Information Processing Letters 112 (2012) 118–123

Contents lists available at SciVerse ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Optimal online pricing with network externalities

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ARTICLE INFO

Article history:
Received 8 November 2010
Received in revised form 2 November 2011
Accepted 3 November 2011
Available online 6 November 2011
Communicated by Ł. Kowalik

Keywords:
Social networks
Pricing
Approximation algorithm
Online algorithm
Market

ABSTRACT

We study the optimal pricing strategy for profit maximization in presence of network externalities where a decision to buy a product depends on the price offered to the buyer and also on the set of her friends who have already bought that product. We model the network influences by a weighted graph where the utility of each buyer is the sum of her initial value on the product, and the linearly additive influence from her friends. We assume that the buyers arrive online and the seller should offer a price to each buyer when she enters the market. We also take into account the manufacturing cost. In this paper, we first assume that the monopolist defines a unique price for the product and commits to it for all buyers. In this case, we present an FPTAS algorithm that approximates the optimal price with a high probability. We also prove that finding the optimum price is NP-hard. Second, we consider a market with positive network externalities and assume that the monopolist could offer a private price to each customer. We prove that this problem is also hard to approximate for linear influences. On the positive side, we present a polynomial time algorithm for the problem when influences are symmetric. At last, we show that the seller has more ability to extract influences with price discrimination.

1. Introduction

Social networks play an important role in providing information and influencing choices. This is realized by the rapid growth of online social networking such as Facebook, MySpace and Twitter, which produce a huge and valuable information for advertising and online businesses. How can companies use this data to design comprehensive business strategies, and use network influences to earn more money?

Literature contains many papers that propose reasonable business models for monetizing social networks for advertising purposes [1,2]. An alternative approach which we follow in this paper, is to use networks influences and to design clever pricing strategies [3–5].

In this paper, we focus on designing intelligent pricing strategies for the seller when products exhibit *network effect*. That is, the valuation of a buyer is affected on her friend's reactions. In many cases, buyers have *positive* influences on others, i.e., the valuation of a buyer for a product increases as more people use it. For example, many software or electronics products evolve over time: operating systems may have more bugs and security holes at the beginning. But, as more people use it, more bugs will be fixed and it becomes more reliable. Therefore the more people use the product, the more inherent value it accrues. As another example of positive effects, consider a cell-phone service that offers extra discounts for calls among people who use it most. The value of such service increases as more friends buy it.

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[1] This author's research was partially supported by the IPM under grant No. CS1389-2-01.

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doi:10.1016/j.ipl.2011.11.004

In this paper, we study this problem in a stylized model, in which a monopolistic seller wishes to design a pricing strategy for a product with network externalities. We also consider the cost of manufacturing the product in our modeling, which has been ignored in the previous works [3,5]. We assume that buyers arrive *online*. This means that the seller should offer a buyer a price when she enters the market. In this situation, the seller faces a dilemma: A low price may attract the current buyer and hence help the seller gain more from future buyers. On the other hand, considering the production cost, it may not worth to sell the product with too low a price.

Our results. Price discrimination, which is useful for revenue maximization in some settings, may result in a negative reaction from buyers [6]. Also, allowing price discrimination makes the implementation of such strategies hard. So, in many markets the seller defines a public price for the product and commits to this price [3].

In Section 3, we study the problem when the monopolist offers a single price for the product and commits to it. In this case, we develop an FPTAS algorithm that approximates the optimal price with a high probability. We then show that the problem of finding the optimum price is indeed NP-hard.

In Section 4, we explore the problem with price discrimination. In this setting, the seller could offer a private price for each upcoming buyer. We show that it is impossible to approximate the optimum price strategy. However, if influences are symmetric, we present a polynomial time algorithm which is based on network flow idea. At last, in Section 5, we compare the seller's profit for the cases of single or multiple prices. We show that the seller has more ability to extract influences with price discrimination.

Related work. Hartline et al. [5] study the problem of designing a marketing strategy for a digital good with no manufacturing cost over social networks. They model the influence among the users as a submodular function, and propose a constant-factor approximation algorithm for the optimal marketing strategy. In their marketing strategy, the seller visits all buyers in some order and offers each of them a private price. A buyer accepts or rejects the offer. The seller is the one who chooses the prices she offers as well as the order in which the buyers are visited. The main difference between this model and ours is the ability of the seller to decide about the order of buyers. They assume that the seller can visit the buyers in any order she wishes. But, in many markets, the seller cannot set the order in advance. We assume that the buyers arrive online and the seller does not have the ability to set the arriving order.

Another work which, in spirit, is close to ours is [3]. In particular, they consider an iterative posted price market in which the seller posts a public price at each time step which is visible to all buyers. Based on her valuation for the product and the offered price, each buyer may decide—in that time step—whether to buy the product. They assume that each buyer acts myopically and buys the product at the first time step in which the offered price is less than her valuation. They study the optimal pricing strategy in this setting and propose approximation algorithms for different versions of the baseline model.

2. Model

Consider the case of selling multiple copies of a good to a set V of n buyers. Each buyer is interested only in a single copy of the item. The cost of manufacturing one unit of the good is c and the seller has an unlimited supply of the good. We assume that the seller is a monopolist and is interested in maximizing her profit. In the presence of network externality, the *valuation* of buyer i for the good is a function of the buyers who already own that item, $v_i : 2^V \to \mathbb{R}^+$, i.e., $v_i(S)$ is the value of the good for buyer i, if the buyers in S already own that item. Several researchers have considered the same valuation function in their studies [5,7,3,8]. They have assumed the value of product depends on earlier buyers.

A common assumption studied in the context of network externalities is the assumption of *additive influence functions* which has been explored and justified in this framework [9,10,8,11]. In this model, the influence of buyer i on buyer j is represented by the weight of edge from i to j ($w_{i,j}$). That is, the value $v_i(S)$ for all i and S is $v_i(S) = v_i(\emptyset) + \sum_{j \in S} w_{j,i}$, where $v_i(\emptyset)$ is the initial value of buyer i and $w_{j,i}$ is the weight of edge from buyer j to buyer i in graph G. Note that if there is no edge from j to i in the graph, we assume $w_{j,i} = 0$. The *additive symmetric* model is similar to the additive model with $w_{i,j} = w_{j,i}$.

We study *optimal online pricing* strategies when buyers arrive online. In particular, at time t a buyer i enters the market and the seller should offer a price p_i to her. Then, she decides to buy the product if and only if $p_i \leqslant v_i(S_t)$, where S_t is the set of buyers who already own the product. Assume $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is a permutation over the buyers and indicates the order in which they enter the market. In other words, buyer π_t enters the market at time t. We assume the seller has *no information* about the order of the new customers and also has no prior knowledge about it. In this case, the most rational or conservative assumption is to assume all permutations of later buyers will happen with the same probability [12]. Now, the seller wishes to find a pricing strategy in order to maximize her expected profit, where the expectation is over all possible permutations of new customers.

We consider both cases of pricing strategies. In UniquePrice problem, the seller wishes to find a unique price p for the product in order to maximize her profit, i.e. for any two buyers $i \neq i'$, we have $p_i = p_{i'}$. On the other hand, in DifferentPrices problem, the seller could offer different prices to different buyers. That is, when a buyer i enters the market the seller will offer her a private price p_i. The prices offered are adaptive, i.e. they can be based on the history of previous accepts and rejects.

A pricing strategy consists of a sequence of offers $\mathbf{p} = (p_1, p_2, \ldots, p_{|V|})$. The result of a pricing strategy is sets of accepted and rejected offers. Let $U(\mathbf{p})$ be the set of buyers who have bought the product. Obviously, in UniquePrice, p can be used for $\mathbf{p}$. The profit of the seller with pricing strategy $\mathbf{p}$ would be $R(\mathbf{p}) = \sum_{i \in U(\mathbf{p})} p_i - c|U(\mathbf{p})|$, which is the sum of the payments from the accepted offers minus the cost of production. The goal is to design a pricing strategy in order to maximize this profit.

Assume that the manufacturing cost is zero. In that case, if the valuation of buyer i for the good was $v'_i(S) = v_i(S) - c$ and the seller offered the price $p'_i = p_i - c$ instead of p_i, the reaction of buyer i remains unchanged. Also, the set of buyers who accept the offers remains unchanged. With this assumption, the profit will be $\sum_{i \in U(\mathbf{p})} p'_i$, which is exactly the same as $\sum_{i \in U(\mathbf{p})} p_i - c|U(\mathbf{p})|$. Therefore, there is a one-to-one correspondence between pricing strategies in these scenarios. In the remaining part of the paper, we thus assume that the production cost is zero but the valuation function $v_i(.)$ could have negative values instead.

3. Unique price

In this section, we study the UniquePrice problem for which we first propose an FPTAS algorithm. We then prove that finding the optimum price for UniquePrice problem is NP-hard.

Let $Lower_i = \min_S \{v_i(S) \mid v_i(S) > 0\}$ be the minimum positive price at which buyer i will buy the product. If we set the price to $p \leqslant p_{\min} = \min_i \{Lower_i\}$, all buyers will accept the offer and buy the item. On the other hand, if customer j is the first customer who buys the product, the price is no more than $v_j(\emptyset)$. This means that no one will buy the product with price more than $p_{\max} = \max_i \{v_i(\emptyset)\}$. Therefore, the optimum price is between $p_{\min}$ and $p_{\max}$.

In order to design an algorithm, we show that if we assume the optimum price the form of $p = p_{\min}(1+\epsilon)^i$, we do not lose too much. Let R_p be a random variable for the seller's profit. Note that the exact value of R_p depends on the order of future buyers that the seller has no information about it. The goal in UniquePrice problem is to find a price p which maximizes $E[R_p]$, where expectation is over all possible permutations of future buyers. It is clear that $R_p = p \times U_p$, which means $E[R_p] = p \times E[U_p]$. So we can find $E[R_p]$ by calculating $E[U_p]$. Note that $E[U_p]$ cannot be computed in polynomial time by calculating U_p over all possible permutations of new buyers. But, $E[U_p]$ and therefore the expected profit of the price p can be estimated with any small error using the sampling technique, in which, for a price p, we calculate U_p in a polynomial number of trials. In each trial, first, we fix the order of future buyers over all their possible permutations, and then calculate the value of U_p in polynomial time. At last, by taking the average of calculated values of U_p in all trials, we can estimate $E[U_p]$. Since $0 \leqslant U_p \leqslant n$, using Chernoff–Hoeffding concentration inequality, we show that $E[U_p]$ can be computed with high probability within an error factor of ϵ (Lemma 1). We are now ready to propose Algorithm 1 for UniquePrice problem.

Chernoff–Hoeffding bound. Let $X_1, \ldots, X_n$ be i.i.d. (independent and identically distributed) random variables over a bounded domain $[0, 1]$ with expectation $E[X_i] = \mu$ (for all i). Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. For all $0 < \epsilon < 1$, $\Pr(|\overline{X} - \mu| > \epsilon\mu) \leqslant 2e^{-\frac{\epsilon^2 n\mu}{3}}$.

Algorithm 1 An FPTAS algorithm for UniquePrice

```
1: p_min ← min_i{Lower_i}
2: p_max ← max_i{v_i(∅)}
3: for i = 0 to log_{1+ε} p_max/p_min do
4:     q ← p_min(1 + ε)^i
5:     Compute the estimated value of E[U_q] using sampling technique.
       Name this value U_q^e.
6: end for
7: return argmax_q q U_q^e
```

Lemma 1. *Given a price $p_{\min} \leqslant p \leqslant p_{\max}$, for every $\epsilon > 0$, and $\delta' < 1$, there is an algorithm with $O(\frac{n}{\epsilon^2}\log\frac{1}{\delta'})$ trials which returns value $\overline{U}_p$ such that $\Pr(|\overline{U}_p - E[U_p]| > \epsilon E[U_p]) \leqslant \delta'$.*

Proof. Assume that $k = \operatorname{argmax}_i\{v_i(\emptyset)\}$. If we set the price $p \leqslant p_{\max}$, the buyer k will buy the product. So the random variable U_p is at least 1 in all trials. Define $E[U_p] = \mu_p$ and $Y_p = \frac{U_p}{n}$. We compute Y_p, m times. Let $\overline{Y}_p$ be equal to mean of these m values and $\overline{U}_p = n\overline{Y}_p$. By Chernoff–Hoeffding bound we have: $\Pr(|\overline{U}_p - \mu_p| > \epsilon\mu_p) = \Pr(|\overline{Y}_p - \frac{\mu_p}{n}| > \epsilon\frac{\mu_p}{n}) \leqslant 2e^{-\frac{\epsilon^2 m\mu_p}{3n}}$. Therefore, by setting $m = \frac{3n}{\epsilon^2\mu_p}\log(\frac{2}{\delta'})$, we conclude the lemma. Note that $\mu_p \geqslant 1$ which implies $m \in O(\frac{n}{\epsilon^2}\log\frac{1}{\delta'})$. □

Theorem 2. *For every $\epsilon > 0$ and $\delta < 1$, Algorithm 1 finds a price p_{ALG} with $O(\frac{nk}{\epsilon^2}\log\frac{k}{\delta})$ trials, where $k = \log_{1+\epsilon}(p_{\max}/p_{\min})$, such that $E[R_{p_{ALG}}] \geqslant E[R_{p_{OPT}}]\frac{1-\epsilon}{(1+\epsilon)^2}$ with probability at least $1-\delta$.*

Proof. Let p_{OPT} be the optimum price and $p_i = p_{\min}(1+\epsilon)^i$. Therefore there is an index j such that $p_j \leqslant p_{OPT} < p_j(1+\epsilon)$. A buyer who has bought the product with price p_{OPT} will buy the product at price p_j. So, for every random sampling we have $U_{p_j} \geqslant U_{p_{OPT}}$ which means $E[U_{p_j}] \geqslant E[U_{p_{OPT}}]$. Note that $p_{OPT} \leqslant p_j(1+\epsilon)$ and $E[R_p] = pE[U_p]$. We can thus conclude that $E[R_{p_j}] \geqslant \frac{1}{1+\epsilon}E[R_{p_{OPT}}]$.

Let $k = \log_{1+\epsilon}^{p_{\max}/p_{\min}}$ and $\delta' = \frac{\delta}{k}$. Let f_i be the estimated value for $E[R_{p_i}] = p_i E[U_{p_i}]$ using Lemma 1. For every i the value of f_i will be out of interval $[(1-\epsilon)E[R_{p_i}], (1+\epsilon)E[R_{p_i}]]$ with probability of at most δ'. So, with probability at least $(1-\delta')^k \geqslant 1 - \delta' k = 1 - \delta$, we have $(1-\epsilon)E[R_{p_i}] \leqslant f_i \leqslant (1+\epsilon)E[R_{p_i}]$ for all i. Now, assume that the algorithm returns p_{ALG} as the best price. So, $E[R_{p_{ALG}}] \geqslant \frac{f_{ALG}}{1+\epsilon} \geqslant \frac{f_j}{1+\epsilon} \geqslant E[R_{p_j}]\frac{1-\epsilon}{1+\epsilon}$ with probability of at least $1-\delta$. Note that $E[R_{p_j}] \geqslant \frac{1}{1+\epsilon}E[R_{p_{OPT}}]$, which means $E[R_{p_{ALG}}] \geqslant E[R_{p_{OPT}}]\frac{1-\epsilon}{(1+\epsilon)^2}$ with probability at least $1-\delta$. □

Theorem 3. *Finding the optimum price for* UniquePrice *problem is* NP-*hard.*

Proof. Assume that, we are given a set of integers $A = \{a_1, a_2, \ldots, a_n\}$, and we are asked whether we can partition A into two disjoint subsets A_1, A_2 such that $\sum_{a_{i \in A_1}} a_i = \sum_{a_{j \in A_2}} a_j$. We call this problem *Perfect Partition* which is known to be NP-complete [13]. We reduce this problem to finding the optimum price in UniquePrice. Let $SUM =$

$\sum_{a_i \in A} a_i$. Without loss of generality, assume that *SUM* and n are even numbers. Let $q = \frac{1}{2} + \frac{(\frac{n}{2})!^2}{(n+1)!}$ and m be the real number such that $\frac{n}{q} < m < 2n$. Note that m can be represented by $O(poly(n))$ bits.

We will build an instance of UniquePrice such that the answer to the *Perfect Partition* instance is YES if and only if the optimum price is m in this instance of UniquePrice. Consider an instance of UniquePrice with $2n+1$ buyers. Let $v_i(\emptyset) = m$ for $1 \leqslant i \leqslant n$, $v_i(\emptyset) = 2m+1$ for $n+1 \leqslant i \leqslant 2n$, and $v_{2n+1}(\emptyset) = m - \frac{SUM}{2}$. Finally, define $w_{i,2n+1} = a_i$, for every $1 \leqslant i \leqslant n$, and set the weight of other edges to 0.

We show that the optimum price p can only be m or $2m+1$. The optimum price will not be more than $2m+1$, since none of the buyers would buy the product. Now, consider the price $m < p < 2m+1$. The initial value of buyers 1 to n is m and there is no influence on them. So, they don't buy the product with price $m < p < 2m+1$. Therefore, buyer $2n+1$ does not receive any positive effect from buyers 1 to n and does not buy the product. It means that only buyers $n+1$ to $2n$ will buy the product and $R(p) = np$. If the seller sets the price to $2m+1$, then the buyers $n+1$ to $2n$ will buy the product and $R(2m+1) = n(2m+1) > R(p)$.

Now, we prove that the optimum price is not less than m. Assume that the optimum price is $m - \epsilon$ and the seller offers it. Because the initial value of buyers 1 to $2n$ is at least m, all buyers' reaction to prices $m - \epsilon$ and m are the same except for the buyer $2n+1$. In order to have $R(m-\epsilon) > R(m)$, there should be a permutation of arrivals in which the buyer $2n+1$ buys the product at price $m - \epsilon$ and does not buy it at price m, which means that $m - \epsilon < v_{2n+1}(S) < m$, where buyers in S own the product when buyer $2n+1$ enters the market. Since, $v_{2n+1}(S) = m - \frac{SUM}{2} + \sum_{i \in S} a_i$ and $-\frac{SUM}{2} + \sum_{i \in S} a_i$ is an integer value less than zero, then we can conclude $\epsilon \geqslant 1$. Now, we compare the profit when the seller offers prices $m-\epsilon$ and $2m+1$. We know $R(m-\epsilon) \leqslant (m-\epsilon)(2n+1)$ and $R(2m+1) = n(2m+1)$. Therefore, according to the facts that $\epsilon \geqslant 1$ and $m < 2n$, we have $R(2m+1) = n(2m+1) > (m-\epsilon)(2n+1) \geqslant R(m-\epsilon)$. So, the only candidates for optimum price are m and $2m+1$.

Finally, we prove that the answer to Perfect Partition is NO if and only if optimum price is $2m+1$ i.e. $R(2m+1) > R(m)$. Note that when the price is m, buyers 1 to $2n$ would buy the product and buyer $2n+1$ may or may not buy. Assume that buyer $2n+1$ enters the market after buyers of set S. The buyer $2n+1$ will buy the product if $\sum_{i \in S, i \leqslant n} a_i \geqslant \frac{SUM}{2}$.

If the answer to Perfect Partition is NO for any set S, exactly one of $\sum_{i \in S, i \leqslant n} a_i$ or $\sum_{i \in \bar{S}, i \leqslant n} a_i$ is greater than $\frac{SUM}{2}$, where $\bar{S}$ is the complement set of S. Therefore, $\sum_{i \in S, i \leqslant n} a_i$ is greater than $\frac{SUM}{2}$ with the probability of $\frac{1}{2}$. Thus, the buyer $2n+1$ will buy the product with the probability of $\frac{1}{2}$ and $R(m) = 2nm + \frac{m}{2}$. On the other hand, if the seller proposes price $2m+1$, buyers $n+1$ to $2n$ would buy the product, so $R(2m+1) = n(2m+1)$. Thus, since $m < 2n$, then $R(2m+1) > R(m)$.

If the answer to Perfect Partition is YES, i.e. we can partition set A into two disjoint subsets A_1 and A_2 such that $\sum_{a_i \in A_1} a_i = \sum_{a_j \in A_2} a_j$. So $\sum_{i \in S, i \leqslant n} a_i$ is greater than or equal to $\frac{SUM}{2}$ with the probability of at least $q' = \frac{1}{2} + \frac{|A_1|! \times |A_2|!}{(n+1)!}$. Therefore, the buyer $2n+1$ will buy the product with the probability of at least q'. Thus, $R(m) \geqslant 2nm + mq'$ and according to the fact that $q \leqslant q'$ and $m > \frac{n}{q}$, we have $m > \frac{n}{q'}$. Therefore, in this case $R(m) \geqslant 2nm + mq' > (2m+1)n = R(2m+1)$ i.e. the optimum price is m. □

4. Different prices

Let the optimal offline profit be the maximum profit one could achieve if the order of buyers was already known to her. In general, *competitive analysis* is a usual way for analyzing online algorithms, where the performance of the online algorithm is compared to the performance of an optimal offline algorithm that can view the sequence of input in advance [14].

In Example 4, we show the ratio of optimal online solution over optimal offline solution could be zero. So any approximation algorithm which approximate the online solution, could not approximate the optimal offline solution. Therefore, competitive analysis doesn't work here. On the positive side, in Section 4.1, we propose a polynomial-time algorithm to find the optimal pricing strategy for the additive symmetric model.

Example 4. Assume a set of buyers $V = \{v_1, v_2, v_3\}$ such that $w_{1,2} = w_{2,3} = w_{3,1} = 5$. The weight of other edges is zero. Let the initial value of each buyer be -3. Suppose the first buyer arrives and without loss of generality, let the first buyer be v_1. Consider the two cases. If the next buyer is v_2, then the maximum profit is 1 which is gained by offering the price -3 to v_1, 2 to v_2 and again 2 to v_3. If the next buyer is v_3, then the maximum profit is 0, which is gained by offering the price 0 to all buyers. So the maximum expected offline profit is $\frac{1}{2}$.

Now, consider the optimum online algorithm. When v_1 arrives, we don't know whether v_2 is the next buyer or not. If we offer a price greater than -3 to v_1 then the maximum profit will be 0. But if we offer the price -3 to v_1 then two cases might happen, each with probability $\frac{1}{2}$. The next buyer might be v_2. In this case, the maximum profit is 1 which is gained by offering the price -3 to v_1, 2 to v_2 and again 2 to v_3. The next buyer might be v_3. In this case, the maximum profit is -1 which is gained by offering the price -3 to v_1, 0 to v_3 and 2 to v_2. So, if we offer the price -3 to v_1, the maximum profit is 1 with probability $\frac{1}{2}$, or -1, again with probability $\frac{1}{2}$. And if we offer a price greater than -3 to v_1, the maximum profit will be 0. So the expected profit of the optimal online algorithm is 0.

4.1. Additive symmetric model

Let $\mathbf{p}$ be the optimal pricing strategy and S_i be the set of buyers who already own the product when the buyer i enters the market in the optimal pricing strategy. If the seller decides to sell the product to the buyer i, she should offer her a price less than or equal to $v_i(S_i)$. Note the seller wants to maximize her profit. Therefore, she will offer the price $v_i(S_i)$. So if we know $U(\mathbf{p})$, the set of

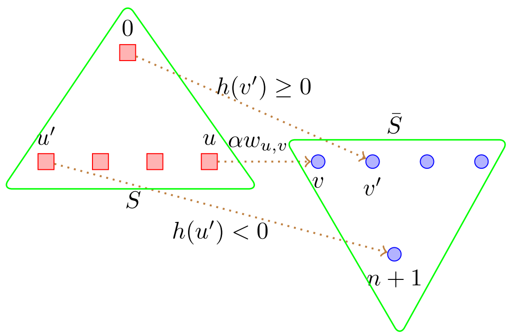

Fig. 1. Network flow for solving MWS.

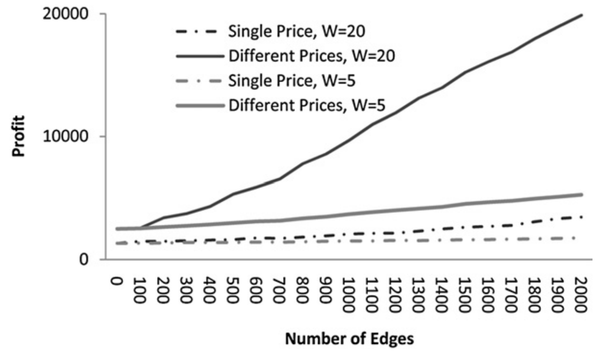

Fig. 2. The profits of seller in UniquePrice and DifferentPrices.

buyers who have bought the product, we can define the price strategy $\mathbf{p}$. The profit of the pricing strategy $\mathbf{p}$ will be $R(\mathbf{p}) = \sum_{i \in U(\mathbf{p})} p_i = \sum_{i \in U(\mathbf{p})} v_i(S_i)$. Note $v_i(S_i)$ could be negative.

Note that in the additive model $v_i(S_i) = v_i(\emptyset) + \sum_{j \in S_i} w_{j,i}$. Therefore, the profit is $R(\mathbf{p}) = \sum_{i \in U(\mathbf{p})} v_i(\emptyset) + \sum_{j,i \in U(\mathbf{p}), \pi^{-1}(j) < \pi^{-1}(i)} w_{j,i}$, where $\pi^{-1}(i)$ is the time when buyer i enters the market. Note that $w_{i,j} = w_{j,i}$. So we can rewrite the profit as $R(\mathbf{p}) = \sum_{i \in U(\mathbf{p})} v_i(\emptyset) + \frac{1}{2} \sum_{j,i \in U(\mathbf{p})} w_{j,i}$. In the remaining part of this section, we propose a polynomial algorithm to find the set $U(\mathbf{p})$ which maximizes $R(\mathbf{p})$. First, we define the *Maximum Weighted Set* problem. In the Maximum Weighted Set problem (MWS) we want to find a subset S of nodes in a weighted directed graph G which maximize $\sum_{i \in S} I_i + \alpha \sum_{(j,i) \in S} w_{j,i}$, where $\alpha \geqslant 0$, I_i is an initial value assigned to node i, and $w_{j,i} \geqslant 0$ is the weight of edge (j,i) in the graph G. We propose an algorithm to solve MWS in Lemma 5. Note that if we set $I_i = v_i(\emptyset)$ and $\alpha = \frac{1}{2}$, then the best set $U(\mathbf{p})$ will be the optimum solution to MWS. So we conclude Theorem 6 by using Lemma 5.

Lemma 5. MWS *is polynomial-time solvable.*

Proof. We propose an algorithm for MWS based on a maximum flow algorithm. We define a network flow F based on graph G and prove that there is a close relation between the minimum cut of the network flow F and the maximum weighted set of graph G.

Define $h_i = I_i + \alpha \sum_j w_{i,j}$. Let F be a network flow with $n+2$ vertices numbered from 0 to $n+1$. For any $1 \leqslant i \leqslant j \leqslant n$, put an edge with capacity $\alpha w_{i,j}$ from vertex i to vertex j. For every $1 \leqslant i \leqslant n$ and $h_i \geqslant 0$, put an edge with capacity h_i from vertex 0 to vertex i. For every $1 \leqslant i \leqslant n$ and $h_i < 0$, put an edge with capacity $-h_i$ from vertex i to vertex $n+1$. Let vertex $s = 0$ be the source and vertex $t = n+1$ be the sink of the network flow F. Network flow F has been shown in Fig. 1.

Let $C(S, \bar{S})$ be the weight of the $(S, \bar{S})$ cut, which is equal to $\sum_{i \in S,\, j \notin S} \alpha w_{i,j} + \sum_{i \notin S,\, h_i \geqslant 0} h_i + \sum_{i \in S,\, h_i < 0} -h_i$. There is a polynomial time algorithm which finds a cut S^* which minimizes $C(S, \bar{S})$. Define $H = \sum_{h_i \geqslant 0} h_i$. The value of H is independent of the cut $(S, \bar{S})$. So the cut $S^* = \operatorname{argmax}_S H - C(S, \bar{S})$ is the minimum cut of the network flow F. We can express the value of $H - C(S, \bar{S})$ as $\sum_{i \in S,\, h_i \geqslant 0} h_i - \sum_{i \in S,\, j \notin S} \alpha w_{i,j} + \sum_{i \in S,\, h_i < 0} h_i$, which is equal to $\sum_{i \in S} h_i - \sum_{i \in S,\, j \notin S} \alpha w_{i,j}$. If we substitute h_i by $I_i + \alpha \sum_j w_{i,j}$, we can rewrite $H - C(S, \bar{S})$ as $\sum_{i \in S} (I_i + \sum_j \alpha w_{i,j}) - \sum_{i \in S, j \notin S} \alpha w_{i,j}$ which is equal to $\sum_{i \in S} I_i + \alpha \sum_{i,j \in S} w_{i,j}$. So the minimum cut of network flow F is the optimum set for MWS. $\square$

Theorem 6. DifferentPrices *problem can be solved in polynomial time for the additive symmetric model.*

5. UniquePrice vs. DifferentPrices

In this section we compare the profits for the optimal solution of UniquePrice vs. DifferentPrices. First, we present an example to show that the profit in DifferentPrices is significantly more than UniquePrice. We come to similar conclusion using random graphs. It seems that the seller has more ability to increase her profit using the influences in DifferentPrices.

Example 7. Consider an additive symmetric model with n buyers. The initial value of each buyer is $c \geqslant 0$, where c is the manufacturing cost. There are m edges between buyers each with the weight of $\beta \geqslant 0$. The optimum profit in UniquePrice is 0, and the optimum profit in DifferentPrices is $nc + m\beta - nc = m\beta$. So, the optimum profit in DifferentPrices is significantly more than UniquePrice.

Finally, we compare the profits using random graphs. We build an undirected random graph with $n = 200$ nodes and $m = 0, 100, 200, \ldots, 2000$ random edges. We assume that the manufacturing cost is 50. The initial value of each node is assumed to be a random variable which is drawn uniformly from $[0, 100]$ and the weight of each edge is a random variable which is drawn uniformly form $[0, W]$, where W is equal to 5 and 20 in our simulations. First, we find a solution for UniquePrice using Algorithm 1 with 2000 trials. Second, we solve the problem for DifferentPrices using Theorem 6. The profit of seller has been shown in Fig. 2 in both models. It has been shown that the seller has more power in DifferentPrices. Also, the figure shows that, in contrast with UniquePrice, the profit of seller highly depends on amount of influences in DifferentPrices.

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