Combinatorial Optimization Course

Maximum Flow Problem

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Topics

- Primal-Dual Approach
- Similar Problems
- Duality
- Applications
- Looking at Algorithms
- Preflow-Push Algorithms

Formulation

- We saw that the problem can be formulated as

$$\begin{aligned} & \max f \\ \sum_j x_{sj} - \sum_j x_{js} - f &\leq 0 \\ f + \sum_j x_{tj} - \sum_j x_{jt} &\leq 0 \\ \sum_j x_{ij} - \sum_j x_{ji} &\leq 0 \quad \forall i \neq s, t \\ x_{ij} &\leq u_{ij} \\ -x_{ij} &\leq 0 \end{aligned}$$

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- Note that Inequalities imply the equalities last seen, but simplify the approach

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x_{ij} &\leq 0 \quad \forall i, j \; where \; x_{ij} = u_{ij} \; in \; \mathbf{D} \\
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x_{ij} &\leq 1 \\
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- What is the meaning of **DRP**?

A Framework

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THEOREM 1 *A flow f is optimal in a graph G iff there is no augmenting path.*

Proof. Very similar to the *Berge Theorem* in matching session. □

1. Initialize $f = 0$
2. **while** there is an augmenting path in G **do**
   (a) augment along the path.

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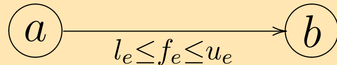

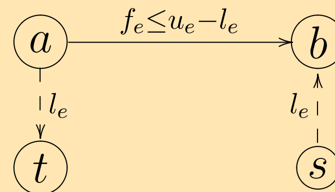

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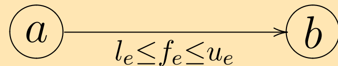

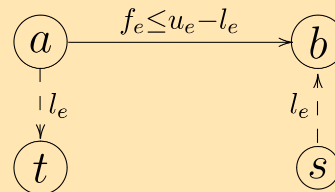

  - ⋆ If all these extra virtual edges are full in the optimal solution, then the solution is *feasible* and we can obtain the original max-flow solution by adding l_e to edge flows

Duality

- An s-t-**cut** is a subset of edges from X to $V - X$, such that $s \in X \subset V$ and $t \notin X$. The value of the cut is the sum of the edge capacities.

  THEOREM 2 *The value of maximum flow in a graph is equal to the value of minimum cut.*

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- For any edge (u, v) from $\overline{R}$ to R the edge is empty
- By flow conservation, the value of the flow is the value of the cut. $\square$

Applications

- *Actors' Problem.* We want to shoot a film. There are n actors we can use; with salaries known to be c_i. There are m companies. Each has a list of actors and will give us p_i dollars if we use *all* the actors they like. Which actors should we choose to maximize the profit?

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  - Build a graph with vertex capacities one.
  - Add edges from boundary to sink and from source to prisoners
  - Use maximum flow.

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  - ⋆ Put an edge from x_0 to x_i and from y_j to y_0.
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  - ⋆ Find a maximum flow from x_0 to y_0

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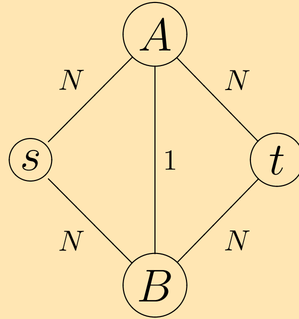

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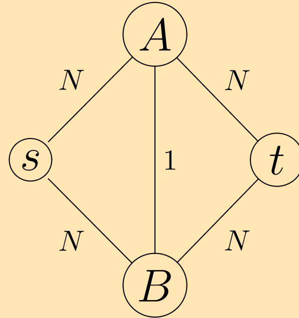

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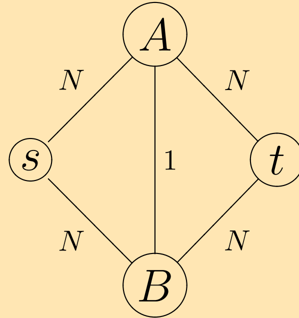

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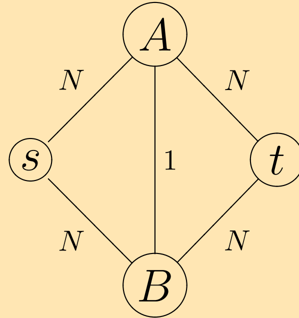

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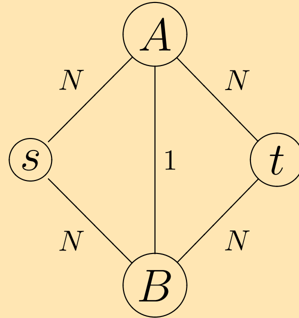

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- But only in case of irrational edge capacities, which fortunately, is not easily represented in computers.

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- Preflow-Push algorithm invented by Goldberg and Tarjan, are a new framework which give more efficient algorithms for Maximum Flow

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- We only **push** flow downhill; but we can have flow going uphill

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- Height h is a function $V \rightarrow \mathbb{N}$

LEMMA 1 *If* $h[u] > h[v] + 1$, *then* (u, v) *is not in the auxiliary graph.*

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Lemma 2 *After a non-saturating push (u, v), u will not be overflowing.*

- A relabel applies when u is overflowing and no push is possible from u.
- By definition, we know that s and t will not ever be overflowing

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LEMMA 3 *An overflowing vertex can be either pushed or relabeled.*

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- If the algorithm terminates, we have a flow and by the previous lemma, no augmenting path

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- We always have a preflow [induction]

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Proof. Assuming the path is simple and writing the inequalities $h[u_i] \leq h[u_{i+1}] + 1$ for the consecutive edges of the path, we turn into a contradiction. $\square$

- If the algorithm terminates, we have a flow and by the previous lemma, no augmenting path
- So a maximum flow is found!

Analysis

Lemma 5 *The height of a vertex is at most* $2|V| - 1$.

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- Variations with $O(|V|^3)$ or better bounds exist.

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- The number of saturating PUSHes is $O(|V|^2(|V| + |E|))$
- The total number of operations is $\Theta(|V|^2|E|)$
- How to implement it with little overhead!?
- Variations with $O(|V|^3)$ or better bounds exist.
- In case of Unit capacities, can be done in $O(\sqrt{|V|}|E|)$

THE END