EE 364a: Convex Optimization I — J. Duchi
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Midterm Quiz Solutions

1. *Convexity of some sets.* Determine if each set below is convex.

   (a) $\left\{(x,y,z) \in \mathbf{R}^3 \mid \begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0\right\}$

   ■ **convex** ☐ not convex

   (b) $\{(x,y,z) \in \mathbf{R}^3 \mid xz - y^2 \geq 0, x \geq 0, z \geq 0\}$

   ■ **convex** ☐ not convex

   (c) $\{(x,y,z) \in \mathbf{R}^3 \mid xz - y^2 \geq 0\}$

   ☐ convex ■ **not convex**

   (d) $\{(x,y,z) \in \mathbf{R}^3 \mid \frac{y^2}{z} \geq x, z < 0, x \leq 0\}$

   ■ **convex** ☐ not convex

   Solution.

   (a) *Convex.* The given set is $\mathbf{S}^2_+$.

   (b) *Convex.* The given set is again $\mathbf{S}^2_+$, which we find by taking a determinant.

   (c) *Not convex.* The given set is the union $\mathbf{S}^2_+ \cup \mathbf{S}^2_-$, where $\mathbf{S}^2_- = \{X \in \mathbf{S}^2 \mid X \preceq 0\}$. Take convex combinations of the vectors $(1,0,0)$ and $(0,0,-1)$.

   (d) *Convex.* The given set is

   $$\left\{(x,y,z) \mid \begin{bmatrix} x & y \\ y & z \end{bmatrix} \preceq 0,\ z < 0\right\},$$

   which again we obtain by taking determinants.

2. *Curvature of some functions.* Determine the curvature of the functions below.

   (a) $f(x) = \max\{2, x, 1/\sqrt{x}, x^3\}$, with $\mathbf{dom}\, f = \mathbf{R}_+$

   ■ **convex** ☐ concave ☐ affine ☐ neither

   (b) $f(x,t) = \frac{\|x\|^{14}}{t^{13}}$ with $\mathbf{dom}\, f = \{x \in \mathbf{R}^n, t > 0\}$

   ■ **convex** ☐ concave ☐ affine ☐ neither

   (c) $f(x) = (1/2)x^2 - (1/12)x^4$, with $\mathbf{dom}\, f = \mathbf{R}$

   ☐ convex ☐ concave ☐ affine ■ **neither**

   (d) $f(x,y,z) = \log(y \log \frac{z}{y} - x) + \log(zy)$, with $\mathbf{dom}\, f = \{(x,y,z) \in \mathbf{R} \times \mathbf{R}^2_{++} \mid ye^{x/y} < z\}$

   ☐ convex ■ **concave** ☐ affine ☐ neither

   Solution.

   (a) *Convex.* The maximum of convex functions is convex, and x^3 and $1/\sqrt{x}$ are convex over $x \geq 0$.

(b) *Convex.* This is the perspective transform of $g(x) = \|x\|^{14}$, which is convex.

(c) *Neither.* We have $f'(x) = x - (1/3)x^3$ and $f''(x) = 1 - x^2$, so $f''(x) \geq 0$ for $|x| \leq 1$ while $f''(x) < 0$ for $|x| > 1$.

(d) *Concave.* The function $h(t) = \log t$ is concave and monotone on $t > 0$, and $g(y, z) = y \log \frac{y}{z}$ is convex (it is the perspective of $y \log y$) on $y, z > 0$, so $c(x, y, z) = \log(-x - g(y, z))$ is concave on $g(y, z) + x < 0$ by composition rules. $\log z + \log y$ is obviously concave.

3. *Convexity of some sets of positive semidefinite matrices.* In each part of the question, n, k are fixed numbers with $k < n$. Determine if each set below is convex.

(a) $\{A \in \mathbf{S}^n_+ \mid \mathbf{Rank}(A) \geq k\}$, where $k < n$.

(b) $\{A \in \mathbf{S}^n_+ \mid \mathbf{Rank}(A) \leq k\}$, where $k < n$.

(c) $\{A \in \mathbf{S}^n_+ \mid \mathbf{Rank}(A) = n\}$.

(d) $\{C \in \mathbf{S}^n_{++} \mid A - B^T C^{-1} B \succeq 0\}$ where A, B are fixed matrices of appropriate size.

Solution.

(a) *Convex.* As in the homework (book exercise 2.13), the convex combination of any two matrices $A, B \in \mathbf{S}^n_+$ with $\min\{\mathbf{Rank}(A), \mathbf{Rank}(B)\} \geq k$ satisfies $\mathbf{Rank}(\theta A + (1-\theta)B) \geq k$.

(b) *Not convex.* Let e_i be the standard basis vectors, and set $A = \sum_{i=1}^k e_i e_i^T$ and $B = \sum_{i=2}^{k+1} e_i e_i^T$. Then $\mathbf{Rank}(A) = \mathbf{Rank}(B) = k$, while $\mathbf{Rank}((A+B)/2) = k + 1 > k$.

(c) *Convex.* For any full rank $A, B \in \mathbf{S}^n_+$, we have $A \succ 0$ and $B \succ 0$, so this set is simply $\mathbf{S}^n_{++}$.

(d) *Convex.* By Schur complements, this is

$$\left\{ C \in \mathbf{S}^n_{++} \mid \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \right\}.$$

4. *DCP rules.* The function

$$f(x) = \log\left(\exp\left(\frac{(a^T x)^2}{c^T x - d}\right) + \exp\left((c^T x - d)^{-1/2}\right)\right)$$

is convex in x over $\{x \in \mathbf{R}^n \mid c^T x - d > 0\}$. Express f using disciplined convex programming (DCP), limited to the following atoms:

`inv_pos(u)`, which is $1/u$, with domain $\mathbf{R}_{++}$

`square(u)`, which is u^2, with domain $\mathbf{R}$

`sqrt(u)`, which is $\sqrt{u}$, with domain $\mathbf{R}_+$

`geo_mean(u)`, which is $(\prod_{i=1}^n u_i)^{1/n}$, with domain $\mathbf{R}^n_+$

`quad_over_lin(u,v)`, which is u^2/v, with domain $\mathbf{R} \times \mathbf{R}_{++}$

`log_sum_exp(u)`, which is $\log(\sum_{i=1}^n \exp(u_i))$, with domain $\mathbf{R}^n$.

`log(u)`, which is $\log u$, with domain $\mathbf{R}_{++}$

`exp(u)`, which is e^u, with domain $\mathbf{R}$

You may also use addition, subtraction, scalar multiplication, and any constant functions. Assume that DCP is sign-sensitive, *e.g.*, `square(u)` increasing in u when $u \geq 0$. Please only write down your composition. *No justification is required.*

Solution. We can write the function as

```
log_sum_exp((quad_over_lin(a' * x, c' * x - d)),
            inv_pos(sqrt(c' * x - d)))
```

The atom `log_sum_exp` is jointly convex on its domain ($\mathbf{R}^n$) and increasing in its arguments, so composition with any demonstrably convex function is valid. As `sqrt(y)` is concave and positive, the composition `inv_pos(sqrt(u))` is DCP convex. The full composition is thus DCP convex.

Note that you may *not* write

```
log_sum_exp((quad_over_lin(a' * x, c' * x - d)),
            sqrt(inv_pos(c' * x - d)))
```

because the square root command `sqrt()` does not guarantee convexity, and so while these are equivalent functions, this is not DCP. For similar reasons, replacing the `quad_over_lin` commands with terms such as `(a' * x)^2 / (c' * x - d)` will fail.