = \frac{1}{s^2 + 0.205s + 0.01295} \cdot \begin{pmatrix} -s^2 - 0.095s + 0.00025 & 0.001267s + 0.00002534 \\ -2.75s - 0.0275 & -s^2 - 0.1100s + 0.00067 \end{pmatrix}. \qquad \textbf{2-123}$$

Now in order to determine the frequency bands of the two links of the control system, we must first choose the weighting matrix W_e. The two controlled variables are the outgoing flow and the outgoing concentration. The flow has the constant nominal value 0.02 m^3/s, while the concentration has the constant nominal value 1.25 $kmol/m^3$. A 10% change in the flow therefore corresponds to 0.002 m^3/s, while a 10% change in the concentration is about 0.1 $kmol/m^3$. Now let us suppose that we make the weighting matrix W_e diagonal, with diagonal entries $W_{e,1}$ and $W_{e,2}$. Let us also assume that 10% changes in either the flow or the concentration make equal contributions to the mean square tracking error. Then we have

$$(0.002)^2 W_{e,1} = (0.1)^2 W_{e,2}, \tag{2-124}$$

or

$$\frac{W_{e,1}}{W_{e,2}} = 2500. \tag{2-125}$$

Let us therefore choose

$$W_e = \text{diag}\,(50, 0.02). \tag{2-126}$$

Since W_e is diagonal, we can use **2-101** to determine the frequency band of the i-th link. The frequency band of the *first* link (the flow link) thus follows from considering the inequality

$$50 \left| \frac{(j\omega)^2 + 0.095(j\omega) - 0.00025}{(j\omega)^2 + 0.205(j\omega) + 0.01295} \right|^2 + 0.02 \left| \frac{2.75(j\omega) + 0.0275}{(j\omega)^2 + 0.205(j\omega) + 0.01295} \right|^2 \le 50\varepsilon^2. \tag{2-127}$$

Dividing by 50 and rearranging, we obtain

$$\frac{|(j\omega)^2 + 0.095(j\omega) - 0.00025|^2 + 0.0004\,|2.75(j\omega) + 0.0275|^2}{|(j\omega)^2 + 0.205(j\omega) + 0.01295|^2} \le \varepsilon^2. \tag{2-128}$$

Figure 2.21 shows a Bode plot of the left-hand side of this inequality, which is precisely the difference function of the first link. It is seen that ε cannot be

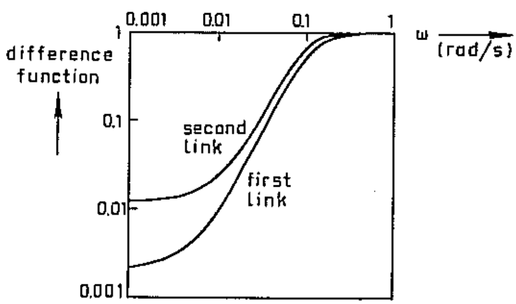

Fig. 2.21. Difference functions of the first and the second link of the stirred-tank control system.

chosen arbitrarily small since the left-hand side of **2-128** is bounded from below. For $\varepsilon = 0.1$ the cutoff frequency is about 0.01 rad/s. The horizontal part of the curve at low frequencies is mainly attributable to the second term in the numerator of **2-128**, which originates from the off-diagonal entry in the first column of $T(j\omega) - I$. This entry represents part of the interaction present in the system.

We now consider the *second* link (the concentration link). Its frequency band follows from the inequality

$$50\left|\frac{0.001267(j\omega) + 0.00002534}{(j\omega)^2 + 0.205(j\omega) + 0.01295}\right|^2 + 0.02\left|\frac{(j\omega)^2 + 0.1100(j\omega) - 0.00067}{(j\omega)^2 + 0.205(j\omega) + 0.01295}\right|^2 \le 0.02\varepsilon^2. \quad \textbf{2-129}$$

By dividing by 0.02 and rearranging, it follows for this inequality,

$$\frac{|(j\omega)^2 + 0.1100(j\omega) - 0.00067|^2 + 2500\,|0.001267(j\omega) + 0.00002534|^2}{|(j\omega)^2 + 0.205(j\omega) + 0.01295|^2} \le \varepsilon^2. \quad \textbf{2-130}$$

The Bode plot of the left-hand side of this inequality, which is the difference function of the second link, is also shown in Fig. 2.21. In this case as well, the horizontal part of the curve at low frequencies is caused by the interaction in the system. If the requirements on ε are not too severe, the cutoff frequency of the second link is somewhere near 0.01 rad/s.

The cutoff frequencies obtained are reasonably close to the 10% cutoff frequencies of 0.011 rad/s and 0.0095 rad/s of the single-loop designs. Moreover, the interaction in the system seems to be limited. In conclusion, Fig. 2.22 pictures the step response matrix of the control system. The plots confirm that the control system exhibits moderate interaction (both dynamic and static). Each link has the step response of a first-order system with a time constant of approximately 10 s.

A rough idea of the resulting input amplitudes can be obtained as follows. From **2-116** we see that a step of 0.002 m^3/s in the flow (assuming that this is a typical value) results in an initial flow change in feed 1 of $k_1 f_1 0.002 = 0.022$ m^3/s. Similarly, a step of 0.1 kmol/m^3 in the concentration results in an initial flow change in feed 2 of $k_2 f_2 0.1 = 0.01267$ m^3/s. Compared to the nominal values of the incoming flows (0.015 m^3/s and 0.005 m^3/s, respectively), these values are far too large, which means that either smaller step input amplitudes must be chosen or the desired transition must be made more gradually. The latter can be achieved by redesigning the control system with smaller bandwidths.

In Problem 2.2 a more sophisticated design of a controller for the stirred tank is considered.

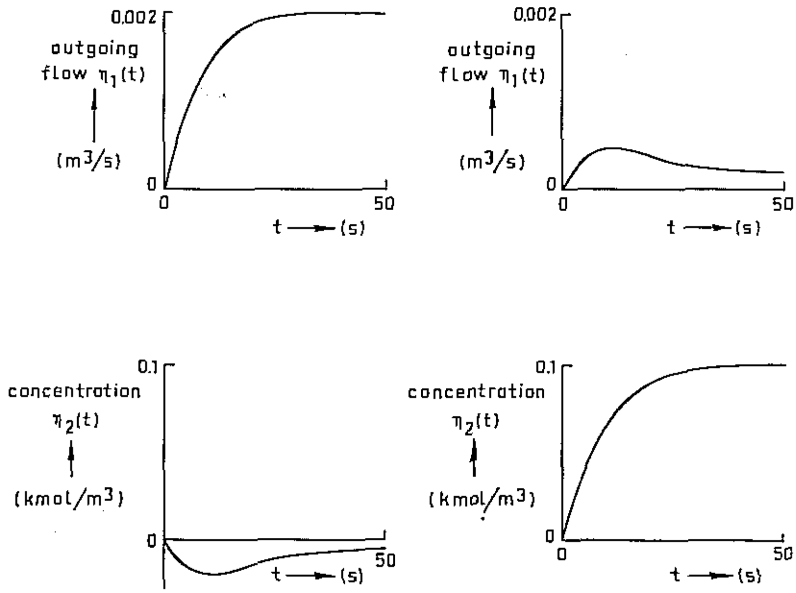

Fig. 2.22. Step response matrix of the stirred-tank control system. Left column: Responses of the outgoing flow and concentration to a step of 0.002 m^3/s in the set point of the flow. Right column: Responses of the outgoing flow and concentration to a step of 0.1 $kmol/m^3$ in the set point of the concentration.

2.6 THE TRANSIENT ANALYSIS OF THE TRACKING PROPERTIES

In the previous section we quite extensively discussed the steady-state properties of tracking systems. This section is devoted to the *transient* behavior of tracking systems, in particular that of the mean square tracking error and the mean square input. We define the *settling time* of a certain quantity (be it the mean square tracking error, the mean square input, or any other variable) as the time it takes the variable to reach its steady-state value to within a specified accuracy. When this accuracy is, say, 1 % of the maximal deviation from the steady-state value, we speak of the 1 % *settling time*. For other percentages similar terminology is used.

Usually, when a control system is started the initial tracking error, and as a result the initial input also, is large. Obviously, it is desirable that the mean square tracking error settles down to its steady-state value as quickly as possible after starting up or after upsets. We thus formulate the following directive.

Design Objective 2.4. *A control system should be so designed that the settling time of the mean square tracking error is as short as possible.*

As we have seen in Section 2.5.1, the mean square tracking error attributable to the reference variable consists of two contributions. One originates from the constant part of the reference variable and the other from the variable part. The transient behavior of the contribution of the variable part must be found by solving the matrix differential equation for the variance matrix of the state of the control system, which is fairly laborious. The transient behavior of the contribution of the constant part of the reference variable to the mean square tracking error is much simpler to find; this can be done simply by evaluating the response of the control system to nonzero initial conditions and to steps in the reference variable. As a rule, computing these responses gives a very good impression of the transient behavior of the control system, and this is what we usually do.

For asymptotically stable time-invariant linear control systems, some information concerning settling times can often be derived from the locations of the closed-loop poles. This follows by noting that *all* responses are exponentially damped motions with time constants that are the negative reciprocals of the real parts of the closed-loop characteristic values of the system. Since the 1% settling time of

$$e^{-t/\theta}, \; t \geq 0, \qquad \textbf{2-131}$$

is 4.6θ, a bound for the 1% settling time t_s of any variable is

$$t_s \leq 4.6 \max_i \left\{\frac{1}{|\text{Re}(\lambda_i)|}\right\}, \qquad \textbf{2-132}$$

where λ_i, $i = 1, 2, \cdots, n$, are the closed-loop characteristic values. Note that for *squared* variables such as the mean square tracking error and the mean square input, the settling time is half that of the variable itself.

The bound **2-132** sometimes gives misleading results, since it may easily happen that the response of a given variable does not depend upon certain characteristic values. Later (Section 3.8) we meet instances, for example, where the settling time of the rms tracking error is determined by the closed-loop poles furthest from the origin and not by the nearby poles, while the settling time of the rms input derives from the nearby closed-loop poles.

Example 2.9. *The settling time of the tracking error of the position servo*

Let us consider Design I of Example 2.4 (Section 2.3) for the position servo. From the steady-state analysis in Example 2.7 (Section 2.5.2), we learned that as the gain λ increases the rms steady-state tracking error keeps decreasing, although beyond a certain value (15–25 V/rad) very little improvement in the rms tracking error is obtained, while the rms input voltage becomes larger and larger. We now consider the settling time of the tracking error. To this end, in Fig. 2.23 the response of the controlled variable to a step in the reference variable is plotted for various values of λ, from zero initial conditions. As can be seen, the settling time of the step response (hence also that of the tracking error) first decreases rapidly as λ increases, but beyond a value of λ of about 15 V/rad the settling time fails to improve because of the increasingly oscillatory behavior of the response. In this case

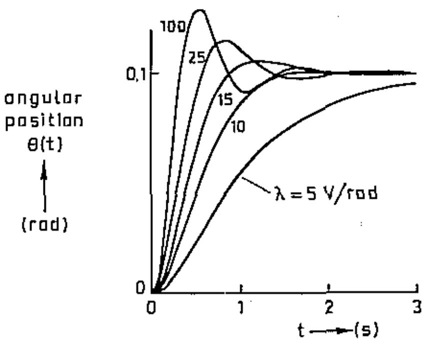

Fig. 2.23. Response of Design I of the position servo to a step of 0.1 rad in the reference variable for various values of the gain λ.

as well, the most favorable value of λ seems to be about 15 V/rad, which corresponds to a relative damping ζ (see Example 2.7) of about 0.7. From the plots of $|T(j\omega)|$ of Fig. 2.18, we see that for this value of the gain the largest bandwidth is achieved without undesirable peaking of the transmission.

2.7 THE EFFECTS OF DISTURBANCES IN THE SINGLE-INPUT SINGLE-OUTPUT CASE

In Section 2.3 we saw that very often disturbances act upon a control system, adversely affecting its tracking or regulating performance. In this section we derive expressions for the increases in the steady-state mean square tracking error and the steady-state mean square input attributable to disturbances, and formulate design objectives which may serve as a guide in designing control systems capable of counteracting disturbances.

Throughout this section the following assumptions are made.

1. *The disturbance variable $v_p(t)$ is a stochastic process that is uncorrelated with the reference variable $r(t)$ and the observation noise $v_m(t)$.*
As a result, we can obtain the increase in the mean square tracking error and the mean square input simply by setting $r(t)$ and $v_m(t)$ identical to zero.

2. *The controlled variable is also the observed variable, that is, $C = D$.*

This means that we can write

$$y(t) = z(t) + v_m(t), \qquad \textbf{2-133}$$

and that in the time-invariant case

$$H(s) = K(s). \qquad \textbf{2-134}$$

The assumption that the controlled variable is also the observed variable is quite reasonable, since it is intuitively clear that feedback is most effective when the controlled variable itself is directly fed back.

3. *The control system is asymptotically stable and time-invariant.*

4. *The input variable and the controlled variable, hence also the reference variable, are scalars.* W_e *and* W_u *are both* 1.

The analysis of this section can be extended to multivariable systems but doing so adds very little to the conclusions of this and the following sections.

5. *The disturbance variable* $v_p(t)$ *can be written as*

$$v_p(t) = v_{p0} + v_{pv}(t), \qquad \textbf{2-135}$$

where the constant part v_{p0} *of the disturbance variable is a stochastic vector with given second-order moment matrix, and where the variable part* $v_{pv}(t)$ *of the disturbance variable is a wide-sense stationary zero mean stochastic process with power spectral density matrix* $\Sigma_{vp}(\omega)$, *uncorrelated with* v_{p0}.

The transfer matrix from the disturbance variable $v_p(t)$ to the controlled variable $z(t)$ can be found from the relation (see Fig. 2.24)

$$\mathbf{Z}(s) = -H(s)G(s)\mathbf{Z}(s) + D(sI - A)^{-1}\mathbf{V}_p(s), \qquad \textbf{2-136}$$

where $\mathbf{Z}(s)$ and $\mathbf{V}_p(s)$ denote the Laplace transforms of $z(t)$ and $v_p(t)$,

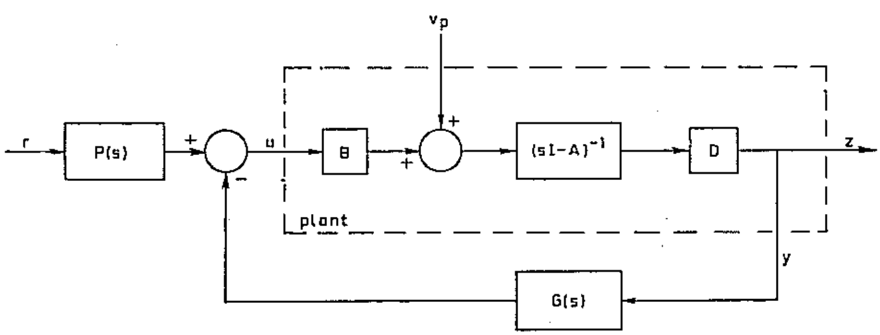

Fig. 2.24. Transfer matrix block diagram of a closed-loop control system with plant disturbance v_p.

respectively, so that

$$\mathbf{Z}(s) = \frac{1}{1 + H(s)G(s)} D(sI - A)^{-1}\mathbf{V}_p(s). \qquad \textbf{2-137}$$

Here we have used the fact that the controlled variable is a scalar so that $1 + H(s)G(s)$ is also a scalar function. We now introduce the function

$$S(s) = \frac{1}{1 + H(s)G(s)}, \qquad \textbf{2-138}$$

which we call the *sensitivity function* of the control system for reasons to be explained later.

We compute the contribution of the disturbance variable to the steady-state mean square tracking error as the sum of two terms, one originating from the constant part and one from the variable part of the disturbance. Since

$$\mathbf{Z}(s) = S(s)D(sI - A)^{-1}\mathbf{V}_p(s), \qquad \textbf{2-139}$$

the steady-state response of the controlled variable to the constant part of the disturbance is given by

$$\lim_{t\to\infty} z(t) = S(0)D(-A)^{-1}v_{p0} = S(0)v_{00}. \qquad \textbf{2-140}$$

Here we have assumed that the matrix A is nonsingular—the case where A is singular is treated in Problem 2.4. Furthermore, we have abbreviated

$$v_{00} = D(-A)^{-1}v_{p0}. \qquad \textbf{2-141}$$

As a result of **2-140**, the contribution of the constant part of the disturbance to the steady-state mean square tracking error is

$$E\{|S(0)v_{00}|^2\} = |S(0)|^2\, V_0, \qquad \textbf{2-142}$$

where V_0 is the second-order moment of v_{00}, that is, $V_0 = E\{v_{00}^2\}$. Furthermore it follows from **2-139** with the methods of Sections 1.10.4 and 1.10.3 that the contribution of the variable part of the disturbance to the steady-state mean square tracking error can be expressed as

$$\int_{-\infty}^{\infty} |S(j\omega)|^2\, D(j\omega I - A)^{-1}\Sigma_{vp}(\omega)(-j\omega I - A^T)^{-1}D^T\, df = \int_{-\infty}^{\infty} |S(j\omega)|^2\Sigma_{v0}(\omega)\, df. \qquad \textbf{2-143}$$

Here we have abbreviated

$$\Sigma_{v0}(\omega) = D(j\omega I - A)^{-1}\Sigma_{vp}(\omega)(-j\omega I - A^T)^{-1}D^T. \qquad \textbf{2-144}$$

Consequently, the increase in the steady-state mean square tracking error attributable to the disturbance is given by

$$C_{e\infty} \text{ (with disturbance)} - C_{e\infty} \text{ (without disturbance)}$$
$$= |S(0)|^2 V_0 + \int_{-\infty}^{\infty} |S(j\omega)|^2 \Sigma_{v0}(\omega)\, df. \qquad \textbf{2-145}$$

Before discussing how to make this expression small, we give an interpretation. Consider the situation of Fig. 2.25 where a variable $v_0(t)$ acts upon the closed-loop system. This variable is added to the controlled variable. It is

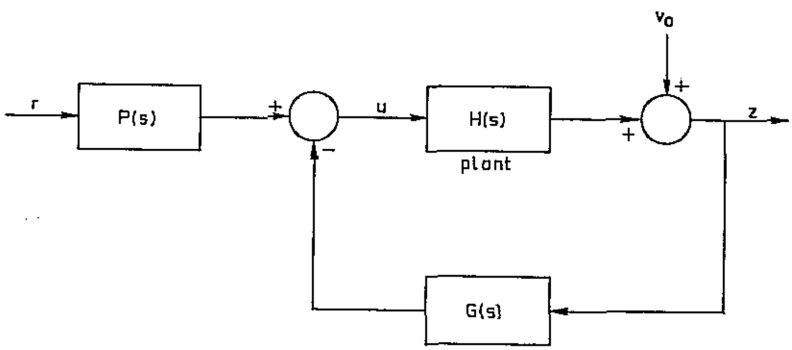

Fig. 2.25. Transfer matrix block diagram of a closed-loop control system with the equivalent disturbance v_0 at the controlled variable.

easily found that in terms of Laplace transforms with the reference variable and the initial conditions identical to zero the controlled variable is given by

$$\mathbf{Z}(s) = S(s)\mathbf{V}_0(s), \qquad \textbf{2-146}$$

where $\mathbf{V}_0(s)$ denotes the Laplace transform of $v_0(t)$. We immediately see that if $v_0(t)$ is a stochastic process with as constant part a stochastic variable with second-order moment V_0 and as variable part a zero-mean wide-sense stationary stochastic process with power spectral density $\Sigma_{v0}(\omega)$, the increase in the steady-state mean square tracking error is exactly given by **2-145**. We therefore call the process $v_0(t)$ with these properties the *equivalent disturbance at the controlled variable.*

An examination of **2-145** leads to the following design rule.

Design Objective 2.5. *In order to reduce the increase of the steady-state mean square tracking error attributable to disturbances in an asymptotically stable linear time-invariant control system with a scalar controlled variable, which is also the observed variable, the absolute value of the sensitivity function $S(j\omega)$ should be made small over the frequency band of the equivalent disturbance at* *the controlled variable. If constant errors are of special concern,* $S(0)$ *should be made small, preferably zero.*

The last sentence of this design rule is not valid without further qualification for control systems where the matrix A of the plant is singular; this case is discussed in Problem 2.4. It is noted that since $S(j\omega)$ is given by

$$S(j\omega) = \frac{1}{1 + H(j\omega)G(j\omega)}, \tag{2-147}$$

a small $S(j\omega)$ generally must be achieved by making the loop gain $H(j\omega)G(j\omega)$ of the control system large over a suitable frequency range. This easily conflicts with Design Objective 2.1 (Section 2.4) concerning the stability of the control system (see Example 2.5, Section 2.4), ~~and with Objective 2.3 (Section 2.5.2) concerning the mean square input~~. A compromise must be found.

Reduction of constant errors is of special importance for regulator and tracking systems where the set point of the controlled variable must be maintained with great precision. Constant disturbances occur very easily in control systems, especially because of errors made in establishing the nominal input. Constant errors can often be completely eliminated by making $S(0) = 0$, which is usually achieved by introducing *integrating action*, that is, by letting the controller transfer function $G(s)$ have a pole at the origin (see Problem 2.3).

Let us now turn to a consideration of the steady-state mean square input. It is easily found that in terms of Laplace transforms we can write (see Fig. 2.24)

$$\mathrm{U}(s) = \frac{-G(s)}{1 + H(s)G(s)} D(sI - A)^{-1}\mathrm{V}_p(s), \tag{2-148}$$

where $\mathrm{U}(s)$ is the Laplace transform of $u(t)$. It follows for the increase in the steady-state mean square input, using the notation introduced earlier in this section,

$$C_{u\infty} \text{ (with disturbance)} - C_{u\infty} \text{ (without disturbance)}$$
$$= \left|\frac{G(0)}{1 + H(0)G(0)}\right|^2 V_0 + \int_{-\infty}^{\infty} \left|\frac{G(j\omega)}{1 + H(j\omega)G(j\omega)}\right|^2 \Sigma_{v0}(\omega)\, df. \tag{2-149}$$

This expression results in the following directive.

Design Objective 2.6. *In order to obtain a small increase in the steady-state mean square input attributable to the disturbance in an asymptotically stable linear time-invariant control system with a scalar controlled variable that is* *also the observed variable and a scalar input,*

$$\left|\frac{G(j\omega)}{1 + H(j\omega)G(j\omega)}\right| \qquad \textbf{2-150}$$

should be made small over the frequency band of the equivalent disturbance at the controlled variable.

In this directive no attention is paid to the constant part of the input since, as assumed in the discussion of Objective 2.3, the plant must be able to sustain these constant deviations.

Design Objective 2.6 conflicts with Objective 2.5. Making the loop gain $H(j\omega)G(j\omega)$ large, as required by Objective 2.5, usually does not result in small values of **2-150**. Again a compromise must be found.

Example 2.10. *The effect of disturbances on the position servo*

In this example we study the effect of disturbances on Design I of the position servo of Example 2.4 (Section 2.3). It is easily found that the sensitivity function of the control system as proposed is given by

$$S(s) = \frac{s(s + \alpha)}{s^2 + \alpha s + \kappa\lambda}. \qquad \textbf{2-151}$$

In Fig. 2.26 Bode plots of $|S(j\omega)|$ are given for several values of the gain λ. It is seen that by choosing λ larger the frequency band over which disturbance suppression is obtained also becomes larger. If the equivalent disturbance at the controlled variable, however, has much power near the frequency where $|S(j\omega)|$ has its peak, then perhaps a smaller gain is advisable.

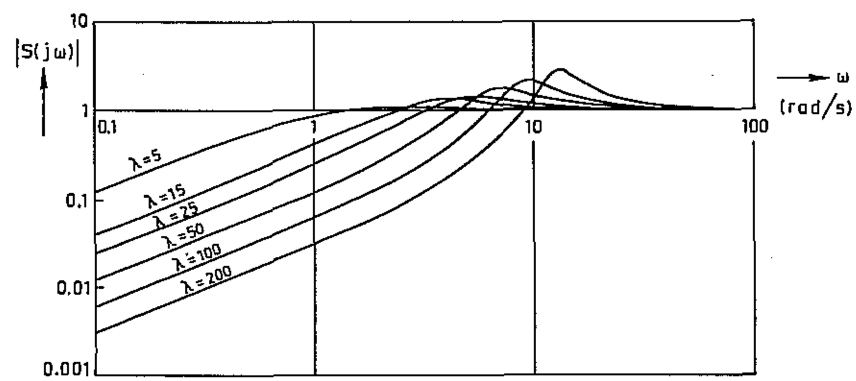

Fig. 2.26. Bode plots of the sensitivity function of the position control system, Design I, as a function of the gain λ.

In Example 2.4 we assumed that the disturbance enters as a disturbing torque $\tau_d(t)$ acting on the shaft of the motor. If the variable part of this disturbing torque has the power spectral density function $\Sigma_{\tau d}(\omega)$, the variable part of the equivalent disturbance at the controlled variable has the power spectral density function

$$\left|\frac{\gamma}{j\omega(j\omega + \alpha)}\right|^2 \Sigma_{\tau d}(\omega). \qquad \textbf{2-152}$$

The power spectral density of the contribution of the disturbing torque to the controlled variable is found by multiplying **2-152** by $|S(j\omega)|^2$ and thus is given by

$$\left|\frac{\gamma}{(j\omega)^2 + \alpha(j\omega) + \kappa\lambda}\right|^2 \Sigma_{\tau d}(\omega). \qquad \textbf{2-153}$$

Let us suppose that the variable part of the disturbing torque can be represented as exponentially correlated noise with rms value $\sigma_{\tau d}$ and time constant $T_{\tau d}$ so that

$$\Sigma_{\tau d}(\omega) = \frac{2\sigma_{\tau d}^2 T_{\tau d}}{1 + \omega^2 T_{\tau d}^2}. \qquad \textbf{2-154}$$

The increase in the steady-state mean square tracking error attributable to the disturbing torque can be computed by integrating **2-153**, or by modeling the disturbance, augmenting the state differential equation, and solving for the steady-state variance matrix of the augmented state. Either way we find

$$C_{e\infty} \text{ (with disturbing torque)} - C_{e\infty} \text{ (without disturbing torque)}$$
$$= \frac{1 + \alpha T_{\tau d}}{1 + \alpha T_{\tau d} + \kappa\lambda T_{\tau d}^2} \frac{\gamma^2 T_{\tau d}}{\alpha\kappa\lambda} \sigma_{\tau d}^2. \qquad \textbf{2-155}$$

From this we see that the addition to $C_{e\infty}$ monotonically decreases to zero with increasing λ. Thus the larger λ the less the disturbing torque affects the tracking properties.

In the absence of the reference variable, we have $\mu(t) = -\lambda\eta(t)$ so that the increase in the mean square input voltage attributable to the disturbing torque is λ^2 times the increase in the mean square tracking error:

$$C_{u\infty} \text{ (with disturbing torque)} - C_{u\infty} \text{ (without disturbing torque)}$$
$$= \frac{(1 + \alpha T_{\tau d})\lambda}{1 + \alpha T_{\tau d} + \kappa\lambda T_{\tau d}^2} \frac{\gamma^2 T_{\tau d}}{\alpha\kappa} \sigma_{\tau d}^2. \qquad \textbf{2-156}$$

For $\lambda \to \infty$, $C_{u\infty}$ monotonically increases to

$$\frac{(1 + \alpha T_{\tau d})\gamma^2}{\alpha T_{\tau d}\kappa^2} \sigma_{\tau d}^2. \qquad \textbf{2-157}$$

It is easily found from **2-25** that a *constant* disturbing torque τ_0 results in a steady-state displacement of the controlled variable of

$$\frac{\gamma\tau_0}{\kappa\lambda}. \tag{2-158}$$

Clearly, this displacement can also be made arbitrarily small by making the gain λ sufficiently large.

2.8 THE EFFECTS OF OBSERVATION NOISE IN THE SINGLE-INPUT SINGLE-OUTPUT CASE

In any closed-loop scheme, the effect of observation noise is to some extent felt. In this section the contribution of the observation noise to the mean square tracking error and the mean square input is analyzed. To this end, the following assumptions are made.

1. *The observation noise $v_m(t)$ is a stochastic process which is uncorrelated with the reference variable $r(t)$ and the plant disturbance $v_p(t)$.*

As a result, the increase in the mean square tracking error and the mean square input attributable to the observation noise may be computed simply by setting $r(t)$ and $v_p(t)$ identical to zero.

2. *The controller variable is also the observed variable, that is, $C = D$, so that*

$$y(t) = z(t) + v_m(t), \tag{2-159}$$

and, in the time-invariant case,

$$H(s) = K(s). \tag{2-160}$$

3. *The control system is asymptotically stable and time-invariant.*
4. *The input variable and the controlled variable, hence also the reference variable, are scalars. W_e and W_u are both* 1.

Here also the analysis can be extended to multivariable systems but again very little additional insight is gained.

5. *The observation noise is a zero-mean wide-sense stationary stochastic process with power spectral density function $\Sigma_{vm}(\omega)$.*

Figure 2.27 gives a transfer function block diagram of the situation that results from these assumptions. It is seen that in terms of Laplace transforms

$$\mathrm{Z}(s) = -H(s)G(s)[\mathrm{V}_m(s) + \mathrm{Z}(s)], \tag{2-161}$$

so that

$$\mathrm{Z}(s) = -\frac{H(s)G(s)}{1 + H(s)G(s)}\mathrm{V}_m(s). \tag{2-162}$$

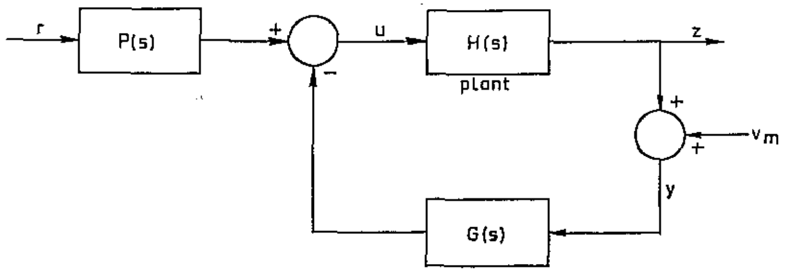

Fig. 2.27. Transfer matrix block diagram of a closed-loop control system with observation noise.

Consequently, the increase in the steady-state mean square tracking error attributable to the observation noise can be written as

$$C_{e\infty} \text{ (with observation noise)} - C_{e\infty} \text{ (without observation noise)}$$

$$= \int_{-\infty}^{\infty} \left| \frac{H(j\omega)G(j\omega)}{1 + H(j\omega)G(j\omega)} \right|^2 \Sigma_{vm}(\omega)\, df. \qquad \textbf{2-163}$$

Our next design objective can thus be formulated as follows.

Design Objective 2.7. *In order to reduce the increase in the steady-state mean square tracking error attributable to observation noise in an asymptotically stable linear time-invariant control system with a scalar controlled variable that is also the observed variable, the system should be designed so that*

$$\left| \frac{H(j\omega)G(j\omega)}{1 + H(j\omega)G(j\omega)} \right| \qquad \textbf{2-164}$$

is small over the frequency band of the observation noise.

Obviously, this objective is in conflict with Objective 2.5, since making the loop gain $H(j\omega)G(j\omega)$ large, as required by Objective 2.5, results in a value of **2-164** that is near 1, which means that the observation noise appears unattenuated in the tracking error. This is a result of the fact that if a large loop gain $H(j\omega)G(j\omega)$ is used the system is so controlled that $z(t) + v_m(t)$ instead of $z(t)$ tracks the reference variable.

A simple computation shows that the transfer function from the observation noise to the plant input is given by

$$\mathrm{U}(s) = -\frac{G(s)}{1 + G(s)H(s)} \mathrm{V}_m(s), \qquad \textbf{2-165}$$

which results in the following increase in the steady-state mean square input attributable to observation noise:

$C_{u\infty}$ (with observation noise) $-$ $C_{u\infty}$ (without observation noise)

$$= \int_{-\infty}^{\infty} \left| \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)} \right|^2 \Sigma_{vm}(\omega)\, df. \tag{2-166}$$

This yields the design rule that to make the increase in the steady-state mean square input attributable to the observation noise small,

$$\left| \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)} \right| \tag{2-167}$$

should be made small over the frequency band of the observation noise. Clearly, this rule is also in conflict with Objective 2.5.

Example 2.11. *The position servo with position feedback only*

Let us once again consider the position servo of Example 2.4 (Section 2.3) with the three different designs proposed. In Examples 2.7 (Section 2.5.2) and 2.9 (Section 2.6), we analyzed Design I and chose $\lambda = 15$ V/rad as the best value of the gain. In Example 2.7 it was found that Design II gives better performance because of the additional feedback link from the angular velocity. Let us now suppose, however, that for some reason (financial or technical) a tachometer cannot be installed. We then resort to Design III, which attempts to approximate Design II by using an approximate differentiator with time constant T_d. If no observation noise were present, we could choose $T_d = 0$ and Design III would reduce to Design II. Let us suppose that observation noise is present, however, and that is can be represented as exponentially correlated noise with time constant

$$T_m = 0.02 \text{ s} \tag{2-168}$$

and rms value

$$\sigma_m = 0.001 \text{ rad}. \tag{2-169}$$

The presence of the observation noise forces us to choose $T_d > 0$. In order to determine a suitable value of T_d, we first assume that T_d will turn out to be small enough so that the gains ρ and λ can be chosen as in Design II. Then we see how large T_d can be made without spoiling the performance of Design II, while at the same time sufficiently reducing the effect of the observation noise.

It is easily found that the transmission of the control system according to Design III is given by

$$T(s) = \frac{\kappa\lambda(T_d s + 1)}{T_d s^3 + (\alpha T_d + 1)s^2 + (\alpha + \kappa\lambda T_d + \rho\lambda\kappa)s + \lambda\kappa}. \tag{2-170}$$

To determine a suitably small value of T_d, we argue as follows. The closed-loop system according to Design II, with the numerical values obtained in Example 2.7 for λ and ρ, has an undamped natural frequency ω_0 of about 20 rad/s with a relative damping of 0.707. Now in order not to impede the behavior of the system, the time constant T_d of the differentiator should be chosen small with respect to the inverse natural frequency, that is, small with respect to 0.05 s. In Fig. 2.28 we have plotted the transmission **2-170** for

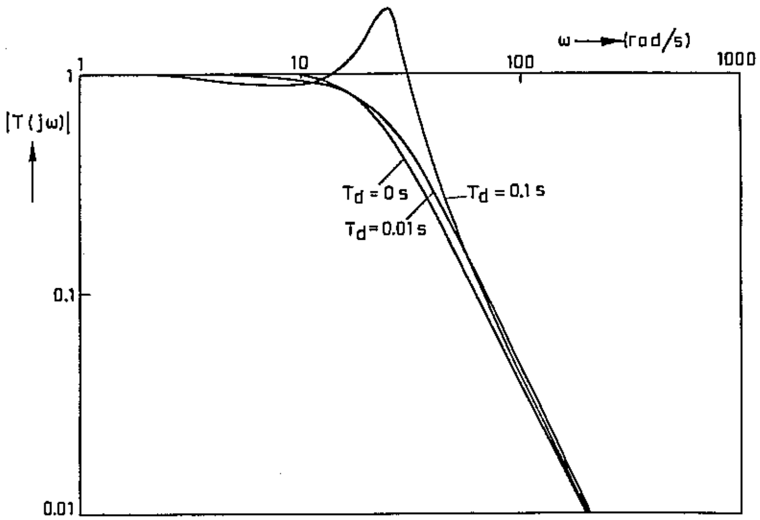

Fig. 2.28. The effect of T_d on the transmission of Design III of the position servo.

various values of T_d. It is seen that for $T_d = 0.01$ s the transmission is hardly affected by the approximate derivative operation, but that for $T_d = 0.1$ s discrepancies occur.

Let us now consider the effect of the observation noise. Modeling $v_m(t)$ in the usual way, the additions to the steady-state mean square tracking error and input attributable to the observation noise can be computed from the variance matrix of the augmented state. The numerical results are plotted in Fig. 2.29. These plots show that for small T_d the steady-state mean square input is greatly increased. An acceptable value of T_d seems to be about 0.01 s. For this value the square root of the increase in the steady-state mean square input is only about 2 V, the square root of the increase in the steady-state mean square tracking error of about 0.0008 rad is very small, and the transmission of the control system is hardly affected.

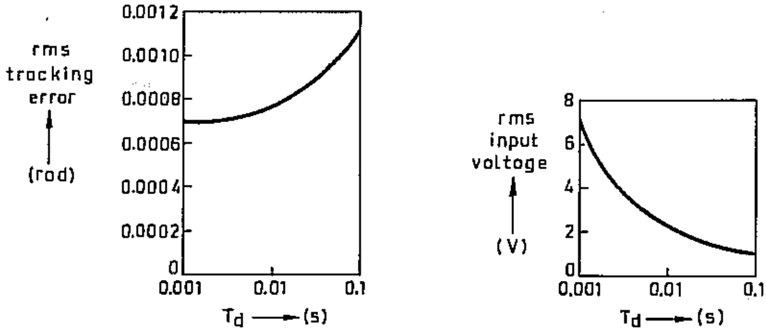

Fig. 2.29. The square roots of the additions to the steady-state mean square tracking error and input voltage due to observation noise as a function of T_d for Design III of the position servo.

2.9 THE EFFECT OF PLANT PARAMETER UNCERTAINTY IN THE SINGLE-INPUT SINGLE-OUTPUT CASE

Quite often a control system must be designed for a plant whose parameters are not exactly known to the designer. Also, it may happen in practice that changes of plant parameters frequently occur and that it is too laborious to measure the plant parameters each time and adjust the controller.

We shall see that closed-loop controllers can be designed so that the performance of the control system deteriorates very little even though there may be quite a large discrepancy between the actual plant parameters and the *nominal* plant parameters, that is, the parameter values that have been used while designing the controller. To this end we investigate the addition to the steady-state mean square tracking error attributable to parameter deviations.

In this section we work with the following assumptions.

1. *The control system is time-invariant and asymptotically stable.*

2. *The controlled variable is also the observed variable, that is, $C = D$, hence $K(s) = H(s)$.*

3. *The input variable and the controlled variable, hence also the reference variable, are scalar. W_e and W_u are both 1.*

Extension to the multivariable case is possible, but does not give much additional insight.

4. *Only the effect of parameter changes on the tracking properties is considered and not that on the disturbance suppression or noise reduction properties.*

5. *The reference variable has a constant part r_0, which is a stochastic vector,*

with second-order moment R_0 and as variable part a zero-mean wide-sense stationary stochastic process with power spectral density function $\Sigma_r(\omega)$.

We denote by $H_0(s)$ the *nominal* transfer function of the plant, and by $H_1(s)$ the *actual* transfer function. Similarly, we write $T_0(s)$ for the transmission of the control system with the nominal plant transfer function and $T_1(s)$ for the transmission with the actual plant transfer function. We assume that the transfer function $G(s)$ in the feedback link and the transfer function $P(s)$ in the link from the reference variable (see the block diagram of Fig. 2.14, Section 2.5.1) are precisely known and not subject to change.

Using **2-63**, we obtain for the nominal transmission

$$T_0(s) = \frac{H_0(s)P(s)}{1 + G(s)H_0(s)}, \qquad \textbf{2-171}$$

and for the actual transmission

$$T_1(s) = \frac{H_1(s)P(s)}{1 + G(s)H_1(s)}. \qquad \textbf{2-172}$$

For the actual control system, the steady-state mean square tracking error is given by

$$C_{e\infty} = |T_1(0) - 1|^2 R_0 + \int_{-\infty}^{\infty} |T_1(j\omega) - 1|^2 \Sigma_r(\omega)\, df. \qquad \textbf{2-173}$$

We now make an estimate of the increase in the mean square tracking error attributable to a change in the transmission. Let us denote

$$\Delta T(s) = T_1(s) - T_0(s). \qquad \textbf{2-174}$$

Inserting $T_1(s) = T_0(s) + \Delta T(s)$ into **2-173**, we obtain

$$\begin{aligned} C_{e\infty} &= |T_0(0) - 1|^2 R_0 + \int_{-\infty}^{\infty} |T_0(j\omega) - 1|^2 \Sigma_r(\omega)\, df \\ &\quad + 2[T_0(0) - 1]\,\Delta T(0) R_0 + 2\mathrm{Re}\left\{\int_{-\infty}^{\infty} [T_0(j\omega) - 1]\,\Delta T(-j\omega)\Sigma_r(\omega)\, df\right\} \\ &\quad + |\Delta T(0)|^2 R_0 + \int_{-\infty}^{\infty} |\Delta T(j\omega)|^2 \Sigma_r(\omega)\, df. \end{aligned} \qquad \textbf{2-175}$$

We now proceed by assuming that the nominal control system is well-designed so that the transmission $T_0(j\omega)$ is very close to 1 over the frequency band of the reference variable. In this case we can neglect the first four terms of **2-175** and we approximate

$$C_{e\infty} \simeq |\Delta T(0)|^2 R_0 + \int_{-\infty}^{\infty} |\Delta T(j\omega)|^2 \Sigma_r(\omega)\, df. \qquad \textbf{2-176}$$

This approximation amounts to the assumption that

$$|T_0(j\omega) - 1| \ll |\Delta T(j\omega)| \tag{2-177}$$

for all ω in the frequency band of the reference variable.

Our next step is to express $\Delta T(s)$ in terms of $\Delta H(s)$, where

$$\Delta H(s) = H_1(s) - H_0(s). \tag{2-178}$$

We obtain:

$$\begin{aligned}\Delta T(s) &= \frac{H_1(s)P(s)}{1 + G(s)H_1(s)} - \frac{H_0(s)P(s)}{1 + G(s)H_0(s)}\\ &= \frac{\Delta H(s)P(s)}{[1 + H_1(s)G(s)][1 + G(s)H_0(s)]}\\ &= S_1(s)\,\Delta H(s)N_0(s),\end{aligned} \tag{2-179}$$

where

$$S_1(s) = \frac{1}{1 + H_1(s)G(s)} \tag{2-180}$$

is the sensitivity function of the actual control system, and where

$$N_0(s) = \frac{P(s)}{1 + G(s)H_0(s)} \tag{2-181}$$

is the transfer function of the nominal control system from the reference variable r to the input variable u. Now with the further approximation

$$S_1(j\omega) \simeq S_0(j\omega), \tag{2-182}$$

where

$$S_0(s) = \frac{1}{1 + H_0(s)G(s)} \tag{2-183}$$

is the sensitivity function of the nominal control system (which is known), we write for the steady-state mean square tracking error

$$C_{e\infty} \simeq |S_0(0)|^2\,|\Delta H(0)N_0(0)|^2\,R_0 + \int_{-\infty}^{\infty} |S_0(j\omega)|^2\,|\Delta H(j\omega)N_0(j\omega)|^2\,\Sigma_r(\omega)\,df. \tag{2-184}$$

We immediately conclude the following design objective.

Design Objective 2.8. *Consider a time-invariant asymptotically stable linear closed-loop control system with a scalar controlled variable that is also the observed variable. Then in order to reduce the steady-state mean square tracking error attributable to a variation $\Delta H(s)$ in the plant transfer function $H(s)$, the control system sensitivity function $S_0(j\omega)$ should be made small over* *the frequency band of* $|\Delta H(j\omega)N_0(j\omega)|^2 \Sigma_r(\omega)$. *If constant errors are of special concern,* $S_0(0)$ *should be made small, preferably zero, when* $\Delta H(0)N_0(0)$ *is different from zero.*

This objective should be understood as follows. Usually the plant transmission $T_0(s)$ is determined by finding a compromise between the requirements upon the mean square tracking error and the mean square input. Once $T_0(s)$ has been chosen, the transfer function $N_0(s)$ from the reference variable to the plant input is fixed. The given $T_0(s)$ and $N_0(s)$ can be realized in many different ways, for example, by first choosing the transfer function $G(s)$ in the feedback link and then adjusting the transfer function $P(s)$ in the link from the reference variable so that the desired $T_0(s)$ is achieved. Now Design Objective 2.8 states that this realization should be chosen so that

$$S_0(j\omega) = \frac{1}{1 + H_0(j\omega)G(j\omega)} \qquad \textbf{2-185}$$

is small over the frequency band of $|\Delta H(j\omega)N_0(j\omega)|^2 \Sigma_r(\omega)$. The latter function is known when some idea about $\Delta H(j\omega)$ is available and $T_0(j\omega)$ has been decided upon. We note that making the sensitivity function $S_0(j\omega)$ small is a requirement that is also necessary to reduce the effect of disturbances in the control system, as we found in Section 2.7. As noted in Section 2.7, $S_0(0)$ can be made zero by introducing integrating action (Problem 2.3).

We conclude this section with an interpretation of the function $S_0(s)$. From **2-179** and **2-171** it follows that

$$\frac{\Delta T(s)}{T_0(s)} = S_1(s)\frac{\Delta H(s)}{H_0(s)}. \qquad \textbf{2-186}$$

Thus $S_1(s)$ relates the relative change in the plant transfer function $H(s)$ to the resulting relative change in the control system transmission $T(s)$. When the changes in the plant transfer function are restricted in magnitude, we can approximate $S_1(j\omega) \simeq S_0(j\omega)$. This interpretation of the function $S_0(s)$ is a classical concept due to Bode (see, e.g., Horowitz, 1963). $S_0(s)$ is called the *sensitivity function* of the closed-loop system, since it gives information about the sensitivity of the control system transmission to changes in the plant transfer function.

Example 2.12. *The effect of parameter variations on the position servo*

Let us analyze the sensitivity to parameter changes in Design I of the position servo (Example 2.4, Section 2.3). The sensitivity function for this design is given by

$$S_0(s) = \frac{s(s+\alpha)}{s^2 + \alpha s + \kappa\lambda}. \qquad \textbf{2-187}$$

Plots of $|S(j\omega)|$ for various values of the gain λ have been given in Fig. 2.26. It is seen that for $\lambda = 15$ V/rad, which is the most favorable value of the gain, protection against the effect of parameter variations is achieved up to about 3 rad/s. To be more specific, let us assume that the parameter variations are caused by variations in the moment of inertia J. Since the plant parameters α and κ are given by (Example 2.4)

$$\alpha = \frac{B}{J}, \qquad \kappa = \frac{k}{J}, \tag{2-188}$$

it is easily found that for small variations ΔJ in J we can write

$$\frac{\Delta H(s)}{H(s)} \simeq -\frac{s}{s+\alpha}\frac{\Delta J}{J}, \tag{2-189}$$

where

$$H(s) = \frac{\kappa}{s(s+\alpha)} \tag{2-190}$$

is the plant transfer function. We note the following.

1. For zero frequency we have

$$\frac{\Delta H(0)}{H(0)} = 0, \tag{2-191}$$

no matter what value ΔJ has. Since $T(0) = 1$, and consequently $\Delta T(0) = 0$, this means that the response to changes in the set point of the tracking system is always correct, independent of the inertial load of the servo.

2. We see from **2-189** that as a function of ω the effect of a variation in the moment of inertia upon the plant transfer function increases up to the break frequency $\alpha = 4.6$ rad/s and stays constant from there onward. From the behavior of the sensitivity function, it follows that for low frequencies (up to about 3 rad/s) the effect of a variation in the moment of inertia upon the transmission is attenuated and that especially for low frequencies a great reduction results.

To illustrate the control system sensitivity, in Fig. 2.30 the response of the closed-loop system to a step in the reference variable is given for the cases

$$\frac{\Delta J}{J} = 0,\ -0.3,\ \text{and}\ +0.3. \tag{2-192}$$

Taking into account that a step does not have a particularly small frequency band, the control system compensates the parameter variation quite satisfactorily.

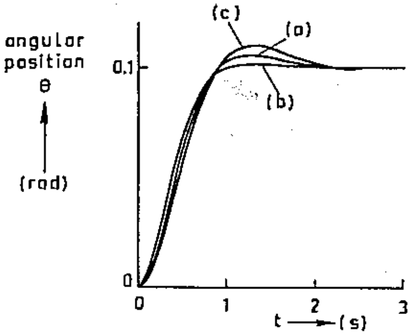

Fig. 2.30. The effect of parameter variations on the response of the position servo, Design I, to a step of 0.1 rad in the reference variable: (*a*) Nominal inertial load; (*b*) inertial load 1.3 of nominal; (*c*) inertial load 0.7 of nominal.

2.10* THE OPEN-LOOP STEADY-STATE EQUIVALENT CONTROL SCHEME

The potential advantages of closed-loop control may be very clearly brought to light by comparing closed-loop control systems to their so-called open-loop steady-state equivalents. This section is devoted to a discussion of such open-loop equivalent control systems, where we limit ourselves to the time-invariant case.

Consider a time-invariant closed-loop control system and denote the transfer matrix from the reference variable r to the plant input u by $N(s)$. Then we can always construct an open-loop control system (see Fig. 2.31) that has the same transfer matrix $N(s)$ from the reference variable r to the plant input u. As a result, the transmission of both the closed-loop system and the newly constructed open-loop control system is given by

$$T(s) = K(s)N(s), \tag{2-193}$$

where $K(s)$ is the transfer matrix of the plant from the plant input u to the controlled variable z. For reasons explained below, we call the open-loop system *steady-state equivalent* to the given closed-loop system.

In most respects the open-loop steady-state equivalent proves to be inferior to the closed-loop control system. Often, however, it is illuminating to

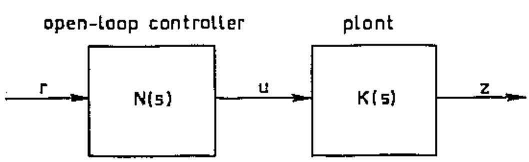

Fig. 2.31. The open-loop steady-state equivalent control system.

study the open-loop equivalent of a given closed-loop system since it provides a reference situation with a performance that should be improved upon. We successively compare closed-loop control systems and their open-loop equivalents according to the following aspects of control system performance: *stability*; *steady-state tracking properties*; *transient behavior*; *effect of plant disturbances*; *effect of observation noise*; *sensitivity to plant variations.*

We first consider *stability.* We immediately see that the characteristic values of the equivalent open-loop control system consist of the characteristic values of the plant, together with those of the controller (compare Section 1.5.4). This means, among other things, that *an unstable plant cannot be stabilized by an open-loop controller.* Since stability is a basic design objective, there is little point in considering open-loop equivalents when the plant is not asymptotically stable.

Let us assume that the plant and the open-loop equivalent are asymptotically stable. We now consider the *steady-state tracking properties* of both control systems. Since the systems have equal transmissions and equal transfer matrices from the reference variable to the plant input, their steady-state mean square tracking errors and mean square input are also equal. This explains the name steady-state equivalent. This also means that *from the point of view of tracking performance there is no need to resort to closed-loop control.*

We proceed to the *transient properties.* Since among the characteristic values of the open-loop equivalent control system the characteristic values of the plant appear unchanged, obviously *no improvement in the transient properties can be obtained by open-loop control*, in contrast to closed-loop control. By transient properties we mean the response of the control system to nonzero initial conditions of the plant.

Next we consider the *effect of disturbances.* As in Section 2.7, we assume that the disturbance variable can be written as the sum of a constant and a variable part. Since in the multivariable case we can write for the contribution of the disturbance variable to the controlled variable in the closed-loop system

$$\mathbf{Z}(s) = [I + H(s)G(s)]^{-1}D(sI - A)^{-1}\mathbf{V}_p(s), \qquad \textbf{2-194}$$

it follows that the contribution of the disturbance variable to the mean square tracking error of the *closed-loop* system can be expressed as

$$C_{e\infty}\text{ (with disturbance)} - C_{e\infty}\text{ (without disturbance)}$$

$$= \operatorname{tr}\left\{S^T(0)V_0S(0)W_e + \int_{-\infty}^{\infty} S(j\omega)\Sigma_{v0}(\omega)S^T(-j\omega)W_e\,df\right\}, \qquad \textbf{2-195}$$

where we have used the results of Sections 1.10.3 and 1.10.4, and where

$$\begin{aligned} S(s) &= [I + H(s)G(s)]^{-1}, \\ \Sigma_{v0}(\omega) &= D(j\omega I - A)^{-1}\Sigma_{vp}(\omega)(-j\omega I - A^T)^{-1}D^T, \\ V_0 &= D(-A)^{-1}E\{v_{p0}v_{p0}^T\}(-A^T)^{-1}D^T. \end{aligned} \qquad \textbf{2-196}$$

In analogy with the single-input single-output case, $S(s)$ is called the *sensitivity matrix* of the system. The matrix A is assumed to be nonsingular.

Let us now consider the equivalent open-loop system. Here the contribution of the disturbance to the controlled variable is given by

$$\mathbf{Z}(s) = D(sI - A)^{-1}\mathbf{V}_p(s). \qquad \textbf{2-197}$$

Assuming that the open-loop equivalent control system is asymptotically stable, it is easily seen that the increase in the steady-state mean square tracking error due to the disturbance in the *open-loop* system can be expressed as

$$C_{e\infty} \text{ (with disturbance)} - C_{e\infty} \text{ (without disturbance)}$$
$$= \text{tr}\left\{W_eV_0 + \int_{-\infty}^{\infty}\Sigma_{v0}(\omega)W_e\,df\right\}. \qquad \textbf{2-198}$$

We see from **2-198** that the increase in the mean square tracking error is completely independent of the controller, hence is not affected by the open-loop control system design. Clearly, *in an open-loop control system disturbance reduction is impossible.*

Since the power spectral density matrix $\Sigma_{v0}(\omega)$ may be ill-known, it is of some interest to establish whether or not there exists a condition that guarantees that in a closed-loop control system the disturbance is reduced as compared to the open-loop equivalent irrespective of Σ_{v0}. Let us rewrite the increase **2-195** in the mean square tracking error of a closed-loop system as follows:

$$C_{e\infty} \text{ (with disturbance)} - C_{e\infty} \text{ (without disturbance)}$$
$$= \text{tr}\left\{S^T(0)W_eS(0)V_0 + \int_{-\infty}^{\infty}S^T(-j\omega)W_eS(j\omega)\Sigma_{v0}(\omega)\,df\right\}, \qquad \textbf{2-199}$$

where $S(s)$ is the sensitivity matrix of the system. A comparison with **2-198** leads to the following statement.

Theorem 2.1. *Consider a time-invariant asymptotically stable closed-loop control system where the controlled variable is also the observed variable and where the plant is asymptotically stable. Then the increase in the steady-state mean square tracking error due to the plant disturbance is less than or* *at least equal to that for the open-loop steady-state equivalent, regardless of the properties of the plant disturbance, if and only if*

$$S^T(-j\omega)W_eS(j\omega) \le W_e \qquad \textit{for all real } \omega. \tag{2-200}$$

The proof of this theorem follows from the fact that, given any two nonnegative-definite Hermitian matrices M_1 and M_2, then $M_1 \ge M_2$ implies and is implied by $\text{tr}\,(M_1N) \ge \text{tr}\,(M_2N)$ for any nonnegative-definite Hermitian matrix N.

The condition **2-200** is especially convenient for single-input single-output systems, where $S(s)$ is a scalar function so that **2-200** reduces to

$$|S(j\omega)| \le 1 \qquad \text{for all real } \omega. \tag{2-201}$$

Usually, it is simpler to verify this condition in terms of the return difference function

$$J(s) = \frac{1}{S(s)} = 1 + H(s)G(s). \tag{2-202}$$

With this we can rewrite **2-201** as

$$|J(j\omega)| \ge 1 \qquad \text{for all real } \omega. \tag{2-203}$$

Also, for multiinput multioutput systems it is often more convenient to verify **2-200** in terms of the return difference matrix

$$J(s) = S^{-1}(s) = I + H(s)G(s). \tag{2-204}$$

In this connection the following result is useful.

Theorem 2.2. *Let $J(s) = S^{-1}(s)$. Then the three following statements are equivalent:*

$$\begin{aligned} &\text{(a)}\ \ S^T(-j\omega)W_eS(j\omega) \le W_e, \\ &\text{(b)}\ \ J^T(-j\omega)W_eJ(j\omega) \ge W_e, \\ &\text{(c)}\ \ J(j\omega)W_e^{-1}J^T(-j\omega) \ge W_e^{-1}. \end{aligned} \tag{2-205}$$

The proof is left as an exercise.

Thus we have seen that open-loop systems are inferior to closed-loop control systems from the point of view of disturbance reduction. In all fairness it should be pointed out, however, that in open-loop control systems the plant disturbance causes no increase in the mean square input.

The next item of consideration is the *effect of observation noise.* Obviously, in open-loop control systems observation noise does not affect either the mean square tracking error or the mean square input, since there is no feedback link that introduces the observation noise into the system. In this respect the open-loop equivalent is superior to the closed-loop system.

Our final point of consideration is the *sensitivity to plant variations*. Let us first consider the single-input single-output case, and let us derive the mean square tracking error attributable to a plant variation for an open-loop control system. Since an open-loop control system has a unity sensitivity function, it follows from **2-184** that under the assumptions of Section 2.9 the mean square tracking error resulting from a plant variation is given by

$$C_{e\infty}\ (\text{open-loop}) \simeq |\Delta H(0)N_0(0)|^2 R_0 + \int_{-\infty}^{\infty} |\Delta H(j\omega)N_0(j\omega)|^2 \Sigma_r(\omega)\, df. \qquad \textbf{2-206}$$

Granting that $N_0(s)$ is decided upon from considerations involving the nominal mean square tracking error and input, we conclude from this expression that the sensitivity to a plant transfer function variation of an open-loop control system is not influenced by the control system design. Apparently, *protection against plant variations cannot be achieved through open-loop control.*

For the closed-loop case, the mean square tracking error attributable to plant variations is given by **2-184**:

$$C_{e\infty}\ (\text{closed-loop}) \simeq |S_0(0)|^2\, |\Delta H(0)N_0(0)|^2 R_0 + \int_{-\infty}^{\infty} |S_0(j\omega)|^2\, |\Delta H(j\omega)N_0(j\omega)|^2 \Sigma_r(\omega)\, df. \qquad \textbf{2-207}$$

A comparison of **2-206** and **2-207** shows that the closed-loop system is always less sensitive to plant variations than the equivalent open-loop system, no matter what the nature of the plant variations and the properties of the reference variable are, if the sensitivity function satisfies the inequality

$$|S_0(j\omega)| \leq 1 \qquad \text{for all } \omega. \qquad \textbf{2-208}$$

Thus we see that the condition that guarantees that the closed-loop system is less sensitive than the open-loop system to disturbances also makes the system less sensitive to plant variations.

In the case of disturbance attenuation, the condition **2-208** generalizes to

$$S_0^T(-j\omega)W_eS_0(j\omega) \leq W_e, \qquad \text{for all } \omega, \qquad \textbf{2-209}$$

for the multivariable case. It can be proved (Cruz and Perkins, 1964; Kreindler, 1968a) that the condition **2-209** *guarantees that the increase in the steady-state mean square tracking error due to (small) plant variations in a closed-loop system is always less than or equal to that for the open-loop steady-state equivalent, regardless of the nature of the plant variation and the properties of the reference variable.*

We conclude this section with Table 2.2, which summarizes the points of agreement and difference between closed-loop control schemes and their open-loop steady-state equivalents.

Table 2.2 Comparison of Closed-Loop and Open-Loop Designs

Feature	Closed-loop design	Open-loop steady-state equivalent
Stability	Unstable plant can be stabilized	Unstable plant cannot be stabilized
Steady-state mean square tracking error and input attributable to reference variable	Identical performance if the plant is asymptotically stable.	
Transient behavior	Great improvement in response to initial conditions is possible	No improvement in response to initial conditions is possible
Effect of disturbances	Effect on mean square tracking error can be greatly reduced; mean square input is increased	Full effect on mean square tracking error; mean square input is not affected
Effect of observation noise	Both mean square tracking error and mean square input are increased	No effect on mean square tracking error or mean square input
Effect of plant variations	Effect on mean square tracking error can be greatly reduced	Full effect on mean square tracking error

2.11 CONCLUSIONS

In this chapter we have given a description of control problems and of the various aspects of the performance of a control system. It has been shown that closed-loop control schemes can give very attractive performances. Various rules have been developed which can be applied when designing a control system.

Very little advice has been offered, however, on the question how to select the precise form of the controller. This problem is considered in the following chapters. We formulate the problem of finding a suitable compromise for the requirement of a small mean square tracking error without an overly large mean square input as a mathematical optimization problem. This optimization problem will be developed and solved in stages in Chapters 3–5. Its solution enables us to determine, explicitly and quantitatively, suitable control schemes.

2.12 PROBLEMS

2.1. *The control of the angular velocity of a motor*

Consider a dc motor described by the differential equation

$$J \frac{dc(t)}{dt} + Bc(t) = m(t), \qquad \textbf{2-210}$$

where $c(t)$ is the angular velocity of the motor, $m(t)$ the torque applied to the shaft of the motor, J the moment of inertia, and B the friction coefficient. Suppose that

$$m(t) = ku(t), \qquad \textbf{2-211}$$

where $u(t)$ is the electric voltage applied to the motor and k the torque coefficient. Inserting **2-211** into **2-210**, we write the system differential equation as

$$\frac{dc(t)}{dt} + \alpha c(t) = \kappa u(t). \qquad \textbf{2-212}$$

The following numerical values are used:

$$\alpha = 0.5\ \text{s}^{-1}, \qquad \kappa = 150\ \text{rad}/(\text{V s}^2), \qquad J = 0.01\ \text{kg m}^2. \qquad \textbf{2-213}$$

It is assumed that the angular velocity is both the observed and the controlled variable. We study the simple proportional control scheme where the input voltage is given by

$$u(t) = -\lambda c(t) + \rho r(t). \qquad \textbf{2-214}$$

Here $r(t)$ is the reference variable and λ and ρ are gains to be determined. The system is to be made into a tracking system.

(a) Determine the values of the feedback gain λ for which the closed-loop system is asymptotically stable.

(b) For each value of the feedback gain λ, determine the gain ρ such that the tracking system exhibits a zero steady-state error response to a step in the reference variable. In the remainder of the problem, the gain ρ is always chosen so that this condition is satisfied.

(c) Suppose that the reference variable is exponentially correlated noise with an rms value of 30 rad/s and a break frequency of 1 rad/s. Determine the feedback gain such that the rms input voltage to the dc motor is 2 V. What is the rms tracking error for this gain? Sketch a Bode plot of the transmission of the control system for this gain. What is the 10% cutoff frequency? Compare this to the 10% cutoff frequency of the reference variable and comment on the magnitude of the rms tracking error as compared to the rms value of the reference variable. What is the 10% settling time of the response of the system to a step in the reference variable?

(d) Suppose that the system is disturbed by a stochastically varying torque on the shaft of the dc motor, which can be described as exponentially correlated noise with an rms value of 0.1732 N m and a break frequency of 1 rad/s. Compute the increases in the steady-state mean square tracking error and mean square input attributable to the disturbance for the values of λ and ρ selected under (c). Does the disturbance significantly affect the performance of the system?

(e) Suppose that the measurement of the angular velocity is afflicted by additive measurement noise which can be represented as exponentially correlated noise with an rms value of 0.1 rad/s and a break frequency of 100 rad/s. Does the measurement noise seriously impede the performance of the system?

(f) Suppose that the dc motor exhibits variations in the form of changes in the moment of inertia J, attributable to load variations. Consider the off-nominal values 0.005 kg m^2 and 0.02 kg m^2 for the moment of inertia. How do these extreme variations affect the response of the system to steps in the reference variable when the gains λ and ρ are chosen as selected under (c)?

2.2. *A decoupled control system design for the stirred tank*

Consider the stirred tank control problem as described in Examples 2.2 (Section 2.2.2) and 2.8 (Section 2.5.3). The state differential equation of the plant is given by

$$\dot{x}(t) = \begin{pmatrix} -0.01 & 0 \\ 0 & -0.02 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ -0.25 & 0.75 \end{pmatrix} u(t), \qquad \textbf{2-215}$$

and the controlled variable by

$$z(t) = \begin{pmatrix} 0.01 & 0 \\ 0 & 1 \end{pmatrix} x(t). \qquad \textbf{2-216}$$

(a) Show that the plant can be completely decoupled by choosing

$$u(t) = Qu'(t), \qquad \textbf{2-217}$$

where Q is a suitable 2×2 matrix and where $u'(t) = \text{col}\,[\mu_1'(t), \mu_2'(t)]$ is a new input to the plant.

(b) Using (a), design a closed-loop control system, analogous to that designed in Example 2.8, which is completely decoupled, where $T(0) = I$, and where each link has a 10% cutoff frequency of 0.01 rad/s.

2.3. *Integrating action*

Consider a time-invariant single-input single-output plant where the controlled variable is also the observed variable, that is, $C = D$, and which has a nonsingular A-matrix. For the suppression of constant disturbances, the sensitivity function $S(j\omega)$ should be made small, preferably zero, at $\omega = 0$. $S(s)$ is given by

$$S(s) = \frac{1}{1 + H(s)G(s)}, \qquad \textbf{2-218}$$

where $H(s)$ is the plant transfer function and $G(s)$ the controller transfer function (see Fig. 2.25). Suppose that it is possible to find a rational function $Q(s)$ such that the controller with transfer function

$$G(s) = \frac{1}{s} Q(s) \qquad \textbf{2-219}$$

makes the closed-loop system asymptotically stable. We say that this controller introduces *integrating action*. Show that for this control system $S(0) = 0$, provided $H(0)Q(0)$ is nonzero. Consequently, controllers with integrating action can completely suppress constant disturbances.

2.4*. *Constant disturbances in plants with a singular A-matrix*

Consider the effect of constant disturbances in a control system satisfying the assumptions 1 through 5 of Section 2.7, but where the matrix A of the plant is singular, that is, the plant contains integration.

(a) Show that the contribution of the constant part of the disturbance to the steady-state mean square tracking error can be expressed as

$$\lim_{s \to 0} E\{v_{p0}^T(-sI - A^T)^{-1} D^T S(-s) S(s) D(sI - A)^{-1} v_{p0}\}. \qquad \textbf{2-220}$$

We distinguish between the two cases (b) and (c).

(b) Assume that the disturbances enter the system in such a way that

$$\lim_{s \to 0} D(sI - A)^{-1} v_{p0} \qquad \textbf{2-221}$$

is always finite. This means that constant disturbances always result in finite, constant equivalent errors at the controlled variable despite the integrating nature of the plant. Show that in this case

(i) Design Objective 2.5 applies without modification, and

(ii) $S(0) = 0$, provided

$$\lim_{s \to 0} sH(s)G(s) \qquad \textbf{2-222}$$

is nonzero.

Here $H(s)$ is the plant transfer function and $G(s)$ the transfer function in the feedback link (see Fig. 2.25). This result shows that in a plant with integration where constant disturbances always result in finite, constant equivalent errors at the controlled variable, constant disturbances are completely suppressed (provided **2-222** is satisfied, which implies that neither the plant nor the controller transfer function has a zero at the origin).

(c) We now consider the case where **2-221** is not finite. Suppose that

$$\lim_{s \to 0} s^k D(sI - A)^{-1} v_{p0} \tag{2-223}$$

is finite, where k is the least positive integer for which this is true. Show that in this case

$$\lim_{s \to 0} \frac{S(s)}{s^k} \tag{2-224}$$

should be made small, preferably zero, to achieve a small constant error at the controlled variable. Show that **2-224** can be made equal to zero by letting

$$G(s) = \frac{1}{s^{k-m_0+1}} Q(s), \tag{2-225}$$

where $Q(s)$ is a rational function of s such that $Q(0) \neq 0$ and $Q(0) \neq \infty$, and where m_0 is the least integer m such that

$$\lim_{s \to 0} s^m H(s) \tag{2-226}$$

is finite.