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THE RESONANT BEHAVIOR OF A MULTI-PHASE COMPOSITE MATERIAL

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We study the resonant behavior of a system consisting of a square array of multi-coated cylinders by calculating the effective dielectric constant of the system. The results were examined numerically using the finite element method.

Keywords: Dielectric constant; resonance; cylinder; multi-coated; finite element method.

1. Introduction

Finding the response of a system consisting of a periodic array of solid cylinders embedded in a homogenous matrix has a long history dating back to Lord Rayleigh.[1] Runge[2] extended Rayleigh's method to coated elements where the geometry was composed of an array of tubes and the core of the tubes was filled with the same material as the matrix. Israelachvili *et al.*[3] reported the solution of the problem when the materials of core and matrix were different.

The inspection of the behavior of the solution as a function of the property of the core and shell divulged a new feature of the system. Nicorovici *et al.*[4–6] discovered that the procedure of coating the cylinders with a material that has a dielectric constant, which is the negative of that of the cylinders ($\varepsilon_{\text{shell}} = -\varepsilon_{\text{core}}$) or matrix ($\varepsilon_{\text{shell}} = -\varepsilon_{\text{matrix}}$) can yield the response of a system with magnified cylinders. This means that a system with diminutive concentration may conditionally give the response of a concentrated system. These authors termed the system in these situations "partially-resonant" and the conditions that put the system in these particular states the "partial resonances" of the system. The term "resonance" may be misleading here. In fact, the above-mentioned features are a generalization of what happens in two-phase composites when $\varepsilon_{\text{core}} = -\varepsilon_{\text{matrix}}$, which is not a resonance.[7,8] Therefore, one needs to consider that the term "partial resonance"

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only refers to a specific condition. The term "partial" was applied due to the fact that the response of the system in these situations is still limited.

In this report, we elucidate how the above-described behavior occurs in a system consisting of arrays of multi-coated cylinders. The motivation for this research stems from the further need to inspect the behavior of systems with components, which have a negative permittivity (ε) or negative permeability (μ) or systems having both ε and μ negative (the so-called left-handed medium[9]). These systems have shown interesting and unexpected results,[4–6,9–11] which may also be of practical interest.[12]

Note that, although periodic arrays are idealized microstructures they may be realizable experimentally. One reason for studying these structures is that their properties can many times be easily computed. Also, the results of periodic systems can be useful for grasping the interplay between the microstructure and macroscopic properties of composites. We numerically show that a disordered structure can also be partially resonant.

The structure of this report is explained here. The following section describes the geometry under study. In Sec. 3, we briefly explain the procedure of deriving the response of the system. Section 4 predicts some results from the theoretical study of the system when it is in resonant states. Section 5 verifies the results of the previous section and finally, in Sec. 6, we summarize the key findings of this study.

2. Geometric Description

Consider a homogenous matrix with a unit dielectric constant surrounding an array of composite cylinders, which have a topology based upon the well-known square lattice. Each lattice point of the square array can be described by a lattice vector, $\boldsymbol{r}_n$, defined as:

$$\boldsymbol{r}_n = h(n_1\mathbf{e}_{x_1} + n_2\mathbf{e}_{x_2})\,, \tag{1}$$

where h is the characteristic length that expresses all the dimensionless distances n_1 and n_2 are arbitrary integers and the two basic vectors, $\mathbf{e}_{x_1}$ and $\mathbf{e}_{x_2}$, form the orthonormal basis of the plane. Let θ be an angle measured from the x_1-axis. Thus, one can write

$$x_1 = r\cos\theta\,, \quad x_2 = r\sin\theta\,. \tag{2}$$

The radiuses of the core and coating layers are determined by $a_1, \ldots, a_{N-1}$, respectively, as is shown Fig. 1. With these considerations, one can show that the volume fractions occupied by the core and coating layers can be obtained from

$$f_i = \pi(a_i^2 - a_{i-1}^2) \quad (i = 1, \ldots, N-1) \tag{3}$$

and the total volume fraction can be expressed as:

$$F = \sum_{i=1}^{N-1} f_i\,. \tag{4}$$

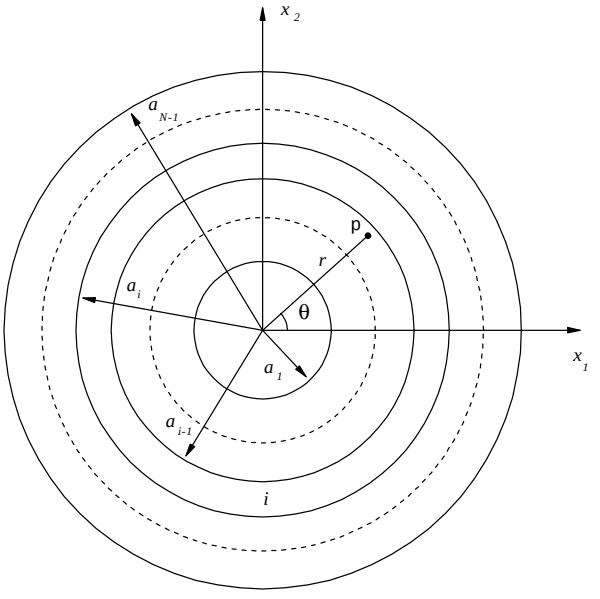


Fig. 1. The multi-coated cylinder under study.

$a_0 = 0$ has been included here and in the following relations in order to reduce the number of mathematical notations. We also use the term "layer" for the core, coating layers and matrix, and the dielectric constant ratio between layers $i-1$ and i is represented by $\varepsilon_{i-1,i}$.

3. Mathematical Modeling

Considering the problem symmetrical allows it to be solved independent of the direction of the applied field, and without compromising generality, we assume a potential gradient of unit magnitude to be applied along the x_1-axis. For the unit cell located at the origin, considering the general solution of the Laplace equation in polar coordinates (r, θ) and following Ref. 1, the potential (V) inside the layers may be given as:

$$V^1 = \sum_{n=1}^{\infty} E_n^1 r^{2n-1} \cos(2n-1)\theta \qquad r \le a_1\,, \tag{5a}$$

$$V^i = \sum_{n=1}^{\infty} [E_n^i r^{2n-1} + F_n^i r^{-2n+1}] \cos(2n-1)\theta \qquad a_{i-1} \le r \le a_i (i = 2, \ldots, N-1)\,, \tag{5b}$$

$$V^N = \sum_{n=1}^{\infty} [E_n^N r^{2n-1} + F_n^N r^{-2n+1}] \cos(2n-1)\theta \quad r \ge a_{n-1}\,, \tag{5c}$$

where E_n^N and F_n^N are unknown coefficients that are to be determined.

At all the surfaces between the layers, the potential and normal component of the electric displacement are continuous, i.e.,

$$\begin{cases} V^1 = V^2 \\ \varepsilon_1 \dfrac{\partial V^1}{\partial r} = \varepsilon_2 \dfrac{\partial V^2}{\partial r} \end{cases} \qquad r = a_1\,, \tag{6a}$$

$$\begin{cases} V^{i-1} = V^i \\ \varepsilon_{i-1} \dfrac{\partial V^{i-1}}{\partial r} = \varepsilon_i \dfrac{\partial V^i}{\partial r} \end{cases} \qquad r = a_{i-1} (i = 2, \ldots, N-1)\,, \tag{6b}$$

$$\begin{cases} V^{N-1} = V^N \\ \varepsilon_{N-1} \dfrac{\partial V^{N-1}}{\partial r} = \varepsilon_N \dfrac{\partial V^N}{\partial r} \end{cases} \qquad r = a_{N-1}\,. \tag{6c}$$

By applying Eqs. (6) E_n^i and F_n^i can be related to each other as:

$$F_n^i + L_n^i \, a_{i-1}^{4n-2} E_n^i = 0 \quad (i \geq 2) \tag{7}$$

where

$$L_n^i = \frac{\varepsilon_{i-1,i} - 1 + (\varepsilon_{i-1,i} + 1) L_n^{i-1} (a_{i-2}/a_{i-1})^{4n-2}}{\varepsilon_{i-1,i} + 1 + (\varepsilon_{i-1,i} - 1) L_n^{i-1} (a_{i-2}/a_{i-1})^{4n-2}}\,. \tag{8}$$

To derive E_n^N and F_n^N, one may follow the method of Zuzovski and Brenner.[13] These authors derived another relation for the potential through the matrix in one unknown A_n. Comparing the resultant relation with Eq. (5c), allows two linear equations to be found relating E_n^N and F_n^N to A_n. These two equations yield a set of linear equations in the unknowns A_n with the help of Eq. (7)

$$\frac{A_n}{a_{N-1}^{4n+2} L_{n+1}^N} = \sum_{J=1}^{\infty} \binom{2n+2J-1}{2n+1} S_{2n+2J} A_{J-1} + \delta_{n0}\,, \tag{9}$$

where δ_{n0} represents the Kronecker delta (1 for $n = 0$ otherwise 0) and S_n are constants characteristic of the array. After finding E_n^N and F_n^N, other coefficients can be found by using Eqs. (6).

The effective dielectric constant can be calculated using the following formula

$$\varepsilon_{\text{eff}} = 1 + 2\pi A_0\,, \tag{10}$$

where A_0 can be derived by solving the system of linear algebraic equations obtained from Eq. (9). As a solution in an explicit form, we present the following simple formula, which gives reasonable results in very low volume fractions

$$\varepsilon_{\text{eff}} = 1 - \frac{2F}{-1/L_1^N + F}\,. \tag{11}$$

In order to obtain a more accurate expression one can use the methods outlined by Manteufel and Todreas.[14]

4. Theoretical Prediction

As can be seen from Eq. (9), a_{N-1} and L_n^N play important roles in the response of the system to the applied field. Due to the form of the mathematical expression of L_n^N [i.e., Eq. (8)], for some cases, there may exist a different number and series of dispersed layers that provide following relation

$$a_{N^*-1}^{4n+2} L_{n+1}^{N^*} \equiv a_{N-1}^{4n+2} L_{n+1}^{N} \tag{12}$$

and, as a result, the response of the system in these situations would be the same. For example, when the coating layer i is perfectly conducting or insulating, L_n^{i+1} would be 1 and -1, respectively. This means that the layers under the layer i will have no effect on the calculation of L_n^N. Thus, there are infinite selections for the number and property of the layers under the layer i.

The particular cases occur when

$$\varepsilon_{i-1} + \varepsilon_i = 0 \quad (2 \le i \le N)\,. \tag{13}$$

From Eq. (8) we find the following

$$L_n^{i+1} = \frac{P+Q}{R+S} \quad (1 \le i \le N-1) \tag{14}$$

where

$$\begin{cases} P = (\varepsilon_{i,i+1} - 1)\left[\varepsilon_{i-1,i} + 1 + (\varepsilon_{i-1,i} - 1)L_n^{i-1}\left(\dfrac{a_{i-2}}{a_{i-1}}\right)^{4n-2}\right] \\ Q = (\varepsilon_{i,i+1} + 1)\left[\varepsilon_{i+1,i} - 1 + (\varepsilon_{i-1,i} + 1)L_n^{i-1}\left(\dfrac{a_{i-2}}{a_{i-1}}\right)^{4n-2}\right]\left(\dfrac{a_{i-1}}{a_i}\right)^{4n-2} \\ R = (\varepsilon_{i,i+1} + 1)\left[\varepsilon_{i-1,i} + 1 + (\varepsilon_{i-1,i} - 1)L_n^{i-1}\left(\dfrac{a_{i-2}}{a_{i-1}}\right)^{4n-2}\right] \\ S = (\varepsilon_{i,i+1} - 1)\left[\varepsilon_{i-1,i} - 1 + (\varepsilon_{i-1,i} + 1)L_n^{i-1}\left(\dfrac{a_{i-2}}{a_{i-1}}\right)^{4n-2}\right]\left(\dfrac{a_{i-1}}{a_i}\right)^{4n-2} \end{cases} \tag{15}$$

Substituting condition (13) into Eq. (14) gives the following statement

$$L_n^{i+1} = \frac{\varepsilon_{i-1,i+1} - 1 + (\varepsilon_{i-1,i+1} + 1)L_n^{i-1}(a_{i-2}/a_i)^{4n-2}(a_i/a_{i-1})^{2(4n-2)}}{\varepsilon_{i-1,i+1} + 1 + (\varepsilon_{i-1,i+1} - 1)L_n^{i-1}(a_{i-2}/a_i)^{4n-2}(a_i/a_{i-1})^{2(4n-2)}}\,. \tag{16}$$

If two layers, $i-1$ and i, have the same dielectric constant, which is equal to ε_{i-1}, one may derive the following

$$L_n^{i+1} = \frac{\varepsilon_{i-1,i+1} - 1 + (\varepsilon_{i-1,i+1} + 1)L_n^{i-1}(a_{i-2}/a_i)^{4n-2}}{\varepsilon_{i-1,i+1} + 1 + (\varepsilon_{i-1,i+1} - 1)L_n^{i-1}(a_{i-2}/a_i)^{4n-2}}\,. \tag{17}$$

A comparison of Eqs. (16) and (17) shows that the field through the continuous phase would be the same if layer i had the same property as layer $i-1$ and all

the layers under layer $i-1$ were magnified by a factor of $(a_i/a_{i-1})^4$. Layer $i-1$ experiences two changes. Magnification by occupying the place of layer i and reduction due to the extension of layer $i-2$. Thus, this case can be materialized only if

$$a_{i-1} > \sqrt{a_i a_{i-2}}\,. \tag{18}$$

The inspection of L_n^{i+1} in condition (13) revealed part of the results. Let us now derive an expression for L_n^i. From Eq. (8) after substituting Eq. (13) and developing a relation for L_n^{i-1} analogous to Eq. (8), the following equation can be derived

$$L_n^i = \frac{\varepsilon_{i-2,i} - 1 + (\varepsilon_{i-2,i}+1)L_n^{i-2}(a_{i-3}/a_{i-1})^{4n-2}(a_{i-1}/a_{i-2})^{4n-2}}{\varepsilon_{i-2,i} + 1 + (\varepsilon_{i-2,i}-1)L_n^{i-2}(a_{i-3}/a_{i-1})^{4n-2}(a_{i-1}/a_{i-2})^{4n-2}} \times \left(\frac{a_{i-1}}{a_{i-2}}\right)^{4n-2} \qquad (2 \le i \le N)\,. \tag{19}$$

When two layers, $i-1$ and $i-2$, have the same dielectric constant equal to ε_{i-2}, the result for L_n^i would be

$$L_n^i = \frac{\varepsilon_{i-2,i} - 1 + (\varepsilon_{i-2,i}+1)L_n^{i-2}(a_{i-3}/a_{i-1})^{4n-2}}{\varepsilon_{i-2,i} + 1 + (\varepsilon_{i-2,i}-1)L_n^{i-2}(a_{i-3}/a_{i-1})^{4n-2}}\,. \tag{20}$$

A comparing of Eqs. (19) and (20) indicates that there is another equivalent system in which all the layers ($i = 1, \ldots, i-3$) have been extended by a factor of $(a_{i-1}/a_{i-2})^4$. The layer $i-1$ now has the property equal to that of layer $i-2$ and its outer radius is $a_{i-1} \times (a_{i-1}/a_{i-2})$. This system can be materialized if

$$a_{i-1} < \sqrt{a_i a_{i-2}}\,. \tag{21}$$

Some conclusions can be drawn on the basis of the above relations. When $i = 2$ in Eq. (13), the second equivalent system cannot occur, but there is always an equivalent system. When $3 \le i \le N-1$, both equivalent systems can occur, although not simultaneously because of the limitations dictated in Eqs. (18) and (21), which contravene each other. Therefore, finding the equivalent system in this situation has been warranted except when $a_{i-1} = \sqrt{a_i a_{i-2}}$. If $i = N$, the first equivalent system has no meaning and the second one can occur if $a_{i-1} < \sqrt{a_{i-2}/2}$.

Since Eq. (13) expresses resonance between two successive layers, one may consider different cases in which several pair layers satisfy relation (13), i.e.,

$$\varepsilon_{i-1} + \varepsilon_i = 0 \quad (i = j, k, l, \ldots)\,. \tag{22}$$

The behavior of the system in these states can be readily detected by successively applying the methods declared above. For example, we consider the following case

$$\varepsilon_{i-1} + \varepsilon_i = 0 \quad (i = 2, \ldots, N-1)\,. \tag{23}$$

Successively employing the methods ultimately indicates that the field inside the matrix would not change if the multi-coated cylinders were replace by solid cylinders

of radius a_{N-1} and dielectric constant ε_1. Therefore it can be shown that L_n^N can be simplified into the following form

$$L_n^N = \frac{\varepsilon_1 - 1}{\varepsilon_1 + 1} \tag{24}$$

and as a result, the potential inside the matrix would be

$$V^N = \sum_{n=1}^{\infty} F_n^N \left[\frac{1 + \varepsilon_1}{(1 - \varepsilon_1) a_{N-1}^{4n-2}} r^{2n-1} + r^{-2n+1} \right] \cos(2n-1)\theta \,. \tag{25}$$

From the above result, it can be understood that in this situation, the sign of ε for the layers between layers 1 and $N (1 < i < N)$ can be arbitrarily chosen and all the cases yield the same response.

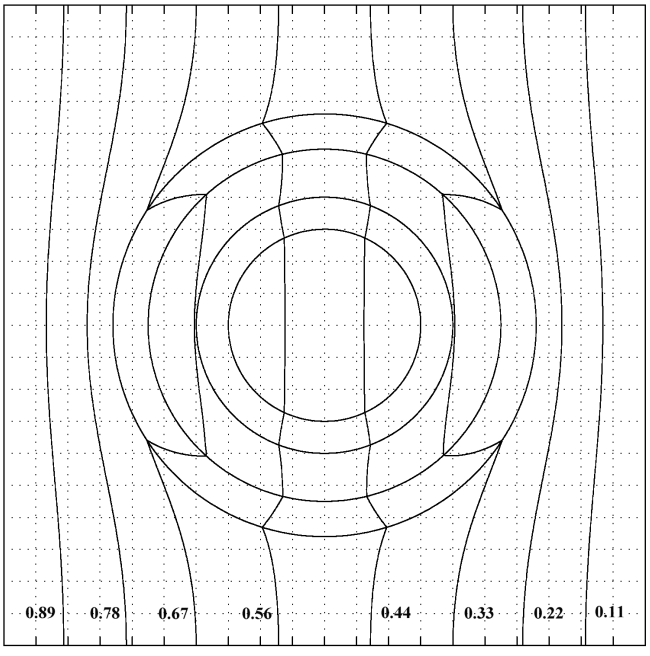
When all the layers $(i = 2, \ldots, N)$ fulfil condition (13) and N is an odd number, the system behaves like a solid medium with the dielectric constant of the matrix.

5. Numerical Verification

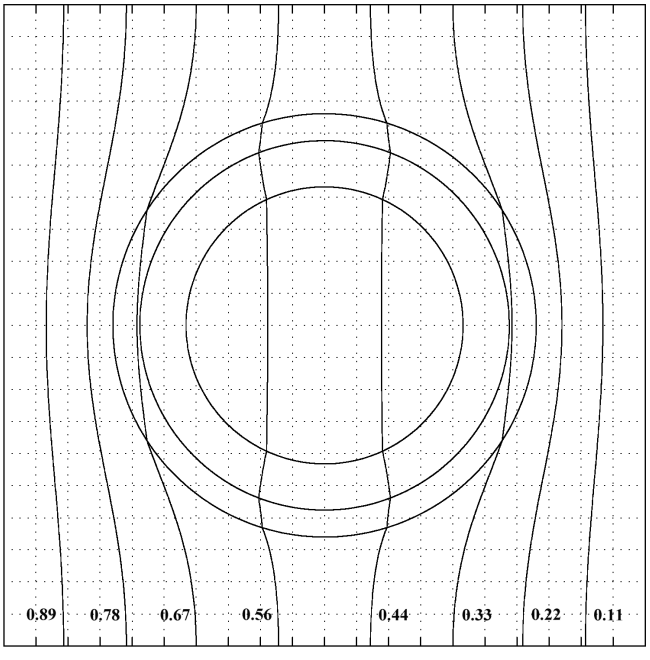
Numerical simulation is nowadays a well-developed tool for inspecting the response of systems. Although the unit cell of the periodic structures has been largely simulated,[15] they are mostly in two-phase with positive transport properties.

For given geometry and solid volume fractions, the Matlab PDE toolbox was utilized and the Laplace equation solved for a unit cell of the system using the finite element method. The unit cell consists of matrix and dispersed layers (core and coating layers). A potential gradient of unit magnitude was applied externally along the x_1-axis and other external boundaries were insulated. At the surfaces between the layers, continuity conditions (6) were implemented. By using solution-adaptive refinement, one can add cells where they are needed in the mesh, thus enabling the features of the potential field to be better resolved. Based on the theoretical findings, the three considerable cases were studied numerically.

- Figure 2(a) shows a unit cell of a system consisting of three-coated cylinders. The dielectric constants of the core and coating layers were selected to be +2.5, +5, +2 and −2 and the radiuses 0.15, 0.2, 0.275 and 0.33, respectively. In order to construct the equivalent system [Fig. 2(b)], second and third coating layers were joined together and considered as one unit layer with a dielectric constant equal to +2. Also, the core and first coating layers are magnified by the factor $(0.33/0.275)^4$. Therefore, the radiuses of the layers are $a_1^* = 0.216$, $a_2^* = 0.288$ and $a_3^* = 0.33$. The distributions of the induced fields are given in the figures show that although the fields inside the dispersed layers of the systems are completely different, they are exactly the same through the matrix.
- Now we consider the system of Fig. 2(a) with one alteration. In order to satisfy condition (21), the radius of the third coating layer has been extended to the value 0.4 as is shown in Fig. 3(a). Based on the predicted scheme for the second equivalent system, we assemble a two-coated cylinder [Fig. 3(b)] with radiuses

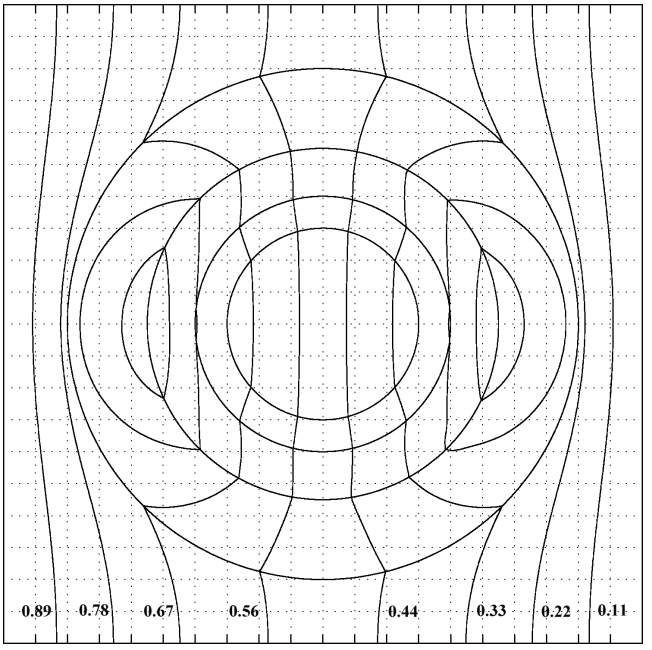


(a)

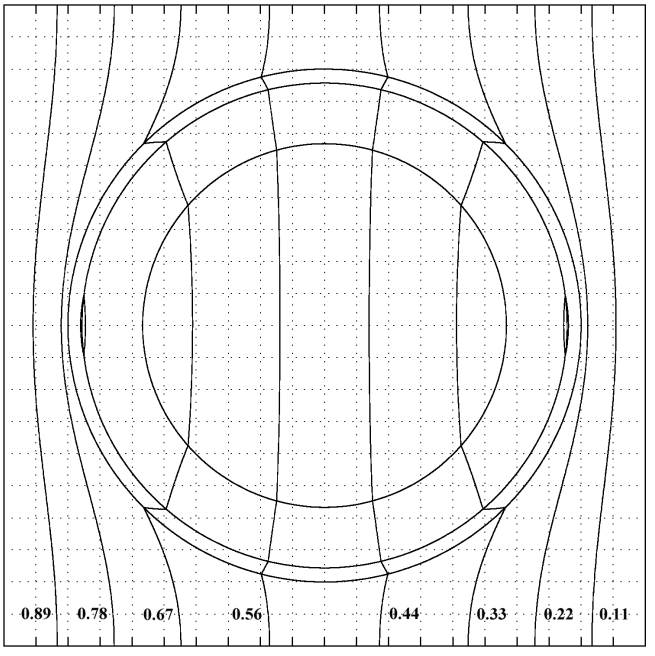


(b)

Fig. 2. Equipotential contours inside the unit cell of the first case. The original system (a) and equivalent system (b).

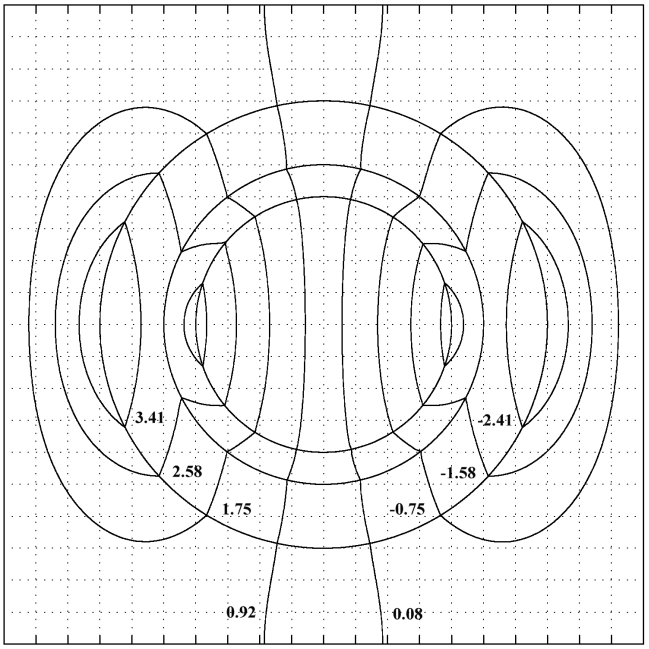


(a)

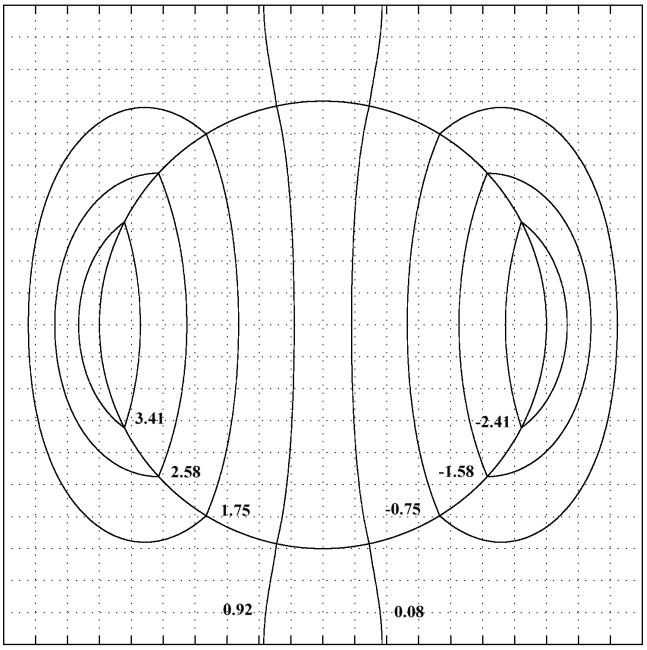


(b)

Fig. 3. Equipotential contours inside the unit cell of the second case. The original system (a) and equivalent system (b).

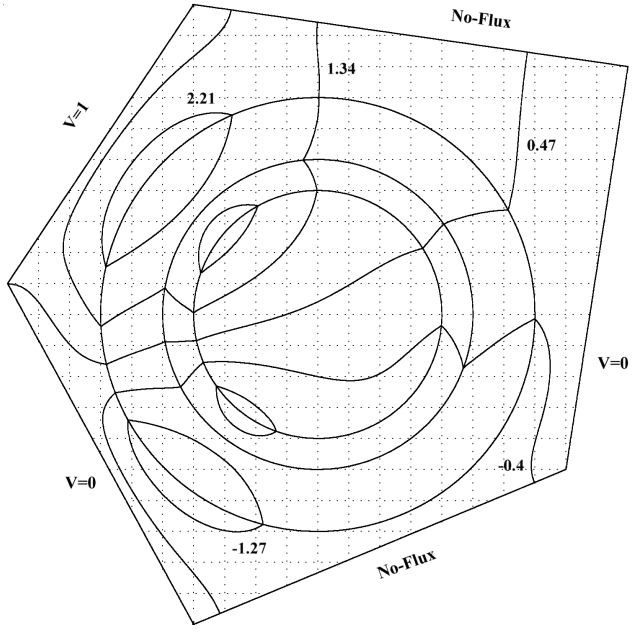


(a)

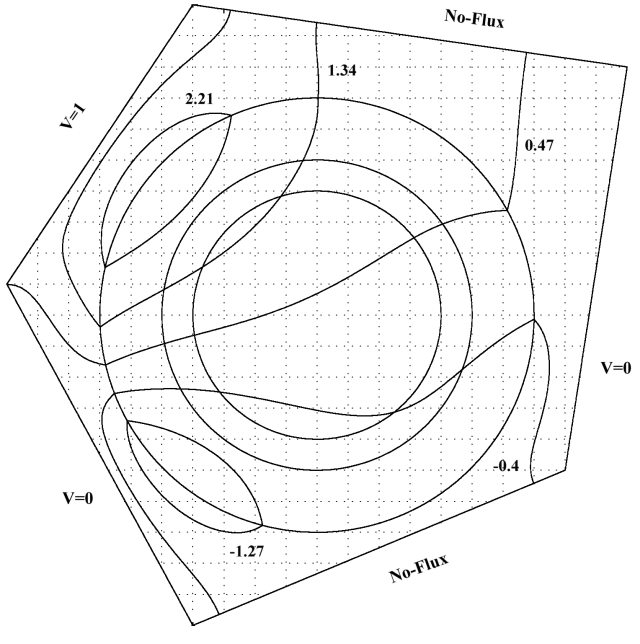


(b)

Fig. 4. Induced potential field inside the layers of the unit cell in third case. The original (a) and equivalent system (b).



(a)



(b)

Fig. 5. Induced potential fields inside the layers of a simple system composed of a two-coated cylinder and arbitrary boundaries. The original (a) and equivalent system (b). Note that the potential fields outside the first coating layers are equal. This is also correct for the third case.

of $a_1^* = 0.28359375$, $a_2^* = 0.378125$ and $a_3^* = 0.4$. The dielectric constants of the layers are $+2.5$ for the core, $+5$ for the first coating layer and -2 for the second coating layer. It is clear from the figures that the potential fields outside a_2^* are equal.

- In third case, a two-coated cylinder has been considered as depicted in Fig. 4(a). The dielectric constants of the core and coating layers are -2, $+2$ and -2, respectively. In the next system [Fig. 4(b)], the two-coated cylinder is replaced by a solid cylinder with the same dielectric constants as the core of the original system. Both systems have the same total volume fraction. Again, by applying the same boundary conditions, the fields through the matrix of both systems are the same.

To further investigate this subject, we consider a simple system consisting of a two-coated cylinder (with details explained in the third case above) covered by another material of unit dielectric constant. Arbitrary boundary and boundary conditions were selected and implemented on the system as is represented in Fig. 5(a). Figure 5(b) shows that changing the dielectric constant of the first coating layer to -2 causes no disturbance in the field through the matrix. Therefore, disordered structures can also be partially resonant. This fact had been predicted by Nicorovici *et al.*[5,6] and the numerical investigation shows the same result.

6. Summary

Inspection of the resonant behavior of the multi-coated structures exposed new results in this field. When the sum of the dielectric constants of two successive layers is equal to zero, a series of layers can be magnified and the magnification is not limited to one layer. The ratio of magnification for the layers detailed. We also explained that, for every resonant state, there may be two types of equivalent system, although only one of them may be of any physical significance. All the numerical investigations were in accordance with the theoretical predictions.

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