Combinatorial Optimization Course

Min-Cost Flow Problem

Mohammad Hossein Bateni

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Topics

- Problem Formulation
- Applications
- Primal-Dual Algorithms
- Hitchcock Problem
- Alphabeta Solution
- Min-Cost Circulation
- Equivalency
Min-Cost Flow Problem

- Given a graph $G$ with edge capacities and edge cost, find the flow with size $v$ with minimum cost.
Min-Cost Flow Problem

- Given a graph $G$ with edge capacities and edge cost, find the flow with size $v$ with minimum cost.

$$\min c^T f$$

$$Af = -v_0 d$$

$$f \leq b$$

$$f \geq 0$$

where

$$d_i = \begin{cases} 
-1 & \text{if } i = s \\
+1 & \text{if } i = t \\
0 & \text{otherwise}
\end{cases}$$
Applications

Caterer Problem

- She needs to supply $r_i$ napkins for $N$ successive days.
- She can buy new napkins at $p$ cents each.
- She can launder them at a fast laundry that takes $m$ days and costs $f$ cents a napkin.
- Or she can do that at a slow one which takes $n > m$ days and costs $s < f$ cents each.
- At the end of each day, she should determine how many to send to each laundry and how many new ones to buy so that she can satisfy the demand of the days.
- Minimize the cost.
Applications (Cont’d)

Capacitated Spanning Tree Problem

• We are given a complete undirected weighted graph $G$
• Each node has a traffic generation rate $A_i$
• And a central node $s$
• We want to force a maximum traffic $B$ on edges
• Traffic is directed to central node
• Minimize the cost, where $A_i$ is zero/one and $B = 1$
Combinatorializing the capacities

- Write it as $D$
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$$\begin{align*}
\max & \quad -c^T f \\
Af & \leq -v_0 d \\
f & \leq b \\
-f & \leq 0
\end{align*}$$
Combinatorializing the capacities

- Write it as $D$

\[
\begin{align*}
\max \quad & -c^T f \\
Af & \leq -v_0 d \\
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-f & \leq 0
\end{align*}
\]

- Writing the DRP we get

\[
\begin{align*}
\max \quad & -c^T f \\
Af & = 0 \\
f & \leq 0 \quad \text{for saturated edges} \\
f & \geq 0 \quad \text{for empty edges} \\
f & \geq -1 \quad \text{for all edges, because } -c \leq 0
\end{align*}
\]
Algorithm CYCLE

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**THEOREM 1** A flow is min-cost iff there is no negative-cost circulation.

**Algorithm**

1. Use max-flow to find a flow of size $\nu$
2. **while** there is a negative-cost circulation **do**
   (a) Augment along it until this is no longer a circulation
- Finding the negative-weight cycle by Bellman-Ford in $\Theta(|V||E|)$
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- By Goldberg’s algorithm in $O(\sqrt{|V||E|\log |C|})$
- Running time is $O(|V||E|^2|C||U|)$
- If we always choose the minimum mean cycle it takes $O(|V|^2|E|^3 \log |V|)$
Combinatorializing the cost

- We shall write it as $P$ and consider duals
- But it has lots of complexities
- Intuitively, we will increase the flow until reaching $v$
- We always choose the minimum weight augmenting path

**Theorem 2** Let $f_1$ be a flow of minimum cost with size $v$ and let $f_2$ be a least cost augmentation. Then, $f_1 + f_2$ is the least cost flow with size $v + 1$.

- Assume we don’t have a negative weight at the beginning.

**Algorithm**

1. Initialize with $f = 0$
2. **while** $f$ is less than $v$ **do**
   (a) Augment along the least cost augmentation
Hitchcock Problem

- There are \( m \) sources which supply \( a_i \) units of something
- And \( n \) sinks which demand \( b_j \) units of that material
- Sending the output of source \( i \) to sink \( j \) has \( c_{ij} \) cost per unit
- We need to minimize the cost, satisfying all the demands
- We write it as \( P \) form

\[
\min \sum_{i,j} c_{ij} f_{ij}
\]
\[
\sum_{j=1}^{n} f_{ij} = a_i
\]
\[
\sum_{i=1}^{m} f_{ij} = b_j
\]
\[
f_{ij} \geq 0
\]

- supposing that \( \sum_i a_i = \sum_j b_j \)
Hitchcock Problem (Cont’d)

• Dual is like this

\[
\text{max } w = \sum_{i=1}^{m} a_i\alpha_i + \sum_{j=1}^{n} b_j\beta_j
\]

\[
\alpha_i + \beta_j \leq c_{ij}
\]

\[
\alpha_i, \beta_j \geq 0
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Hitchcock Problem (Cont’d)

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\[\alpha_i + \beta_j \leq c_{ij}\]

\[\alpha_i, \beta_j \geq 0\]

• An initial feasible solution is

\[
\alpha_i = 0
\]

\[
\beta_j = \min_{1 \leq i \leq m} \{c_{ij}\}
\]
Hitchcock Problem (Cont’d)

- Dual is like this

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\begin{align*}
\text{max } w &= \sum_{i=1}^{m} a_i \alpha_i + \sum_{j=1}^{n} b_j \beta_j \\
\alpha_i + \beta_j &\leq c_{ij} \\
\alpha_i, \beta_j &\geq 0
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\alpha_i &= 0 \\
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\end{align*}
\]

- Let \( IJ = \{(i, j) : \alpha_i + \beta_j = c_{ij}\} \)
Hitchcock Problem (Cont’d)

- **RP** is as follows

\[
\begin{align*}
\min \quad \zeta &= \sum_{i=1}^{m+n} x_i^a \\
\sum_j f_{ij} + x_i^a &= a_i \\
\sum_i f_{ij} + x_{m+j}^a &= b_j \\
x_i^a &\geq 0 \\
f_{ij} &\geq 0 \quad (i, j) \in IJ \\
f_{ij} &= 0 \quad (i, j) \notin IJ
\end{align*}
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Hitchcock Problem (Cont’d)

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\]

\[
\zeta = \sum_{i} a_i + \sum_{j} b_j - 2 \sum_{(i, j) \in IJ} f_{ij}
\]
Hitchcock Problem (Cont’d)

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\zeta = \sum_{i} a_i + \sum_{j} b_j - 2 \sum_{(i,j) \in IJ} f_{ij}
\]

- So, we are maximizing the flow through admissible edges
Alphabeta

- We will reach an optimal flow for RP called nonbreakthrough situation
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\[ I^* = \{ i : \text{source } i \text{ is labeled} \} \]
\[ J^* = \{ j : \text{sink } j \text{ is labeled} \} \]

\[ \theta = \min_{i \in I^*, j \notin J^*} \left[ \frac{c_{ij} - \alpha_i - \beta_j}{2} \right] \]

Algorithm

1. choose \( \alpha, \beta \) feasible in D
2. while flow is not maximum do
   (a) Solve the maximum flow in RP using only admissible edges
   (b) Find \( I^* \) and \( J^* \) and nonbreakthrough
   (c) Calculate \( \theta \) and update \( \alpha \) and \( \beta \)
Equivalency

• Hitchcock and Min-Cost Flow are equivalent
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  - Hitchcock can be solved using MCF
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• Min-Cost Flow and Min-Cost Circulation are equivalent
Equivalency

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• Min-Cost Flow and Min-Cost Circulation are equivalent
  ★ MCC can be solved using MCF
Equivalency

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  - MCF can be solved using Hitchcock
- Min-Cost Flow and Min-Cost Circulation are equivalent
  - MCC can be solved using MCF
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The End