1. The Probability Transformation Concept and Formulation

A general reformulation of the probabilistic constraints in the stochastic analyses propagation can be written as below:

$$P_i\left[G_q(\mathbf{X})\le 0\right]\ge \beta_i^o \quad q=1,2,\ldots,m \tag{1}$$

where β_i^o is the probability of the constraint $G_i(\mathbf{X})\le 0$ being true, chosen a priori by the designer, and P_i is the probability of the ith constraint that should be satisfied. The probability of each constraint may be obtained by evaluating the integral in Eq. (2), which is the fundamental expression of the probabilistic problem:

$$P_i=\int_{G_q(\mathbf{X})} f_{\mathrm{X}}(\mathbf{X})d\mathbf{X} \quad q=1,2,\ldots,m \tag{2}$$

where $f_{\mathrm{X}}(\mathbf{X})$ is the PDF of random vector X. In practice, it is difficult to use the joint PDF because of scarcity of joint observations for a large number of random variables. At best, what is known are the marginal probability distributions of each random variable and possibly correlations between pairs of random variables. Another difficulty in solving Eq. (2) is the fact that the constraints, $G_q(\mathbf{X})$, may not represent an implicit form after adding random variables. Such difficulties have motivated the development of various approximate techniques. The general methods are the First and Second Order Reliability Methods (FORM and SORM, respectively), Neural Network Surface Responses (NNSR), and Mean Value First Order Second Moments (FOSM). FORM and SORM are said to be transformation methods, because the integral in Eq. (2) is not solved in the original space (X), but is mapped to the Standard Gaussian space (U). In fact, determining the probability of the constraints with random variables is achieved by mapping the problem from the physical space to the normal space by utilizing the probabilistic transformation (PT). Probability transformation is used to transform various random parameters into the new space in which the probability calculation can be easily implemented. The main advantages of this approach are its computational cost and ease of probability calculation compared to other methods of this class [1]. The mathematical representation of this transformation is expressed in the following. Fig. 1 represents a schematic of the PT. As it represents, any random parameter with its specified PDF and CDF could be transformed into the standard normal space.

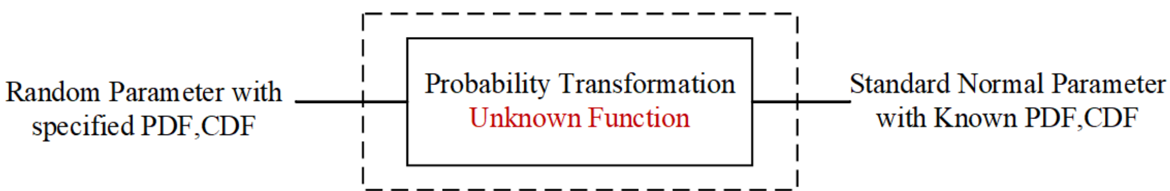

Fig. 1 *A schematic of the probability Transformation*

Fig. 2, depicts schematic steps of transforming input random variables into the independent standard normal space. In Step 1, non-standard input variables X_j^{rp} are transformed to random variables Z_i with standard PDF. However, Z_i^{rp} are still correlated and therefore, another transformation is performed, in step 2, to convert Z_i^{rp} to statistically independent standard-normal variables U_i^{rp}.

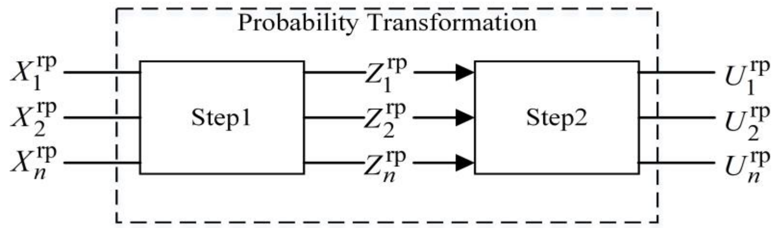

***Fig. 2** Probability transformation the Random Parameters to the standard normal space*

To formulate the step 1 transformation, it has been assumed that the X_i^{rp} and Z_i^{rp} must have similar cumulative density functions (CDF) As below:

$$F_{X_i^{\mathrm{rp}}}\left(x_i^{\mathrm{rp}}\right)=\Phi\left(z_i^{\mathrm{rp}}\right) \quad i=1,2,\ldots,n \tag{3}$$

where $F_{X_i^{\mathrm{rp}}}$ is the CDF of the random variable X_i^{rp} and Φ is the CDF of the random variable Z_i^{rp} which, by definition, is standard normal. Accordingly, random variables Z_i^{rp} can be extracted from:

$$Z_i^{rp}=\Phi^{-1}\left(F_{X_i^{\mathrm{rp}}}\left(x_i\right)\right) \quad i=1,2,\ldots,n \tag{4}$$

Standard normal random variables Z_i^{rp} are correlated. However, new correlation coefficients between Z_i^{rp} and Z_j^{rp} can be assessed by the following equation [1]:

$$\begin{aligned}\rho_{X_i^{\mathrm{rp}},X_j^{\mathrm{rp}}}=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{X_i^{\mathrm{rp}}-\mu_{X_i^{\mathrm{rp}}}}{\sigma_{X_i^{\mathrm{rp}}}}\right)\left(\frac{X_j^{\mathrm{rp}}-\mu_{X_j^{\mathrm{rp}}}}{\sigma_{X_j^{\mathrm{rp}}}}\right)\\&\times\varphi\left(Z_i^{rp},Z_j^{rp},\rho_{Z_i^{rp}Z_j^{rp}}\right)dz_i^{rp}dz_j^{rp}\end{aligned} \tag{5}$$

where φ is the joint binormal PDF of Z_i^{rp} and Z_j^{rp} with the following formula:

$$\begin{aligned}\varphi(Z_i^{rp},Z_j^{rp},\rho_{Z_i^{rp}Z_j^{rp}})=&\frac{1}{2\pi\sqrt{1-\rho_{Z_i^{rp}Z_j^{rp}}^2}}\times\\&\exp\left(-\frac{(z_i^{rp})^2+(z_j^{rp})^2-2z_i^{rp}z_j^{rp}\rho_{Z_i^{rp}Z_j^{rp}}}{2(1-\rho_{Z_i^{rp}Z_j^{rp}}^2)}\right)\end{aligned} \tag{6}$$

The equation **Error! Reference source not found.** can be simply solved by numerical approaches to find $\rho_{Z_i^{rp}Z_j^{rp}}$. In the next step, statistically dependent variables Z_i^{rp} are transformed into independent and standard-normal variables U_i^{rp}. In this regard, the proposed transformation, in [2], is implemented to eliminate the dependencies of Z_i^{rp}. This transformation is formulated as:

$$\begin{aligned}\mathbf{U}^{\mathrm{rp}}=&\mathbf{l}^{\mathbf{-1}}\times\mathbf{Z}^{\mathrm{rp}}=\\&\mathbf{l}^{\mathbf{-1}}\times\left[\Phi^{-1}\left(F_{X_1^{\mathrm{rp}}}\left(x_1^{\mathrm{rp}}\right)\right)\;\cdots\;\Phi^{-1}\left(F_{X_n^{\mathrm{rp}}}\left(x_n^{\mathrm{rp}}\right)\right)\right]\end{aligned} \tag{7}$$

where $\mathbf{l}$ is an upper triangular matrix which is factorized through Cholesky decomposition of the coefficient correlation matrix $\mathbf{R_{ZZ}}$. The coefficient correlation matrix $\mathbf{R_{ZZ}}$ is also defined as:

$$\mathbf{R_{ZZ}} = \begin{bmatrix} 1 & \cdots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \cdots & 1 \end{bmatrix} \quad (8)$$

Based on the proposed transformation, input random parameters X_i^{rp} are now replaced with their equivalent standard and normally distributed parameters U_i^{rp}.

2. Probability of Constraints

For the sake of demonstration, we consider a simple constraint in this section and seek to achieve the probability of it with the proposed approach. Let the constraint be:

$$G = S - R \leq 0 \quad (9)$$

where *S* and *R* show demand and capacity, respectively. Obviously, we desired our capacity to be equal or greater than the demand, therefore, the probability of $P[S - R \leq 0]$ being true is:

$$P[S - R \leq 0] = \iint_{S-R\leq 0} f_{R,S}(r,s)drds \quad (10)$$

To further clarify the situation, Fig. 3 represents all of the possible states that *R* and *S* could happen simultaneously. Having two parameters for the constraint, the joint PDF of *R* and *S* has three dimension which is represented in the figure by its contours. *R*=*S* limit state is of great importance because it separates the joint PDF into two regions; Feasible and Failure. The probability of the constraint being true is sum of all probabilities in the feasible area which is shown in the gray color.

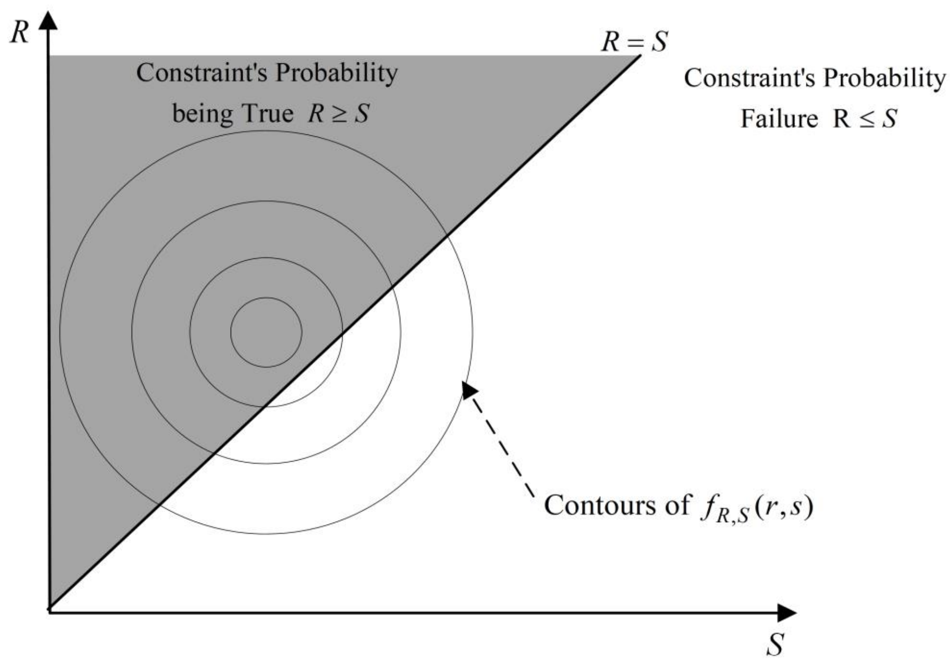

Fig. 3 *Schematic of constraints probability surface, feasible, and failure Region*

Equation 10 can also be written as:

$$P[S - R \le 0] = \iint_{S-R\le 0} f_{R,S}(r,s)drds = F_g(0) \tag{11}$$

where F_g is the joint CDF. Now, assuming that *S* and *R* are independent and standard normal, the probability could be written as:

$$\int_0^\infty \int_0^s f_S(S) f_R(r)drds = F_g(0) = \Phi\left(\frac{g-\mu_g}{\sigma_g}\right) \tag{12}$$

where Φ is the standard normal CDF of *R* and *S*. Considering the constraint in the point $g = 0$, we have:

$$P[g \le 0] = \Phi\left(\frac{0-\mu_g}{\sigma_G}\right) \tag{13}$$

In this equation, $\frac{\mu_g}{\sigma_g}$ is of represents an important indicator β is called safety factor index. From analytical geometry point of view, β is the minimum distance between the origin and the limit state in the standard normal space. Fig. 4 represents this concept.

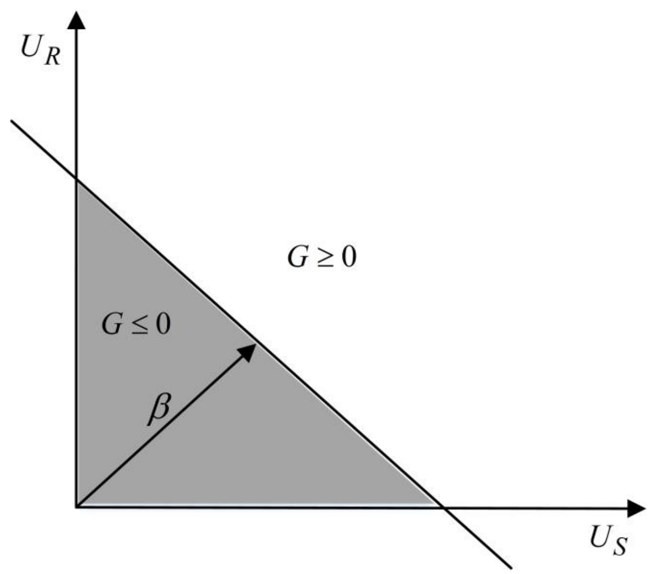

Fig. 4 *geometrical representation of the safety index*

This figure also shows another incentive for using standard normal space. Since β is a distance, the standard normal transformation provides a dimensionless space in which we can measure any distance, easily, while in the physical space the distances are a combination of different dimensions which makes them meaningless.

3. First Order Reliability Method (FORM)

In the FORM, the probability integral is attained in the standard normal space at a point with maximum probability density. The FORM is obtained at constraints $G_i(x)$ at a point $u*$ defined by the optimization problem:

$$u^* = \arg\min \left\{ \|u\| \mid G(u) = 0 \right\} \quad (14)$$

where "arg min" signifies the argument of the minimum of a function. It is proved that $u*$ is located on the limit state, $G(u) = 0$, and has minimum distance from the origin in the standard normal space. Because equal probability density contours in the standard normal space are concentric circles centered at the origin, $u*$ has the highest probability density [3]. This point is known as the design point, most probable point (MPP), and also beta point.

4. References

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3 Ditlevsen, O.: 'Uncertainty modeling with applications to multidimensional civil engineering systems' (McGraw-Hill New York, 1981)