Chapter 3

Finite Element Analysis of Plane Elasticity

1. Basic Equations of Solid Mechanics (3-D)

1.1- Stress and strain

State of stress in a volume (3D)

$$\{\sigma\}^T = \{\sigma_x \ \ \sigma_y \ \ \sigma_z \quad \tau_{xy} \ \ \tau_{yz} \ \ \tau_{zx}\}$$

$\sigma_x, \sigma_y, \sigma_z =$ Normal components of stress

$\tau_{xy}, \tau_{yz}, \tau_{zx} =$ Components of shear stress

Strain $\{\varepsilon\}^T = \{\varepsilon_x \ \ \varepsilon_y \ \ \varepsilon_z \quad \gamma_{xy} \ \ \gamma_{yz} \ \ \gamma_{zx}\}$

1.2- Strain-displacement equations (Kinematics equations)

u , v and w are displacements in x,y and z directions, respectively.

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

Retaining only the first order term and neglecting second order:

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \ \varepsilon_z = \frac{\partial w}{\partial z} \qquad .$$

Only for small deformation, that each derivative is much smaller than unity

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad .$$

for large deformation higher terms must be retained " geometric nonlinearity "

1.3- Linear Constitutive Equations

The stress tensor and strain tensor are related. These relations depend on the nature of the material and are called constitutive laws. We shall be concerned in most of this text with linear elastic behavior, wherein each stress components is linearly related, in the general case, to all the strains by equations of the form(and vice versa):

$$\tau_{ij} = C_{ijkl} \ \varepsilon_{kl}$$

C's are at most functions of position. This law is called generalized Hook's law.

Since τ_{ij} and ε_{kl} are second-order tensor fields and symmetric, C_{ijkl} must be symmetric in ij and kj and also fourth-order tensor field. We may assume that the material is homogeneous(same composition throughout) so C_{ijkl} is constant for a given reference.

Hook's law in one dimensional space can be written as: $(\sigma = E\varepsilon)\ one-D$

The generalized Hook's law in matrix notation can be written as.

$$3-D \qquad \{\sigma\}=[C]\,\{\varepsilon\}$$

$$\{\varepsilon\}=[D]\,\{\sigma\}$$

$$[C]=\begin{bmatrix} c_{11} & c_{12} \,\ldots\ldots\, c_{16} \\ c_{21} & c_{22} \,\ldots\ldots\, c_{26} \\ c_{61} & c_{62} \,\ldots\ldots\, c_{66} \end{bmatrix}$$

Starting with 81 (3^4) terms for C_{ijkl} due to symmetry only 21 terms are independent for Linear elastic, anisotropic and homogeneous material. Therefore, [C] and [D] are symmetric with $\underline{\underline{21}}$ experimental evaluation of elastic constant.

For linear orthotropic, [C] becomes:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & . & . & . \\ & c_{22} & c_{23} & . & . & . \\ & & c_{33} & . & . & . \\ & & & c_{44} & . & . \\ & & & & c_{55} & \\ & & & & & c_{66} \end{bmatrix} \text{ 9 constants}$$

The stress-strain equations for orthotropic materials may be written in terms of the young's moduli and poisson's ratios as dollows:

$$\varepsilon_x = \frac{1}{E_x}\sigma_x - \frac{\upsilon_{yx}}{E_y}\sigma_y - \frac{\upsilon_{zx}}{E_z}\sigma_z$$

$$\varepsilon_y = -\frac{\upsilon_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_y - \frac{\upsilon_{zy}}{E_z}\sigma_z$$

$$\varepsilon_z = -\frac{\upsilon_{xz}}{E_x}\sigma_x - \frac{\upsilon_{yz}}{E_y}\sigma_y + \frac{1}{E_z}\sigma_z$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \quad \gamma_{yz} = \frac{\tau_{yz}}{G_{yz}} \quad \gamma_{zx} = \frac{\tau_{zx}}{G_{zx}}$$

There are 12 material parameters which only 9 are independent. This is because of the followings:

$$\frac{E_x}{\upsilon_{xy}} = \frac{E_y}{\upsilon_{yx}} \quad \frac{E_y}{\upsilon_{yz}} = \frac{E_z}{\upsilon_{zy}} \quad \frac{E_z}{\upsilon_{zx}} = \frac{E_x}{\upsilon_{xz}}$$

Linear Isotropic Elasticity

Isotropic material are those that have point of symmetry, that is, every plane is a plane of symmetry of material behavior. This property requires that mechanical properties of a material at a point are not dependent on direction. Thus a stress such as τ_{xx} must be related to all the strains ε_{ij} for reference xyz exactly as the stresses $\tau_{x'x'}$ is related to all the strains ε'_{ij} for a reference x'y'z' rotated relative to xyz. Accordingly, C_{ijkl} must have the same components for all references. A tensor such as C_{ijkl} whose components are invariant wrt a rotation of axes is called isotropic tensor.

Only two independent elastic constants are necessary to represent the behavior for linear, isotropic and elastic material, the stress-strain relation in this case can be written as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\upsilon}{E} & \frac{-\upsilon}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & \frac{-\upsilon}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \frac{2(1+\upsilon)}{E} & 0 & 0 \\ & Symmetric & & & \frac{2(1+\upsilon)}{E} & 0 \\ & & & & & \frac{2(1+\upsilon)}{E} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

or in terms of stress components:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\upsilon)(1-2\upsilon)} \begin{bmatrix} 1-\upsilon & \upsilon & \upsilon & 0 & 0 & 0 \\ & 1-\upsilon & \upsilon & 0 & 0 & 0 \\ & & 1-\upsilon & 0 & 0 & 0 \\ & & & \frac{1-2\upsilon}{2} & 0 & 0 \\ & Symmetric & & & \frac{1-2\upsilon}{2} & 0 \\ & & & & & \frac{1-2\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

It is well to know that a material may be both isotropic and inhomogeneous or, conversely, anisotropic and homogeneous. These two characteristics are independent.

1.4 Potential Energy for a Linear Elastic Body (general form)

The potential energy can be written as:

$$\pi = U - W$$

$$U = Strain\ Energy$$

$$W = Potential\ Energy\ of\ the\ Applied\ Loads(\ body\ forces\ and\ surface\ traction)$$

$$\pi = \iiint_V du(u,v,w) - \iiint_V \left(\bar{f}_x u + \bar{f}_y v + \bar{f}_z w \right) dV - \iint_{S_1} \left(\bar{T}_x u + \bar{T}_y v + \bar{T}_z w \right) dS_1$$

S_1 is surface of the body on which surface tractions are prescribed.
d(u,v,w) is strain energy per unit volume (strain energy density).
The last two integrals represent the work done by the constant external forces, that is, the body forces $\bar{f}_x, \bar{f}_y, and\ \bar{f}_z$ and surface tractions $\bar{T}_x, \bar{T}_y\ and\ \bar{T}_z$. A bar at the top of a letter indicates that the quantity is specified.

$$du(u,v,w) = \frac{1}{2}\{\varepsilon\}^T\{\sigma\}dV = \frac{1}{2}\{\varepsilon\}^T[C]\{\varepsilon\}dV \quad (1-D : \frac{1}{2}\varepsilon\sigma dl = \frac{1}{2}\frac{\sigma^2}{E} dl\ strain\ energy)$$

$$\pi = \frac{1}{2}\iiint_V \left(\{\varepsilon\}^T[C]\{\varepsilon\} - 2\{u\}^T\left\{\bar{f}\right\} \right) dV - \iint_{S_1} \{u\}^T \left\{\bar{T}\right\} dS_1$$

$$where \quad \{u\}^T = \{u \quad v \quad w\}$$

$$\left\{\bar{f}\right\}^T = \left\{\bar{f}_x \quad \bar{f}_y \quad \bar{f}_z\right\}$$

$$\left\{\bar{T}\right\}^T = \left\{\bar{T}_x \quad \bar{T}_y \quad \bar{T}_z\right\}$$

For springs on the boundary we can write for a 2-D case:

$$du = du + \frac{1}{2}\{u\}^T[\alpha]\{u\}$$

$$springs\ on\ the\ boundary\ must\ be\ added\ to\ du = du + \frac{1}{2}\alpha_{ij}u_i u_j$$

$$du = du + \frac{1}{2}\{u \quad v\}\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$effects\ of\ coupled\ springs$$

2-Dimensional Specializations of Elasticity

Some times due to geometry and loading configurations a 3-D problem may reduce to problem in one or 2-D.

2.1- Plane Strain

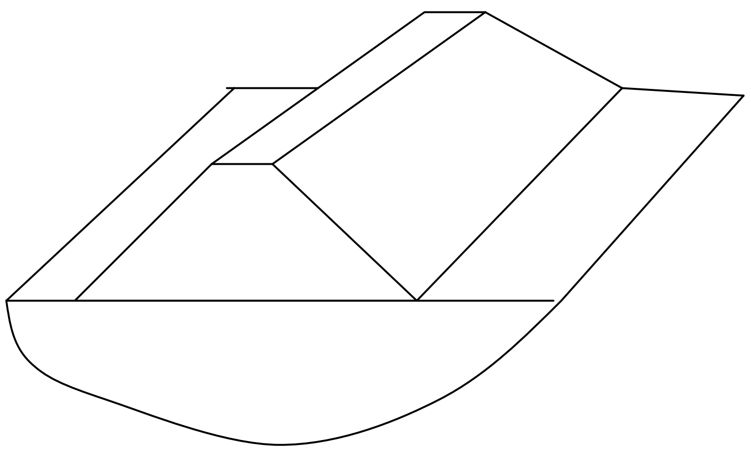
In this case, strain normal to the plane is zero. Long body whose geometry and loading do not vary significantly in the longitudinal direction are taken as plain strain cases, such as dams, retaining walls, etc.

In plain stress problems, we may consider only a slice of unit thickness.
If f(x,y) is variable at a cross section some distance away from the ends, we may assume w=0 (displacement along the z direction. Then

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \qquad then \quad \sigma_z = \nu(\sigma_x + \sigma_y)$$

nonzero strains are ε_x , ε_y and ε_z. Then we can write:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$



Earth dam

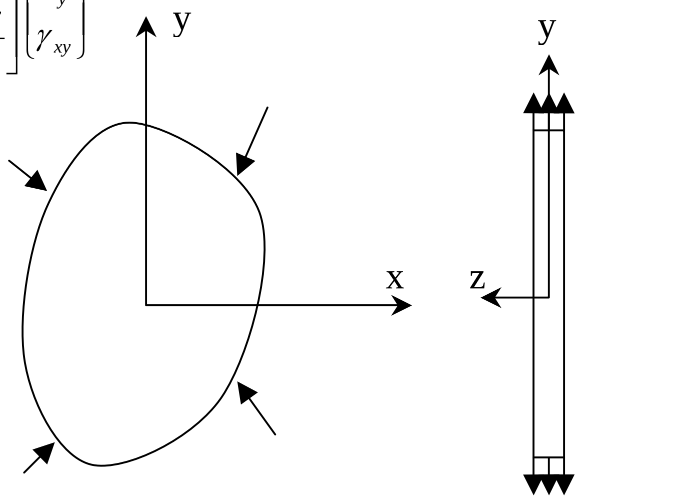
2.2- Plane Stress

In this case, stress normal to the plane is zero such that problem can be characterized by very small dimensions in the z direction for example thin plate loaded in its plane. No loading are applied on the surface of the plate, then:

$$\tau_{yz} = \tau_{zx} = \sigma_z = 0 \quad then \quad \varepsilon_z = \frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)$$

σ_x, σ_y *and* τ_{xy} are averaged over the thickness and independent of z.

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$



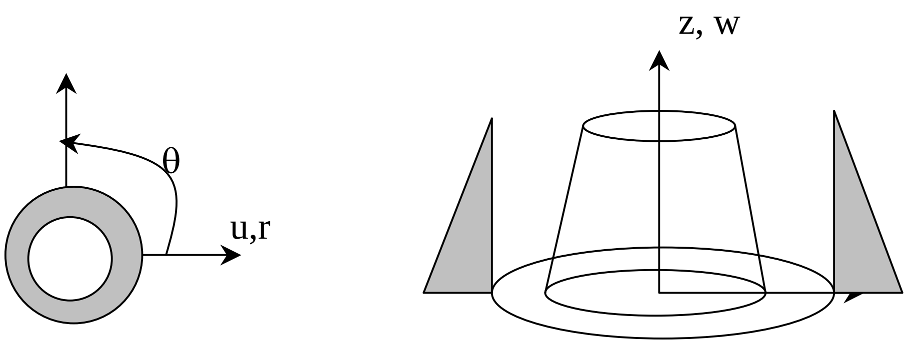
Plane stress: thin plate with in-plane loading

2.3- Axisymmetric Problem

In this case, axisymmetric solids subjected to axially symmetric loading. Due to symmetry, stress components are independent of angular coordinates. Hence, all derivatives wrt θ vanish and component $v, \gamma_{r\theta}, \gamma_{\theta z}, \tau_{r\theta}, \tau_{\theta_z}$ become zeros. Then nonzero components relation can be written as:

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad \varepsilon_\theta = \frac{u}{r} \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{rz}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0\\ & 1-\nu & \nu & 0\\ & & 1-\nu & 0\\ symmetry & & & \frac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_r\\ \varepsilon_z\\ \varepsilon_\theta\\ \gamma_{rz}\end{Bmatrix}$$



Cylinder under axisymmetric loading

3. Plane Elasticity

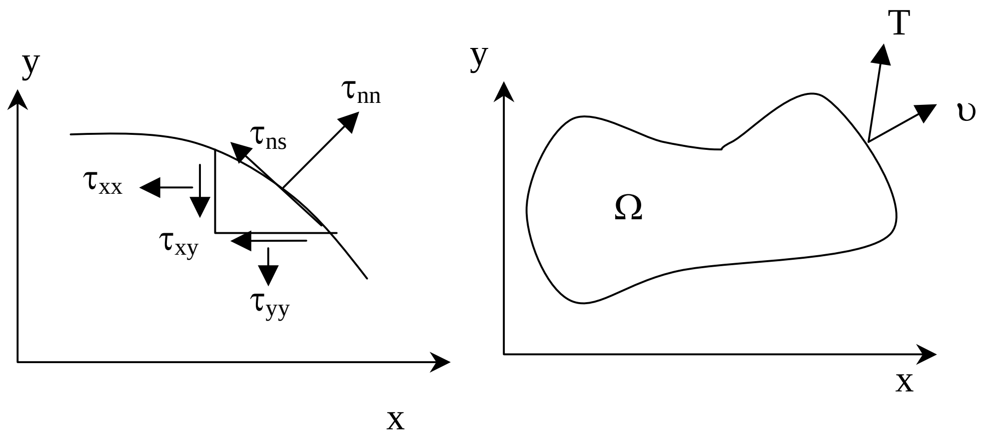
In a domain of Ω for i=1,2,3 and j=1,2,3 we have:

$\tau_{ij,j} = f_i$ *Equilibrium Equations*

$\tau_{ij} = E_{ijkl}\varepsilon_{kl}$ *Constitutive Equations*

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ *Kinematics Equations*

These three set of equations have to be satisfied within the two dimensional domain of Ω. Further, these are also subjected to some boundary conditions.



Following three sets of boundary conditions occur commonly in plane elasticity.

a) Homogeneous boundary conditions

$\tau_{ij}\upsilon_j = 0$ *on* S_T

$u_i = 0$ *on* S_u

υ_j =components of unit outward normal

S_T=Stress free part of the boundary

S_u=Part of the boundary where displacements are specified to be zero

b) Mixed homogeneous boundary conditions

$\tau_{ij}\upsilon_j = 0$ *on* S_T

$u_i = 0$ *on* S_u

$\tau_{ij}\upsilon_j + \alpha_{ij}u_i = 0$ *on* S_M

α_{ij} =Constants such as spring constant

S_M=Part of the boundary where mixed conditions are specified (e.g. springs or elastic foundation etc.)

c) Non homogeneous boundary conditions (Mixed)

$\tau_{ij}\upsilon_j = \bar{T}_i \qquad on\ S_T$

$u_i = \bar{u}_i \qquad on\ S_u$

$\tau_{ij}\upsilon_j + \alpha_{ij}u_i = \bar{C}_i \quad on\ S_M$

Where quantities with the bar on the top are specified.

Let us consider the following condition for now:

$\tau_{ij}\upsilon_j = \bar{T}_i \qquad on\ S_T$

$u_i = 0 \qquad on\ S_u$

$\tau_{ij}\upsilon_j + \alpha_{ij}u_i = 0 \quad on\ S_M$

Potential energy expression is then given by (t=thickness):

$$\pi = t\left[\frac{1}{2}\int_{\Omega}\tau_{ij}\varepsilon_{ij}d\Omega - \int_{\Omega} f_i u_i d\Omega - \int_{S_T}\bar{T}_i\, u_i dS + \frac{1}{2}\int_{S_M}\alpha\, u_i u_i dS\right]$$

equation 1

where the last two integrals are the boundary integrals (line integrals).

3.1- Strain Energy in Plane Elasticity

Previously, the strain energy had been written either in tensor notation or matrix notation.
In x, y coordinates, strain energy can be written as:

$U = Strain\ Energy$

$$U = \frac{1}{2}\iiint_V \left(\tau_{xx}\varepsilon_{xx} + \tau_{yy}\varepsilon_{yy} + \gamma_{xy}\varepsilon_{xy}\right)dV$$

$if\ \ t\ is\ constant\ then:$

$$U = \frac{1}{2}t\iiint_{\Omega} \left(\tau_{xx}\varepsilon_{xx} + \tau_{yy}\varepsilon_{yy} + \gamma_{xy}\varepsilon_{xy}\right)dxdy$$

Substituting for stresses from constitutive equations of plane stress:

$$U = \frac{Et1}{2(1-\nu^2)}\iiint_{\Omega} \left(\varepsilon_{xx}{}^2 + \varepsilon_{yy}{}^2 + 2\nu\varepsilon_{xx}\varepsilon_{yy} + \frac{1-\nu}{2}\gamma_{xy}{}^2\right)dxdy$$

Substituting for strains from kinematic relations:

$$U = \frac{Et1}{2(1-\nu^2)}\iiint_{\Omega} \left(u_x{}^2 + u_y{}^2 + 2\nu u_x u_y + \frac{1-\nu}{2}\left(u_y + v_x\right)^2\right)dxdy$$

In plane stress formulation, if we change ν with ν/(1-ν) and E with E/(1-ν^2) then the constitutive equations would be for plane strain.
You can compare the different boundary conditions of above with the boundary conditions obtained in assignment no.1 problem 4.

3.2- Kinematic Relations for FE Analysis of Plane Stress and Strain Problems

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \{\varepsilon\} = [L]\{u\}$$

In simplified notation:

$$\{\tau\} = [D]\{\varepsilon\}$$

Where [D] is the elasticity matrix obtained previously. [D] for plane stress and strain problems are different. [L] is the linear operator matrix.

Next consider approximations for u(x,y) and v(x,y). Suppose those are given by (within an element):

$$u(x,y) = \phi_1 u_1 + \phi_2 u_2 + \phi_3 u_3$$

$$v(x,y) = \phi_1 v_1 + \phi_2 v_2 + \phi_3 v_3$$

$$\{u^e\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{u^e\} = [\phi]\{\delta^e\}$$

$$\{\varepsilon\} = [L][\phi]\{\delta^e\}$$

$$\{\tau\} = [D]\{\varepsilon\} = [D][L][\phi]\{\delta^e\}$$

$$\{\tau\}^T = [D]\{\varepsilon\} = \{\delta^e\}^T \left([L][\phi]\right)^T [D] \quad note: \quad [D]^T = [D]$$

Substituting above equations into the strain energy, we get strain energy within an element:

$$U_e = \frac{t}{2}\iint_{\Omega^e} \left[\{\delta^e\}^T \left([L][\phi]\right)^T [D]\right] \left[[L][\phi]\{\delta^e\}\right] d\Omega^e$$

$$[B] = [L][\phi]$$

$$U_e = \frac{t}{2}\{\delta^e\}^T \iint_{\Omega^e} [B]^T [D][B] d\Omega^e \ \ \{\delta^e\}$$

$\{\delta^e\}$ *is independent of x and y*

Assuming homogeneous boundary conditions i.e. the last two integrals in equation 1are zero. The body force term then yields:

$$t\int_{\Omega^e} \left\{\bar{f}\right\}^T \{u\} d\Omega^e = t\int_{\Omega^e} \{u\}^T \left\{\bar{f}\right\} d\Omega^e$$

$$\text{where} \quad \left\{\bar{f}\right\} = \begin{Bmatrix} \bar{f}_x \\ \bar{f}_y \end{Bmatrix}$$

$$t\int_{\Omega^e} \{u\}^T \left\{\bar{f}\right\} d\Omega^e = t\{\delta^e\}^T \int_{\Omega^e} [\phi]^T \begin{Bmatrix} \bar{f}_x \\ \bar{f}_y \end{Bmatrix} d\Omega^e = W^e$$

i.e. work done by body forces or PE of body forces

$$\pi_e = \frac{t}{2}\{\delta^e\}^T \iint_{\Omega^e} [B]^T [D][B] d\Omega^e \ \ \{\delta^e\} - t\{\delta^e\}^T \int_{\Omega^e} [\phi]^T \left\{\bar{f}\right\} d\Omega^e$$

$$\delta\pi_e = \delta\{\delta^e\}^T \left[\left(t\iint_{\Omega^e} [B]^T [D][B] d\Omega^e \right)\{\delta^e\} - \left(t\int_{\Omega^e} [\phi]^T \left\{\bar{f}\right\} d\Omega^e\right)\right]$$

$$[K^e]\{\delta^e\} - \left\{\bar{F}_e\right\} = 0$$

3.3- Initial Stresses and Strains

We showed that stresses are given by:

$$\{\tau\}=[D]\{\varepsilon\}$$

In general, material within the element boundaries may be subjected to initial strain $\{\varepsilon_0\}$such as may be due to temperature changes, shrinkage etc.. In addition, the body may be stresses by some known stresses $\{\tau_0\}$ (residual stresses, …), such stresses, for instance could be measured but the prediction of which, without the full knowledge of the material history is impossible. When both, initial stresses and strains are taken into account, then the stresses due to externally applied loads are given by:

$$\{\tau\}=[D](\{\varepsilon\}-\{\varepsilon_0\})+\{\tau_0\}$$
$$\{\tau\}=[D]\{\varepsilon\}-[D]\{\varepsilon_0\}+\{\tau_0\}$$

now the strain energy in the jth element is given by :

$$U_e=\frac{1}{2}t\int_{\Omega^e}\left\{\bar{\varepsilon}\right\}^T\{\tau\}d\Omega^e$$

$$\left\{\bar{\varepsilon}\right\}=\text{Strain causing stress}=\{\varepsilon\}-\{\varepsilon_0\}$$

$$U_e=\frac{1}{2}t\int_{\Omega^e}(\{\varepsilon\}-\{\varepsilon_0\})^T[D](\{\varepsilon\}-\{\varepsilon_0\})d\Omega^e+t\int_{\Omega^e}(\{\varepsilon\}-\{\varepsilon_0\})^T\{\tau_0\}d\Omega^e$$

Expanding the above equation yields:

$$U_e=\frac{1}{2}t\int_{\Omega^e}\left(\{\varepsilon\}^T[D]\{\varepsilon\}-\{\varepsilon\}^T[D]\{\varepsilon_0\}-\{\varepsilon_0\}^T[D]\{\varepsilon\}+\{\varepsilon_0\}^T[D]\{\varepsilon_0\}\right)d\Omega^e+$$
$$t\int_{\Omega^e}\{\varepsilon\}^T\{\tau_0\}d\Omega^e-t\int_{\Omega^e}\{\varepsilon_0\}^T\{\tau_0\}d\Omega^e$$

The first term on the LHS is the same as that we obtained before:

$$U_e=\frac{t}{2}\int_{\Omega^e}\{\varepsilon\}^T[D]\{\varepsilon\}d\Omega^e$$

The second and third terms are equal:

$$U_e = \frac{1}{2} t \int_{\Omega^e} \{\varepsilon\}^T [D]\{\varepsilon\} d\Omega^e \underbrace{- t \int_{\Omega^e} \{\varepsilon\}^T [D]\{\varepsilon_0\} d\Omega^e}_{*} + \frac{t}{2} \int_{\Omega^e} \{\varepsilon_0\}^T [D]\{\varepsilon_0\} d\Omega^e +$$

$$\underbrace{t \int_{\Omega^e} \{\varepsilon\}^T \{\tau_0\} d\Omega^e}_{*} - t \int_{\Omega^e} \{\varepsilon_0\}^T \{\tau_0\} d\Omega^e$$

$$* = t \int_{\Omega^e} \{\tau_0\}^T \{\varepsilon\} d\Omega^e - t \int_{\Omega^e} \{\varepsilon_0\}^T [D]\{\varepsilon\} d\Omega^e = t \int_{\Omega^e} \left(\{\tau_0\}^T - \{\varepsilon_0\}^T [D]\right)[L][\phi] d\Omega^e \left\{\delta^e\right\}$$

$$= \left\{\bar{F}_e\right\}^T \left\{\delta^e\right\} = t \int_{\Omega^e} \left\{\bar{f}\right\}^T [L][\phi] d\Omega^e \left\{\delta^e\right\} = t \int_{\Omega^e} \left\{\bar{f}\right\}^T [B] d\Omega^e \left\{\delta^e\right\}$$

where

$$\left\{\bar{f}\right\}^T = \{\tau_0\}^T - \{\varepsilon_0\}^T [D]$$

Hence:

$$\left\{\bar{F}_e\right\}^T = t \int_{\Omega^e} [B]^T \left(\{\tau_0\} - [D]\{\varepsilon_0\}\right) d\Omega^e$$

Now if we rewrite the potential energy expression ,

$$\pi_e = \frac{1}{2} \left\{\delta^e\right\}^T [K^e] \left\{\delta^e\right\} + \left\{\bar{f}\right\}^T \left\{\delta^e\right\} - \{F_e\}^T \left\{\delta^e\right\} + Cons\tan ts$$

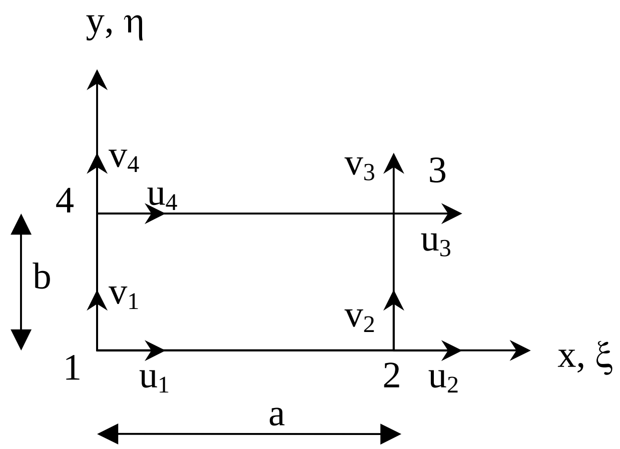
Minimizing Yields:

$$[K^e]\left\{\delta^e\right\} = \left\{\bar{f}\right\} - \{F_e\}^T$$

Initial strains and stresses (know priori) contribute to the load vector.

4- Plane Stress Rectangular Element

We have four corner nodes and it is logical to have u and v displacements as dof at each node. Hence, there are 8 dof per elements.



Let the generalized forces at the i[th] node be U_i and V_i. Assume the polynomials (within the element) for u and v:

$$u(\xi,\eta) = a + b\xi + c\eta + d\xi\eta$$

$$v(\xi,\eta) = \underbrace{e + f\xi + g\eta}_{Linear} + \underbrace{h\xi\eta}_{bilinear}$$

The ξη terms are bilinear, because for constant η, u and v are linear in ξ, and for constant ξ, u and v are linear in η.

Check for convergence can be done by following requirements:

a) Rigid body modes

Two translation $u = a \quad v = e \quad (Cons\tan t)$

One rotation $\frac{\partial u}{\partial \eta} - \frac{\partial v}{\partial \xi} = c - f \quad (Cons\tan t)$

b) Constant strain

This implies that u and v should be arbitrary linear functions i.e. $u = a + bx + cy \quad v = e + fx + gy$

c) Continuity

Both displacements u and v are required to be continuous between elements. "*continuity of disp. And its derivative to the order of (n-1) where n is the highest derivatives in the PE function*".

d) Spatial Isotrophy

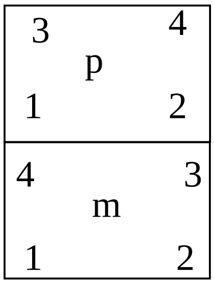
Inclusion of xy instead of x^2 or y^2 (ξη instead of ξ^2 or η^2) satisfies this requirement.

Requirements a, b, c and d are all satisfied by the equation of the displacements. Note along the edges u and v are linear. e.g. edge 1-2:

$$u(\xi,\eta) = a + b\xi$$
$$v(\xi,\eta) = e + f\xi$$

Therefore, we have to match displacements at only two nodes (1 and 2) to make u and v continuity along an edge.

$$u_1^p = u_4^m \qquad u_2^p = u_3^m$$
$$v_1^p = v_4^m \qquad v_2^p = v_3^m$$



Therefore, displacements equations for u and v satisfy the convergence requirement.
Now find constants a to h in displacement equations in terms of generalized degree of freedom (nodal dof) u_i's and v_i's.

$$u_1 = u(0,0) = a$$
$$u_2 = u(1,0) = a + b$$
$$u_3 = u(1,1) = a + b + c + d$$
$$u_4 = u(0,1) = a + c$$

Solving the above equations for a to h in terms of u_i's to get:
Similarly for v_i's .

$$u(\xi,\eta) = (1-\xi)(1-\eta)\, u_1 + \xi(1-\eta)\, u_2 + \xi\eta\, u_3 + (1-\xi)\eta\, u_4$$
$$v(\xi,\eta) = (1-\xi)(1-\eta)\, v_1 + \xi(1-\eta)\, v_2 + \xi\eta\, v_3 + (1-\xi)\eta\, v_4$$

or

$$u(\xi,\eta) = \sum_{i=1}^{4} \phi_i u_i \qquad v(\xi,\eta) = \sum_{i=1}^{4} \phi_i v_i$$

where :

$$\phi_1 = (1-\xi)(1-\eta) \quad \phi_2 = \xi(1-\eta) \qquad \phi_3 = \xi\eta \qquad \phi_4 = (1-\xi)\eta$$

$$u^p(\xi,\eta) = (1-\xi)\, u_1^p + \xi\, u_2^p \quad \text{along edge } 1-2\ (\eta = 0)$$
$$u^m(\xi,\eta) = (1-\xi)\, u_4^m + \xi\, u_3^m \quad \text{along edge } 3-4\ (\eta = 1)$$

Then matching $u_1^p \text{ and } u_4^m, u_2^p \text{ and } u_3^m$ will make u continuous along the boundary between the two elements. Similarly for v displacement.

4.1- Stiffness Matrix

Substitute u and v in the expression for strain energy:

$$U = \frac{Et1}{2(1-\nu^2)} \iiint_{\Omega} \left(u_x{}^2 + u_y{}^2 + 2\nu u_x u_y + \frac{1-\nu}{2}\left(u_y + v_x\right)^2 \right) dxdy = \frac{1}{2}\{\delta\}^T [K^e]\{\delta\}$$

$$\{\delta\}^T = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\}$$

$$U = \frac{Et1}{2(1-\nu^2)} \int_0^1\int_0^1 \left[\begin{array}{c} \frac{1}{a^2}\phi_{i\xi}\phi_{j\xi}u_i u_j + \frac{1}{b^2}\phi_{i\eta}\phi_{j\eta}v_i v_j + \frac{2\nu}{ab}\phi_{i\xi}\phi_{j\eta}u_i v_j + \\ \left(\frac{1-\nu}{2}\right)\left\{ \frac{1}{b^2}\phi_{i\eta}\phi_{j\eta}u_i u_j + \frac{2}{ab}\phi_{i\eta}\phi_{j\xi}u_i v_j + \frac{1}{a^2}\phi_{i\xi}\phi_{j\xi}v_i v_j \right\} \end{array} \right] d\xi\, d\eta$$

$$summation\ over\ i = 1,2,3,4 \quad j = 1,2,3,4$$

$$\phi_{i\eta} = \frac{\partial\phi_i}{\partial\eta} \quad \phi_{i\xi} = \frac{\partial\phi_i}{\partial\xi} \quad u_x{}^2 = \left(\frac{1}{a}\frac{\partial\phi_i}{\partial\xi}u_i\right)\left(\frac{1}{a}\frac{\partial\phi_j}{\partial\xi}u_j\right)$$

Castigliano's Theorem (1st) $F_i = \frac{\partial U_e}{\partial u_i}$ U_e is the strain energy of the element, u_i is the generalized displacement and F_i is the generalized force corresponding to the generalized displacement u_i.

Then the first row of the stiffness matrix is given by:

$$K_{ei,j}\delta_j = \frac{\partial U_e}{\partial u_1} = \frac{Eabt}{2(1-\nu^2)} \int_0^1\int_0^1 \left[\begin{array}{c} \frac{2}{a^2}\phi_{1\xi}\phi_{j\xi}u_j + \frac{2\nu}{ab}\phi_{1\xi}\phi_{j\xi}v_j + \\ \left(\frac{1-\nu}{2}\right)\left\{ \frac{2}{b^2}\phi_{1\eta}\phi_{j\eta}u_j + \frac{2}{ab}\phi_{1\eta}\phi_{j\xi}v_j \right\} \end{array} \right] d\xi\, d\eta$$

$$K_{e1,1}U_1 = \frac{Eabt}{2(1-\nu^2)} \int_0^1\int_0^1 \left[\frac{1}{a^2}\phi_{1\xi}\phi_{1\xi}u_1 + \left(\frac{1-\nu}{2b^2}\right)\left\{\phi_{1\eta}\phi_{1\eta}u_1\right\} \right] d\xi\, d\eta$$

$$\phi_{1\xi} = -(1-\eta) \quad \phi_{1\eta} = -(1-\xi) \quad and\ after\ integration$$

$$K_{e1,1} = \frac{Et}{12(1-\nu^2)} \left[4\frac{b}{a} + 2(1-\nu)\left(\frac{1-\nu}{2b^2}\right)\frac{a}{b} \right]$$

$$call\ \frac{b}{a} = S\ aspect\ ratio$$

Similarly we can determine the other components, finally:

$$[K_e] = \frac{Et}{12(1-\nu^2)} \begin{bmatrix} k_1 & & & & & & & \\ k_2 & k_3 & & & SYMM. & & & \\ k_8 & -k_5 & k_1 & & & & & \\ k_5 & k_{10} & -k_2 & k_3 & & & & \\ k_7 & -k_2 & k_4 & -k_5 & k_1 & & & \\ k_2 & k_9 & k_5 & k_6 & k_2 & k_3 & & \\ k_4 & k_5 & k_7 & k_2 & k_8 & -k_5 & k_1 & \\ k_5 & k_6 & k_2 & k_9 & k_5 & k_{10} & -k_2 & k_3 \end{bmatrix}$$

$$k_1 = 4S + \frac{2}{S}(1-\nu) \qquad k_2 = \frac{3}{2}(1+\nu) \qquad k_3 = \frac{4}{3} + 2(1-\nu)S$$

$$k_4 = 2S - \frac{2}{S}(1-\nu) \qquad k_5 = -\frac{3}{2}(1-3\nu) \qquad k_6 = -\frac{4}{S} + (1-\nu)S$$

$$k_7 = -2S - \frac{1}{S}(1-\nu) \qquad k_8 = -4S + \frac{1}{S}(1-\nu) \qquad k_9 = -\frac{2}{S} - (1-\nu)S$$

$$k_{10} = \frac{2}{S} - 2(1-\nu)S$$

4.2- Load Vector

$$\{F^e\} = [K^e]\{\delta^e\}$$

$$\{F^e\}^T = \{U_1 \quad V_1 \quad U_2 \quad V_2 \quad U_3 \quad V_3 \quad U_4 \quad V_4\}$$

recall:

$$\{F^e\}^T = \frac{\partial W^e}{\partial \{\delta^e\}} \qquad \{F^e\} \text{ is not known yet}$$

In order to determine the load vector for an element we need to know the loading, i.e. body forces, boundary stresses etc..
Consider the case of gravity loading: $f_y=-\gamma$ where γ is the specific weight. The work done by gravity is:

$$W_g{}^e = t\iint -\gamma\, v(\xi,\eta)dA = -t\int_0^1\int_0^1 \gamma\, \phi_i\, v_i abd\xi d\eta$$

$$F_2{}^e = \frac{\partial W_g{}^e}{\partial V_1} = -t\int_0^1\int_0^1 \gamma\, \phi_1\, abd\xi d\eta = -\frac{\gamma At}{4}$$

similarly,

$$F_4{}^e = \frac{\partial W_g{}^e}{\partial V_2} = -\frac{\gamma At}{4} = F_6{}^e = F_8{}^e$$

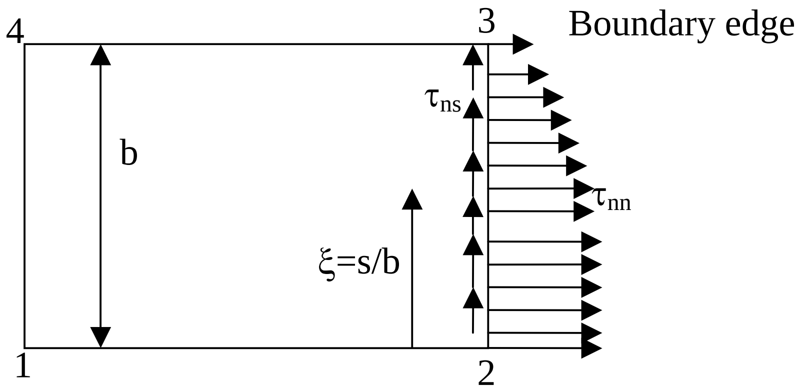
gravity load is given by

$$\{F_g{}^e\}^T = -\frac{\gamma At}{4}\{0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1\}$$

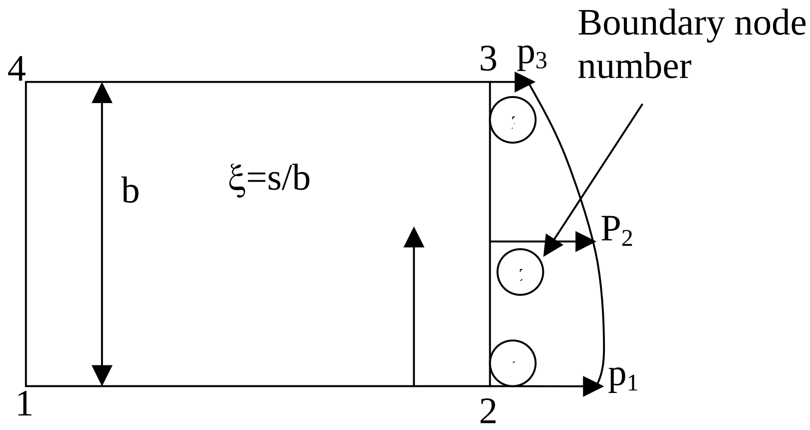
Next considering an element p[th] with one edge coinciding with the boundary of the domain as shown on the figure:

Suppose both normal and shear stresses are prescribed on this boundary. Therefore, work done by the boundary stresses is:

$$W_T = t\int_{S_T} \bar{T}_i\, u_i ds \qquad \text{int}\,egration\ is\ counter\ clockwise\ manner$$



Further suppose the normal and tangential stresses vary along edge 2-3 boundary stresses may then be approximated by a parabolic expression:



$$p(s) = c_1 + c_2 s + c_3 s^2$$

$$p_1 = p(0) = c_1$$

$$p_2 = p(\frac{b}{2}) = c_1 + c_2\frac{b}{2} + c_3\frac{b^2}{4}$$

$$p_3 = p(b) = c_1 + c_2 b + c_3 b^2$$

$solve\ for\ c_i's\ in\ terms\ of\ b_i's$

$$p(\xi) = (1-3\xi+2\xi^2)p_1 + (4\xi-4\xi^2)p_2 + (-\xi+2\xi^2)p_3$$

$$p(\xi) = \sum_{i=1}^{3} \varphi_i(\xi)p_i$$

$$\varphi_1 = (1-3\xi+2\xi^2) \quad \varphi_2 = (4\xi-4\xi^2) \quad \varphi_3 = (-\xi+2\xi^2)$$

Normal and tangential (or shear) stresses can now be approximated by above equation using the stress values at the temporary boundary nodes 1,2 and 3 i.e. let:

$$p_{x1} = \tau_{nn1} \qquad p_{x2} = \tau_{nn2} \qquad p_{x3} = \tau_{nn3}$$

$$p_{y1} = \tau_{ns1} \qquad p_{y2} = \tau_{ns2} \qquad p_{y3} = \tau_{ns3}$$

In order to use them in the equation of potential energy, we also need to know u and v distribution along the edge2-3. From before we know it is linear:
If any of the other edges have prescribed stresses on it, i.e. more than one edge coincides with the stress boundary, then we can generate another load

$$u(\xi) = (1-\xi)\, u_2 + \xi\, u_3$$
$$v(\xi) = (1-\xi)\, v_2 + \xi\, v_3$$

$$W_T = bt\int_0^1 p_x(\xi)\, u(\xi)\, d\xi + bt\int_0^1 p_y(\xi)\, v(\xi)\, d\xi$$

on integration, we obtain :

$$W_T = \frac{bt}{6}\left[(p_{x1} + 2p_{x2})u_2 + (2p_{x2} + p_{x3})u_3 + (p_{y1} + 2p_{y2})v_2 + (2p_{y2} + p_{y3})v_3\right]$$

$$F_1^{eT} = F2_3^{eT} = F_7^{eT} = F_8^{eT} = 0$$

$$F_3^{eT} = \frac{\partial W_T}{\partial u_2} = \frac{bt}{6}(p_{x1} + 2p_{x2})$$

$$F_4^{eT} = \frac{\partial W_T}{\partial v_2} = \frac{bt}{6}(p_{y1} + 2p_{y2})$$

$$F_5^{eT} = \frac{\partial W_T}{\partial u_3} = \frac{bt}{6}(2p_{x2} + p_{x3})$$

$$F_6^{eT} = \frac{\partial W_T}{\partial v_3} = \frac{bt}{6}(2p_{y2} + p_{y3})$$

due to stresses on edge 2 – 3 only :

$$\{F_T{}^e\}^T = \frac{bt}{6}\left[0 \quad 0 \quad (p_{x1} + 2p_{x2}) \quad (p_{y1} + 2p_{y2}) \quad (2p_{x2} + p_{x3}) \quad (2p_{y2} + p_{y3}) \quad 0 \quad 0\right]$$

then for this element, total load vector is then given by :

$$\{F^e\} = \{F_g{}^e\} + \{F_T{}^e\}$$

vector similar to above vectors, with non zero entry indifferent positions of the load vector. For example, if edge 3-4 also has prescribed stresses, then $F_1^{eT}= F_2^{eT} =F_3^{eT} =F_4^{eT}=0$ the new { F_T^e} for 3-4 is then added to { F_T^e} for edge2-3.

4.3- Strains and Stresses in Rectangular Element

Recall:

$$u(\xi,\eta) = (1-\xi)(1-\eta)\, u_1 + \xi(1-\eta)\, u_2 + \xi\eta\, u_3 + (1-\xi)\eta\, u_4$$

$$v(\xi,\eta) = (1-\xi)(1-\eta)\, v_1 + \xi(1-\eta)\, v_2 + \xi\eta\, v_3 + (1-\xi)\eta\, v_4$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{a}\frac{\partial u}{\partial \xi} = \frac{1}{a}\left[-u_1 + u_2 + (u_1 - u_2 + u_3 - u_4)\eta\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{1}{b}\frac{\partial v}{\partial \eta} = \frac{1}{b}\left[-v_1 + v_4 + (v_1 - v_2 + v_3 - v_4)\xi\right]$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{b}\frac{\partial u}{\partial \eta} + \frac{1}{a}\frac{\partial v}{\partial \xi} = \begin{Bmatrix} \frac{1}{b}\left(-u_1 + u_2\right) + \frac{1}{a}\left(-v_1 + v_2\right) + \\ \frac{1}{b}\left(u_1 - u_2 + u_3 - u_4\right)\xi + \frac{1}{a}\left(v_1 - v_2 + v_3 - v_4\right)\eta \end{Bmatrix}$$

Stresses are then obtained from equations $\{\tau\} = [D]\{\varepsilon\}$. [D] can be etermined based on the problem at hand is plane stress or plane strain.

Note strain ε_{xx} is linear in η and ε_{yy} linear in ξ where as γ_{xy} is complete linear. Further, these are not continuous across the interelement boundaries. It is convenient to obtain strains at the centroid of the element, i.e. at $\xi=\eta=0.5$

$$\varepsilon_{xx} = \frac{1}{2a}\left[-u_1 + u_2 + u_3 - u_4\right]$$

$$\varepsilon_{yy} = \frac{1}{2b}\left[-v_1 - v_2 + v_3 + v_4\right]$$

$$\gamma_{xy} = \frac{1}{2b}\left(-u_1 - u_2 + u_3 + u_4\right) + \frac{1}{2a}\left(-v_1 + v_2 + v_3 - v_4\right)$$

Then proceed to obtain stresses at the centroid from theses strains.

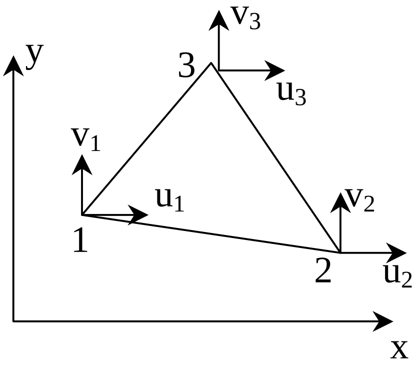
5- Plane Stress Triangular Elements

5.1- Constant Stress Triangular Element (C.S.T)

Note: cheap element

Assume linear displacements, i.e.

$$u(x,y) = a + bx + cy$$
$$v(x,y) = d + ex + fy$$

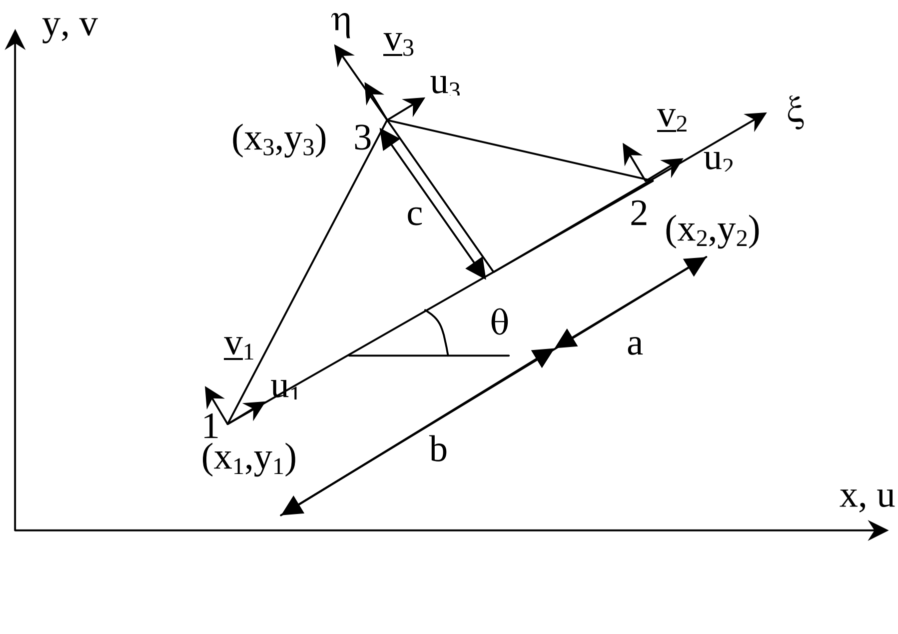


Here, we have three node triangle with two dof per node- six dof per element. We have six parameters a, b, c, d, e, and f to go with six dof.

We can proceed in exactly the same manner as we did for the plane stress rectangle. However, we will follow a more general approach involving transformation matrix.

Define x and y as the global coordinates and ξ and η and the local coordinates for a triangular element as shown in the figure. Specify the nodal coordinate as (x_1,y_1), (x_2,y_2) and (x_3,y_4) for nodes 1, 2 and 3, respectively.

The calculated a, b, c and θ are:



$$\cos\theta = \frac{x_2 - x_1}{(a+b)} = \frac{x_2 - x_1}{r}$$

$$\sin\theta = \frac{y_2 - y_1}{(a+b)} = \frac{y_2 - y_1}{r} \quad r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$a = (x_2 - x_3)\cos\theta - (y_3 - y_2)\sin\theta = \frac{1}{r}[(x_2 - x_3)(x_2 - x_1) - (y_3 - y_2)(y_2 - y_1)]$$

$$b = (x_3 - x_1)\cos\theta + (y_3 - y_1)\sin\theta = \frac{1}{r}[(x_3 - x_1)(x_2 - x_1) + (y_3 - y_1)(y_2 - y_1)]$$

$$c = (y_3 - y_1)\cos\theta - (x_3 - x_1)\sin\theta = \frac{1}{r}[(y_3 - y_1)(x_2 - x_1) - (x_3 - x_1)(y_2 - y_1)]$$

Now assume displacement in ξ and η system:

$$\bar{u}(\xi,\eta) = a_1 + a_2\xi + a_3\eta$$

$$\bar{v}(\xi,\eta) = a_4 + a_5\xi + a_6\eta$$

now find $a_1, a_2 \ldots, a_6$ in terms of nodal displacements $\bar{u}_1, \bar{v}_1 \ldots, \bar{v}_3$

$$\bar{u}_1 = \bar{u}(-b,0) = -a_1 - ba_2 \qquad \bar{v}_1 = a_4 - ba_5$$

$$\bar{u}_2 = \bar{u}(a,0) = a_1 + aa_2 \qquad \bar{v}_2 = a_4 + aa_5$$

$$\bar{u}_3 = \bar{u}(0,c) = a_1 + ca_3 \qquad \bar{v}_3 = a_4 + ca_6$$

$$\{\bar{\delta}\} = [T]\{A\}$$

$$\{\bar{\delta}\}^T = \left\{ \bar{u}_1 \quad \bar{v}_1 \quad \bar{u}_2 \quad \bar{v}_2 \quad \bar{u}_3 \quad \bar{v}_3 \right\}$$

$$\{A\}^T = \{a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6\}$$

$$[T] = \begin{bmatrix} 1 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -b & 0 \\ 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & 0 \\ 1 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & c \end{bmatrix}$$

Note that det[T]=$-c^2$(a+b) ≠0 for practical triangles, hence [T] is nonsingular and can be inverted {A}=$[T]^{-1}\{\bar{\delta}\}$.

Strain Energy Calculation

$$U_e = \frac{Et}{2(1-\nu^2)}\iint_A \left(\bar{u}_{,\xi}{}^2 + \bar{v}_{,\eta}{}^2 + 2\nu\,\bar{u}_{,\xi}\,\bar{v}_{,\eta} + \frac{1-\nu}{2}\left(\bar{u}_{,\eta} + \bar{v}_{,\xi}\right)^2\right) d\xi d\eta$$

$$\bar{u}_{,\xi} = a_2 \qquad \bar{u}_{,\eta} = a_3 \qquad \bar{v}_{,\xi} = a_5 \qquad \bar{v}_{,\eta} = a_6$$

$$U_e = \frac{Et}{2(1-\nu^2)}\iint_A \left(a_2{}^2 + a_6{}^2 + 2\nu a_2 a_6 + \frac{1-\nu}{2}\left(a_3{}^2 + 2a_3 a_5 + a_5{}^2\right)\right) d\xi d\eta$$

Integrand consists of constant terms and $\iint_A d\xi d\eta = \frac{1}{2}(a+b)c$

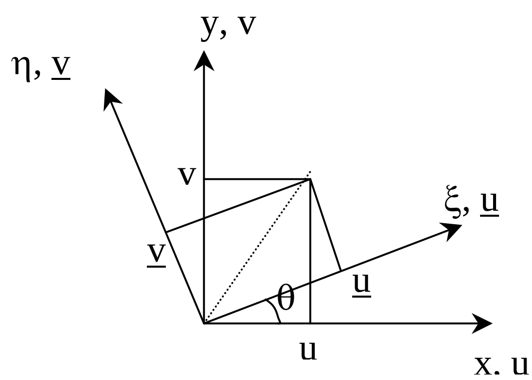
$$U_e = \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} \bar{k}_{ij}\; a_i a_j = \frac{1}{2}\{A\}^T[\bar{k}]\{A\}$$

$$k_{ij} a_j = \frac{\partial U_e}{\partial a_i} \quad i.e. \quad k_{1j} a_j = \frac{\partial U_e}{\partial a_1} \quad k_{2j} a_j = \frac{\partial U_e}{\partial a_2} \quad etc.$$

$$[\bar{k}] = \frac{Et(a+b)c}{2(1-\nu^2)}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \nu \\ 0 & 0 & \frac{1-\nu}{2} & 0 & \frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & \frac{1-\nu}{2} & 0 \\ 0 & \nu & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now transform to generalized displacement in local axes. Note that $U_e = \frac{1}{2}\{\bar{\delta}\}^T[\underline{k}]\{\bar{\delta}\} = \frac{1}{2}\{\bar{\delta}\}^T\left([T]^{-1}\right)^T[\bar{k}][T]^{-1}\{\bar{\delta}\}$ from this equation $[\underline{k}] = \left([T]^{-1}\right)^T[\bar{k}][T]^{-1}$
Where $[\underline{k}]$ is the stiffness matrix in local coordinate system. Next transform $[\underline{k}]$ from $\bar{u}, \bar{v}\ in\ \xi, \eta$ coordinates to u and v in x, y coordinate or global coordinates.

Consider rotation between ξ, η and x, y.



$$u = \bar{u}\cos\theta - \bar{v}\sin\theta \qquad v = \bar{u}\sin\theta + \bar{v}\cos\theta$$

$$or$$

$$\bar{u} = u\cos\theta + v\sin\theta \qquad \bar{v} = -u\sin\theta + v\cos\theta$$

Why do we neglect translation between x, y and ξ, η?
Because rigid body translation of an element does not contribute to strain energy, hence we neglect it.

$$\{\delta\}^T = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3\}$$

$$\{\bar{\delta}\} = [R]\{\delta\}$$

$$[R] = \begin{bmatrix} [r] & [0] & [0] \\ [0] & [r] & [0] \\ [0] & [0] & [r] \end{bmatrix} \quad [r] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad [0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Again strain energy is invariant and:

$$U_e = \frac{1}{2}\{\bar{\delta}\}^T [\underline{k}]\{\bar{\delta}\} = \frac{1}{2}\left([R]\{\delta\}\right)^T [\underline{k}]\left([R]\{\delta\}\right)$$

$$= \frac{1}{2}\{\delta\}^T [R]^T [\underline{k}][R]\{\delta\} = \frac{1}{2}\{\delta\}^T [k]\{\delta\}$$

$$[k] = [R]^T [\underline{k}][R] = [R]^T [T]^{-1^T} [\bar{k}][T]^{-1}[R]$$

[k] is the final stiffness matrix in the global coordinate system.

Computing Steps

1- Specifying global coordinates of element nodes (x_1,y_1), (x_2,y_2) and (x_3,y_3)

2- Calculate a, b, c and θ

3- Calculate [T] and invert numerically

4-Calculate $[\bar{k}]$, multiply $[T]^{-1}[R]=[Q]$

5- Calculate $[k]=[Q]^T[\bar{k}][Q]$

6- Return [k] to the main program

Accuracy of Constant Stress Triangles

We have u and v linear in x and y. Error in u and v from taylor's series is $f(l^2)$ where l is some typical size of an element. Therefore, error in strain(constant) is f(l). Hence, errot in strain energy is given by $f(l^2)$, if l=L/N where N is number of elements along a typical length of a problem, then the error in strain energy is given by $f(1/N^2)$.

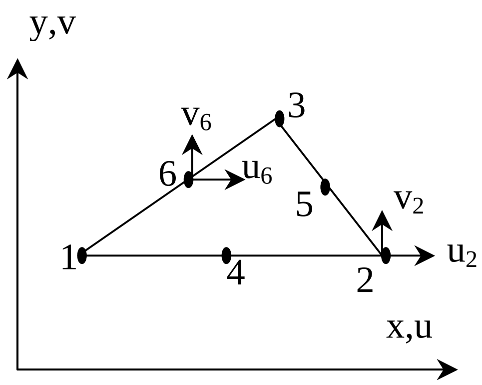
Unfortunately, we do not know the constant of proportionality α in the error $=\alpha/N^2$.

5.2- Linear Stress Triangular Element (L.S.T.)

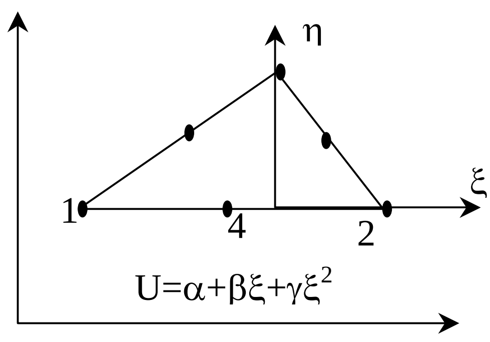
Linear stress implies, we need quadratic or parabolic displacement field:

$$u(x,y)=a_1+a_2x+a_3y+a_4x^2+a_5xy+a_6y^2$$
$$v(x,y)=b_1+b_2x+b_3y+b_4x^2+b_5xy+b_6y^2$$

Note that six parameters $a_1,\dots,a_6$ and six $u_1,\dots,u_6$ and similarly, six $b_1,\dots,b_6$ and six $v_1,\dots,v_6$ are equivalents for 12 dof per element.
We have assumed complete polynomial of degree 2.
A complete polynomial is invariant regarding rotation and allows order of error to be determined from Taylor expression, with expression in displacement function, we also satisfy the convergence requirement i.e. rigid body modes, constant strains, and spatial isotrophy of the displacement field.

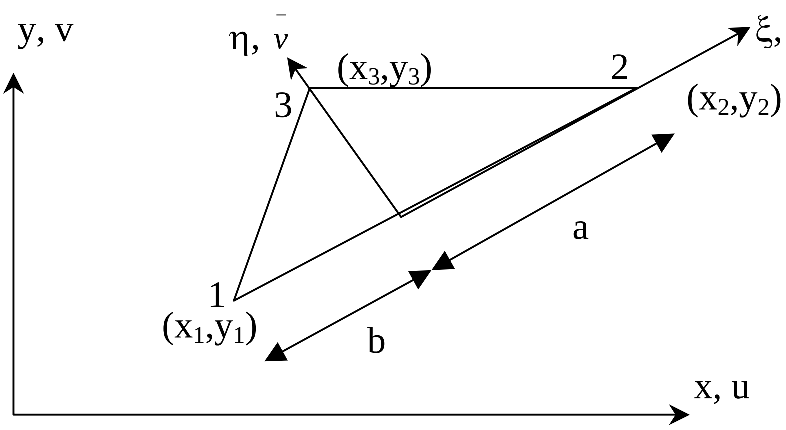


What about continuity?
Along and edge, u and v will be quadratic function of edge coordinate. For the quadratic shown along the edge, we have 3 parameters α, β and γ and 3 dof u_1, u_4 and u_2. Therefore, equating, these u's along the edge will satisfy the continuity requirement.



5.3- Quadratic Stress Triangular Element (QST)

Again use the same coordinate system (local) as for CST, i.e. ξ, η ($\bar{u}, \bar{v}$).
For stress to be quadratic within the element, need a polynomial which is at least cubic:



$$\bar{u}(x,y) = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3$$

$$\bar{v}(x,y) = a_{11} + a_{12}\xi + a_{13}\eta + a_{14}\xi^2 + a_{15}\xi\eta + a_{16}\eta^2 + a_{17}\xi^3 + a_{18}\xi^2\eta + a_{19}\xi\eta^2 + a_{20}\eta^3$$

$$\bar{u} = \sum_{i=1}^{10} a_i \xi^{m_i}\eta^{n_i} \qquad \{m\} = [0\ \ 1\ \ 0\ \ 2\ \ 1\ \ 0\ \ 3\ \ 2\ \ 1\ \ 0]$$

$$\bar{v} = \sum_{i=1}^{10} a_{i+10} \xi^{m_i}\eta^{n_i} \qquad \{n\} = [0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 2\ \ 0\ \ 1\ \ 2\ \ 3]$$

To transform the generalized coordinates $a_1, \ldots, a_{20}$ into the nodal degrees of freedom, we need to decide on discretization in terms of dof and the number of nodes.

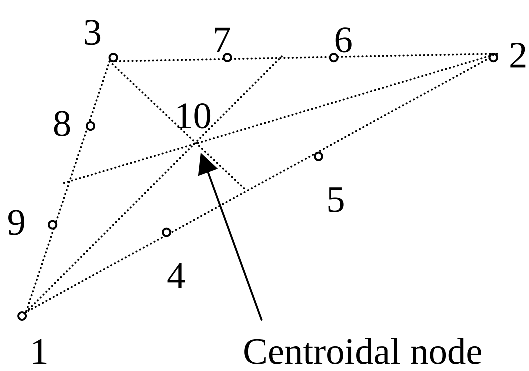
a) At each nodes the dof are $\bar{u}, \bar{v}$. We require 10 dof in $\bar{u}$ and 10 dof in $\bar{v}$ for $a_1, \ldots, a_{20}$. Then obtain:

$$\{\bar{\delta}\} = [T_a]\{A\}$$

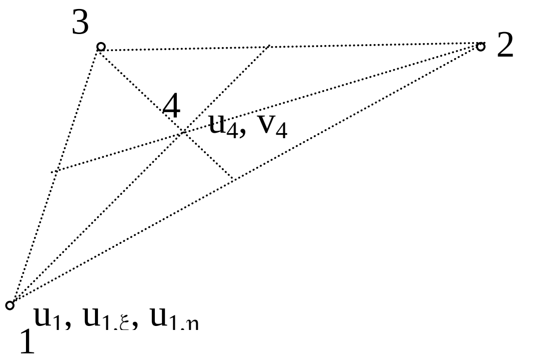
$$where:$$

$$\{\bar{\delta}\}^T = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & . & . & . & . & \overbrace{u_{10}}^{u_c} & \overbrace{v_{10}}^{v_c} \end{bmatrix}$$

$$\{A\}^T = [a_1 \quad a_2 \quad a_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad a_{20}]$$



b) at each corner nodes take $\bar{u}, \bar{u}_{,\xi}, \bar{u}_{,\eta}, \bar{v}, \bar{v}_{,\xi}, \bar{v}_{,\eta}$ as the dof and $\bar{u}, \bar{v}$ at the centroid. Thus, we have six dof per corner nodes and 18+ 2=20 dof per element.



Then obtain:

$$\{\bar{\delta}\} = [T_b]\{A\}$$

where:

$$\{\bar{\delta}\}^T = \left[\bar{u}_1 \quad \bar{u}_{1\xi} \quad \bar{u}_{1\eta} \quad \bar{v}_1 \quad \bar{v}_{1\xi} \quad \bar{v}_{1\eta} \quad . \quad . \quad u_4 \quad v_4\right]$$

$$\{A\}^T = \left[a_1 \quad a_2 \quad a_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad a_{20}\right]$$

Determinant of $[T_b]=c^{14}(a+b)^{14}/729$ which is nonzero for all practical problems.

Regardless of which discretization we use, we always compute $[\bar{k}]$ in generalized coordinates $a_1, \ldots, a_{20}$.

$$U_e = \frac{Et}{2(1-\nu^2)}\iint_A \left(\bar{u}_{,\xi}{}^2 + \bar{v}_{,\eta}{}^2 + 2\nu\,\bar{u}_{,\xi}\,\bar{v}_{,\eta} + \frac{1-\nu}{2}\left(\bar{u}_{,\eta} + \bar{v}_{,\xi}\right)^2\right) d\xi d\eta$$

$$\bar{u}_{,\xi} = \sum_{i=1}^{10} a_i m_i \xi^{m_i-1}\eta^{n_i} \qquad \bar{u}_{,\xi}{}^2 = \sum_{i=1}^{10}\sum_{i=1}^{10} a_i a_j m_i m_j \xi^{m_i+m_j-2}\eta^{n_i+n_j}$$

$$\iint_A \bar{u}_{,\xi}{}^2\, d\xi d\eta = \sum_{i=1}^{10}\sum_{i=1}^{10} a_i a_j m_i m_j \iint_A \xi^{m_i+m_j-2}\eta^{n_i+n_j}\, d\xi d\eta$$

Get general integral like

$$F(m,n) = \iint_A \xi^m \eta^n d\xi d\eta = c^{n+1}[a^{m+1} - (-b)^{m+1}]\frac{m!\,n!}{(m+n+2)!}$$

After evaluation of all terms:

$$U_e = \frac{1}{2}\{A\}^T[\bar{k}]\{A\}$$

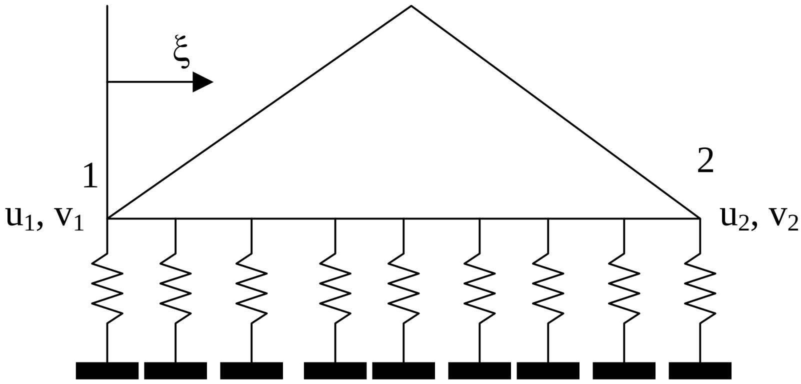
Now transform to nodal variables (DOF) in local coordinates, then transform to global coordinates i.e. into u, v etc.

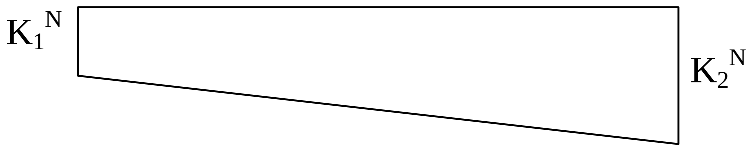
5.4- Boundary with Springs

Recall

$$\pi = t\left[\frac{1}{2}\int_{\Omega}\tau_{ij}\varepsilon_{ij}d\Omega - \int_{\Omega} f_i u_i d\Omega - \int_{S_T}\bar{T}_i\, u_i dS + \frac{1}{2}\int_{S_M}\alpha\, u_i u_i dS\right]$$

suppose edge 1-2 of the element shown is on an elastic foundation. Foundation modulus of spring stiffness can be approximated by: $K^N(\xi) = K_1^{\,N}(1-\xi) + K_2^{\,N}\xi$





For CST, v displacement along the edge is given by:

$$v\ (\xi) = v_1(1-\xi) + v_2\xi$$

the last term in equation of the PE (on the RHS) is:

$$U_B = \frac{1}{2} l \int_0^l K^N(\xi)\, v^2(\xi) d\xi$$

Contribution to the nodal forces is then given by:

$$F_i^{\,B} = \frac{\partial U_B}{\partial \delta_i} \quad where \quad \{\delta\} = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3] = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5 \quad \delta_6]$$

$$F_2^{\,B} = \frac{\partial U_B}{\partial \delta_2} = \frac{\partial U_B}{\partial v_1} = l\int_0^l K(\xi)[v_1(1-\xi) + v_2\ \xi](1-\xi)d\xi$$

$$F_2^{\,B} = l\left\{\left[\frac{K_1^{\,N}}{4} + \frac{K_2^{\,N}}{12}\right]v_1 + \left[\frac{K_1^{\,N}}{12} + \frac{K_2^{\,N}}{12}\right]v_2\right\}$$

$$F_4^{\,B} = \frac{\partial U_B}{\partial \delta_4} = \frac{\partial U_B}{\partial v_2} = l\left\{\left[\frac{K_1^{\,N}}{12} + \frac{K_2^{\,N}}{12}\right]v_1 + \left[\frac{K_1^{\,N}}{12} + \frac{K_2^{\,N}}{4}\right]v_2\right\}$$

$$\begin{Bmatrix} F_2^{\,B} \\ F_4^{\,B} \end{Bmatrix} = l\begin{bmatrix} \left(\frac{K_1^{\,N}}{4} + \frac{K_2^{\,N}}{12}\right) & \left(\frac{K_1^{\,N}}{12} + \frac{K_2^{\,N}}{12}\right) \\ \left(\frac{K_1^{\,N}}{12} + \frac{K_2^{\,N}}{12}\right) & \left(\frac{K_1^{\,N}}{12} + \frac{K_2^{\,N}}{4}\right) \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

For tangential springs along edge 1-2:

$$K^T(\xi) = K_1^{\,T}(1-\xi) + K_2^{\,T}\xi$$

and:

$$u(\xi) = u_1(1-\xi) + u_2\xi$$

and:

$$\begin{Bmatrix} F_1^{\,B} \\ F_2^{\,B} \end{Bmatrix} = l\begin{bmatrix} \left(\frac{K_1^{\,T}}{4} + \frac{K_2^{\,T}}{12}\right) & \left(\frac{K_1^{\,T}}{12} + \frac{K_2^{\,T}}{12}\right) \\ \left(\frac{K_1^{\,T}}{12} + \frac{K_2^{\,T}}{12}\right) & \left(\frac{K_1^{\,T}}{12} + \frac{K_2^{\,T}}{4}\right) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$addition\ to\ the\ stiffness\ matrix\ is\ as\ follows:$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{bmatrix} X & - & X & - & - & - \\ - & Y & - & Y & - & - \\ X & - & X & - & - & - \\ - & Y & - & Y & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}$$

6- Natural Coordinates and Shape Functions "C^0"

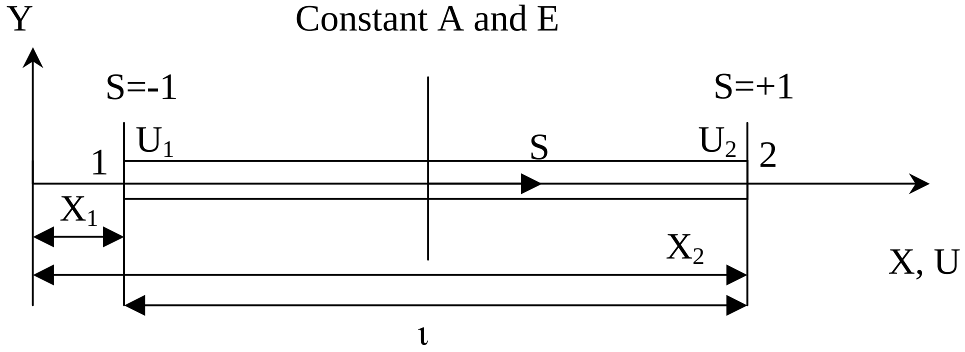
Natural Coordinates are dimensionless, homogeneous and independent of size and shape of the elements.

6.1- Line and Rectangular Elements

Relation between natural and global coordinates:

$$X = \frac{1}{2}(1-s)X_1 + \frac{1}{2}(1+s)X_2$$

$$X = \sum_{i=1}^{2} N_i X_i \quad N_1(s) = \frac{1}{2}(1-s) \quad N_2(s) = \frac{1}{2}(1+s)$$



Line element in global and natural coordinate system

Here N_1 and N_2 are functions of the natural coordinates s and are called element shape functions or interpolation functions. These shape functions or interpolation functions can be used in describing the linear displacement field u within the bar or line element.

$$U = \frac{1}{2}(1-s)U_1 + \frac{1}{2}(1+s)U_2$$

$$U = \sum_{i=1}^{2} N_i U_i \quad U(-1) = U_1 \quad U(+1) = U_2$$

$$\varepsilon = \frac{dU}{dX} = \frac{dU}{ds}\frac{ds}{dX} = \frac{U_2 - U_1}{2}\frac{ds}{dX}$$

Note above equation for U provides only C_0 continuity since only U is continuous across the node between two adjacent elements.

$$\frac{dX}{ds} = \frac{X_2 - X_1}{2} = \frac{l}{2} \quad or \quad \frac{ds}{dX} = \frac{2}{l}$$

$$\varepsilon = \frac{U_2 - U_1}{l} \; as \; expected$$

Hence, strain-displacement transformation matrix [B] is given by:

$$[B] = \frac{1}{l}\begin{bmatrix}-1 & 1\end{bmatrix}$$

$$\varepsilon = \frac{1}{l}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}U_1 \\ U_2\end{Bmatrix}$$

$$[K] = A\int_0^l [B]^T[D][B]dX \quad Where \quad [D] = E \text{ mod}ulus\ of\ elasticity$$

$$[K] = \frac{A}{l^2}\int_{-1}^{+1} \begin{Bmatrix}-1 \\ +1\end{Bmatrix} E\begin{bmatrix}-1 & +1\end{bmatrix} \left|[J]\right| ds$$

Where [J] is the Jacobian relating the element length in global coordinate system to an element length in natural sys. For one dimensional problem:

$$dX = Jds = \frac{l}{2}ds$$

$$[K] = \frac{AE}{l^2}\int_{-1}^{+1} \begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}\frac{l}{2}ds = \frac{AE}{l}\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}$$

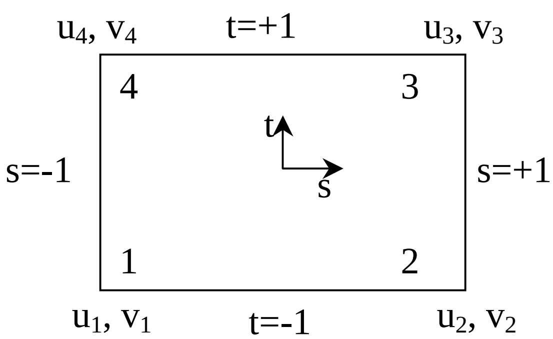
This is the stiffness matrix of the line element.

For rectangular elements, the natural coordinates take in the values of E1 on the edges of the rectangle as shown.

For four node rectangles, the interpolation functions or shape functions are:

$$N_1 = \frac{1}{4}(1-s)(1-t) \qquad N_2 = \frac{1}{4}(1+s)(1-t)$$

$$N_3 = \frac{1}{4}(1+s)(1+t) \qquad N_4 = \frac{1}{4}(1-s)(1+t)$$



Recall bilinear polynomials used for u and v in rectangular plane stress element. If the origin of x and y co-ordinates for the element is chosen at the centroid, the shape functions ϕ_1 , .., ϕ_4 are the same as N_1, .., N_4 except we have s and t coordinates instead of ξ and η (ξ=x/a, η=y/b)
We can use natural coordinates to express u and v within the element as:

$$u = \sum_{i=1}^{4} N_i u_i \quad v = \sum_{i=1}^{4} N_i v_i$$

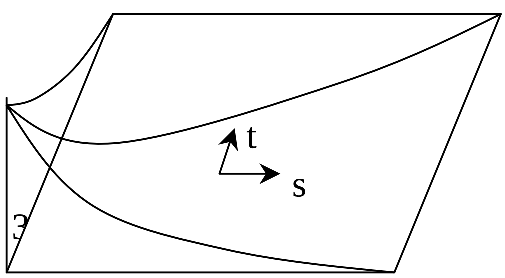
$$u(-1,-1) = u_1 \quad u(+1,-1) = u_2 \quad etc.$$

$$v(-1,-1) = v_1 \quad v(+1,-1) = v_2 \quad etc.$$

$$further\ along\ edge\ 1-2,\ t = -1$$

$$u = \frac{1}{2}(1-s)u_1 + \frac{1}{2}(1+s)u_2$$

$$v = \frac{1}{2}(1-s)v_1 + \frac{1}{2}(1+s)v_2$$



i.e. u and v vary linearly along edge 1-2, therefore equating the nodal displacements u and v at nodes along an edge provides continuity of the displacements i.e. C_0 continuity.

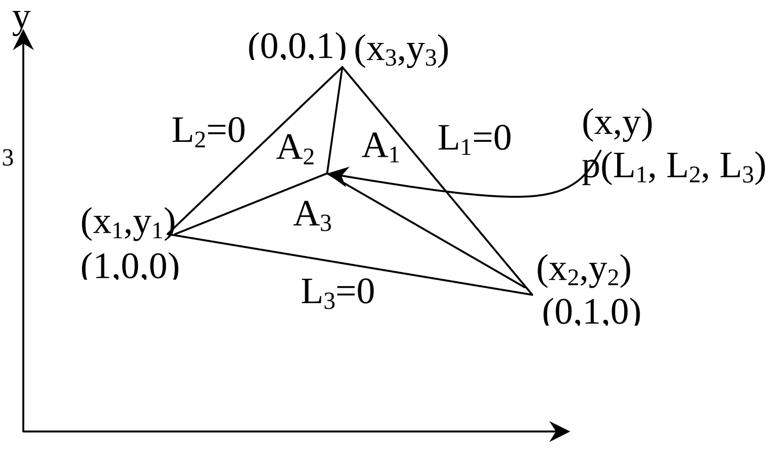
6.2- Triangular elements

Use area coordinates L_1, L_2 and L_3

$$L_1 = \frac{A_1}{A} \quad L_2 = \frac{A_2}{A} \quad L_3 = \frac{A_3}{A}$$

$$A = Area\ of\ trianngle\ 1-2-3$$

$$L_1 + L_2 + L_3 = 1$$



The only two of the area coordinates L_1, L_2 and L_3 are independent. Relation between area coordinates and the global coordinates of any point p is given by:

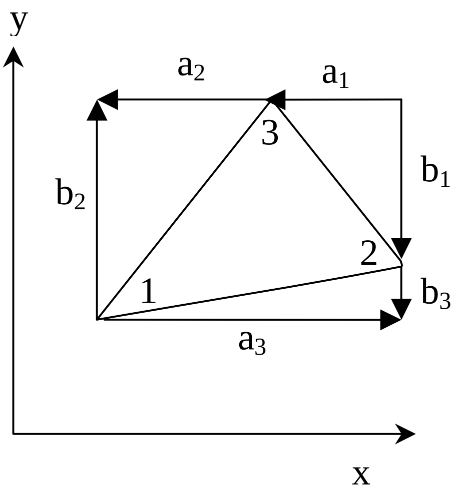
$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \qquad inverting$$

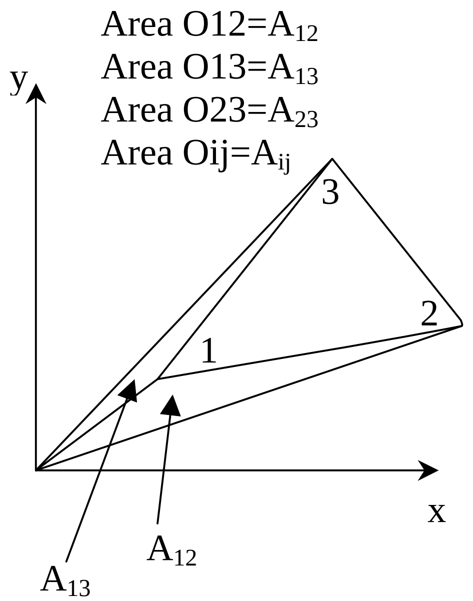
$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_2 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} 2A_{23} & b_1 & a_1 \\ 2A_{31} & b_2 & a_2 \\ 2A_{12} & b_3 & a_3 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\{\Im\} = [T]\{Z\} \quad \{\Im\}^T = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \quad \{Z\}^T = \begin{bmatrix} 1 & x & y \end{bmatrix}$$

Geometric interpolation of the terms in above equation is as follows:





Differentiation
By chain rule:

$$\frac{\partial}{\partial x}=\sum_{i=1}^{3}\frac{\partial L_i}{\partial x}\frac{\partial}{\partial L_i}=\frac{\partial L_1}{\partial x}\frac{\partial}{\partial L_1}+\frac{\partial L_2}{\partial x}\frac{\partial}{\partial L_2}+\frac{\partial L_3}{\partial x}\frac{\partial}{\partial L_3}$$

$$\frac{\partial}{\partial x}=\frac{1}{2A}\sum_{i=1}^{3}b_i\frac{\partial}{\partial L_i}\quad \sin ce\quad \frac{\partial L_i}{\partial x}=\frac{b_i}{2A}$$

similarly,

$$\frac{\partial}{\partial y}=\frac{1}{2A}\sum_{i=1}^{3}a_i\frac{\partial}{\partial L_i}\quad \sin ce\quad \frac{\partial L_i}{\partial y}=\frac{a_i}{2A}$$

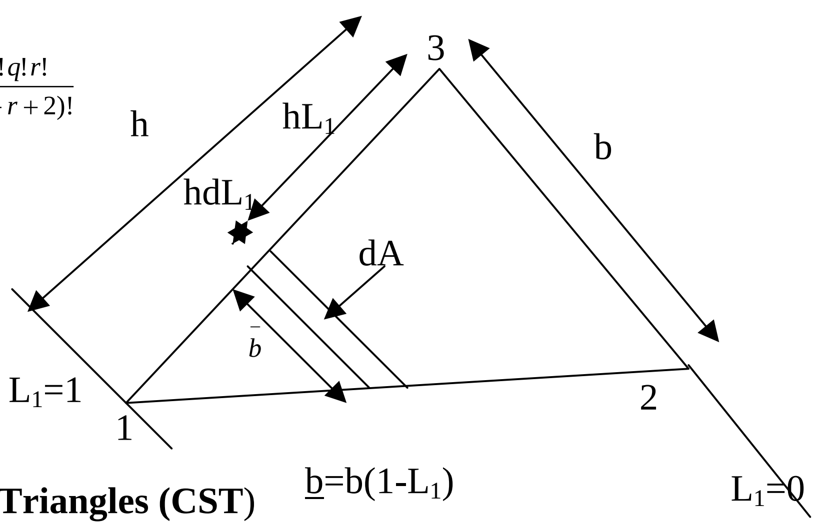
Integration:

$$dA=\bar{b}\,h\,dL_1=bh(1-L_1)dL_1$$

$$\int_A L_1 dA=\int_0^1 bhL_1(1-L_1)dL_1=bh\left(\frac{L_1^{\,2}}{2}-\frac{L_1^{\,3}}{3}\right)\Big|_0^1=\frac{bh}{3}=\frac{A}{3}$$

ingeneral:

$$\iint_A L_1^{\,p}L_2^{\,q}L_3^{\,r}\,dA=\frac{2A\;p!q!r!}{(p+q+r+2)!}$$

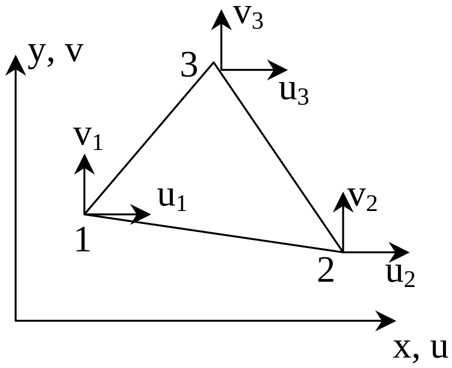


A. Constant Stress Triangles (CST)

Three dof in u and v for a total of six dof (3 nodes)

$$u = L_1 u_1 + L_2 u_2 + L_3 u_3$$
$$v = L_1 v_1 + L_2 v_2 + L_3 v_3$$

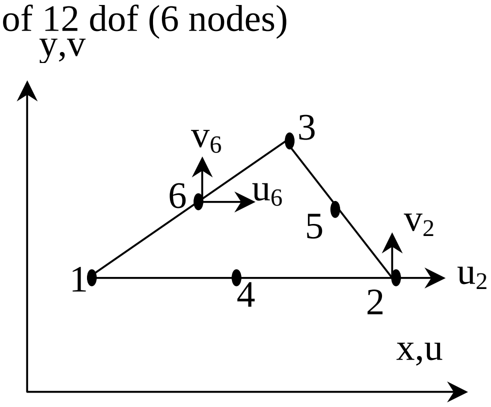
This gives a linear approximation for u and v within the element. Or $N_1=L_1$, $N_2=L_2$ and $N_3=L_3$. Note that L_1, L_2 and L_3 are linear function of x and y.



B. Linear Stress Triangles (LST)

Six dof in u and v for a total of 12 dof (6 nodes)

$$N_1 = L_1(2L_1 - 1) \qquad N_4 = 4L_1L_2$$
$$N_2 = L_2(2L_2 - 1) \qquad N_5 = 4L_2L_3$$
$$N_3 = L_3(2L_3 - 1) \qquad N_6 = 4L_3L_1$$
$$u = \sum_{i=1}^{3} N_i u_i \quad v = \sum_{i=1}^{3} N_i v_i$$

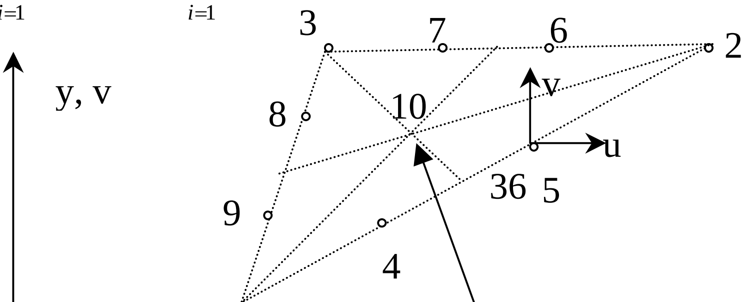


C_0 continuity

C. Quadratic Stress Triangles (QST)

Ten dof in u and v for a total of 20 (10 nodes). Again C_0 continuity.

$$N_1 = L_1(3L_1 - 1)(3L_1 - 2)/2 \qquad N_6 = 9L_2L_3(3L_2 - 1)/2$$
$$N_2 = L_2(3L_2 - 1)(3L_2 - 2)/2 \qquad N_7 = 9L_2L_3(3L_3 - 1)/2$$
$$N_3 = L_3(3L_3 - 1)(3L_3 - 2)/2 \qquad N_8 = 9L_3L_1(3L_3 - 1)/2$$
$$N_4 = 9L_1L_3(3L_1 - 1)/2 \qquad N_9 = 9L_3L_1(3L_1 - 1)/2$$
$$N_5 = 9L_1L_2(3L_2 - 1)/2 \qquad N_{10} = 27L_1L_2L_3$$
$$u = \sum_{i=1}^{10} N_i u_i \quad v = \sum_{i=1}^{10} N_i v_i$$



D. Tetrahedrals

As we saw in triangles:

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} \qquad 0 \le L_i \le 1$$

$$Volume\ V = \det[\ \Uparrow]$$

$$L_i = \frac{V_i}{V} \quad volume\ coordinates$$

$$V_i = V_{ijklp} \quad volume\ of\ Tetrahedral\ surrounded\ by\ vertices\ jkl\ at\ point\ p$$

$$e.g.\ V_1 = V_{234p} \quad V_2 = V_{134p} \quad etc.$$

$$Integration: \iiint_V L_1^P L_2^q L_3^r L_4^s dV = \frac{6Vp!q!r!s!}{(p+q+r+s+3)!}$$

E. Derivation of Stiffness Matrix for Plane Stress CST element

Strain energy U_e given by:

$$U_e = \frac{Et}{2(1-\nu^2)} \iint_A \left(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + 2\nu\varepsilon_{xx}\varepsilon_{yy} + \frac{1-\nu}{2}\gamma_{xy}^2 \right) dxdy \quad t = thickness$$

$$u = L_1 u_1 + L_2 u_2 + L_3 u_3$$
$$v = L_1 v_1 + L_2 v_2 + L_3 v_3$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{2A} \sum_{i=1}^{3} b_i u_i$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{1}{2A} \sum_{i=1}^{3} a_i v_i$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A} \sum_{i=1}^{3} (a_i u_i + b_i v_i)$$

$$Substituting\ above\ equations\ into\ U_e$$

$$U_e = \frac{Et}{8A^2(1-\nu^2)} \iint_A \sum_{i=1}^{3} \sum_{j=1}^{3} \begin{pmatrix} b_i b_j u_i u_j + a_i a_j v_i v_j + 2\nu b_i a_j u_i v_j + \\ \frac{1-\nu}{2} \{ a_i a_j u_i u_j + 2a_i b_i u_i v_j + b_i b_j v_i v_j \} \end{pmatrix} dxdy$$

The integrand in above eqn. consistes of all constant terms and integration is simply multiplication with area A:

$$U_e = \frac{Et\sum_{i=1}^{3}\sum_{j=1}^{3}}{8A(1-\nu^2)}\left(\begin{array}{l}\left\{b_i b_j + \frac{1-\nu}{2}a_i a_j\right\}u_i u_j + \left\{a_i a_j + \frac{1-\nu}{2}b_i b_j\right\}v_i v_j + \\ 2\left\{\nu b_i a_j + \frac{1-\nu}{2}a_i b_j\right\}u_i v_j\end{array}\right)$$

$\{\delta\}^T = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3] = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5 \quad \delta_6]$ *i.e.* $\delta_1 = u_1$ $\delta_2 = v_1$ $\delta_3 = u_2$

therefore

$$k_{11} = \frac{\partial^2 U_e}{\partial \delta_1^{\,2}} = \frac{\partial^2 U_e}{\partial u_1^{\,2}} = \frac{Et}{4A(1-\nu^2)}\left[b_1^{\,2} + (\frac{1-\nu}{2})a_1^{\,3}\right]$$

$$k_{12} = \frac{\partial^2 U_e}{\partial \delta_1 \partial \delta_2} = \frac{\partial^2 U_e}{\partial u_1 \partial v_1} = \frac{Et}{4A(1-\nu^2)}\left[\nu b_1 a_1 + (\frac{1-\nu}{2})a_1 b_1\right]$$

$$k_{13} = \frac{\partial^2 U_e}{\partial \delta_1 \partial \delta_3} = \frac{\partial^2 U_e}{\partial u_1 \partial u_2} = \frac{Et}{4A(1-\nu^2)}\left[b_1 b_2 + (\frac{1-\nu}{2})a_1 a_2\right]$$

$$k_{22} = \frac{\partial^2 U_e}{\partial \delta_2 \partial \delta_2} = \frac{\partial^2 U_e}{\partial v_1 \partial v_1} = \frac{Et}{4A(1-\nu^2)}\left[a_1^{\,2} + (\frac{1-\nu}{2})b_1^{\,2}\right]$$

$$k_{24} = \frac{\partial^2 U_e}{\partial \delta_2 \partial \delta_4} = \frac{\partial^2 U_e}{\partial v_1 \partial v_2} = \frac{Et}{4A(1-\nu^2)}\left[a_1 a_2 + (\frac{1-\nu}{2})b_1 b_2\right]$$

let $\alpha = \frac{Et}{4A(1-\nu^2)} \quad \beta = \nu \quad \gamma = \frac{1-\nu}{2}$

$$[k] = \begin{bmatrix} b_1b_1 + \gamma a_1a_1 & \beta b_1a_1 + \gamma a_1b_1 & b_1b_2 + \gamma a_1a_2 & \beta b_1a_2 + \gamma a_1b_2 & b_1b_3 + \gamma a_1a_3 & \beta b_1a_3 + \gamma a_1b_3 \\ & a_1a_1 + \gamma b_1b_1 & \beta a_1b_2 + \gamma b_1a_2 & a_1a_2 + \gamma b_1b_2 & \beta a_1b_3 + \gamma b_1a_3 & a_1a_3 + \gamma b_1b_3 \\ & & b_2b_2 + \gamma a_2a_2 & \beta b_2a_2 + \gamma a_2b_2 & b_2b_3 + \gamma a_2a_3 & \beta b_2a_3 + \gamma a_3b_2 \\ & & & a_2a_2 + \gamma b_2b_2 & \beta a_2b_3 + \gamma b_2a_3 & a_2a_3 + \gamma b_2b_3 \\ \mathit{Symmetric} & & & & b_3b_3 + \gamma a_3a_3 & \phi b_3a_3 + \gamma a_3b_3 \\ & & & & & a_3a_3 + \gamma b_3b_3 \end{bmatrix}$$

to convert plane stress to plane strain replace E by $\underline{E}$ and ν by $\underline{\nu}$

$\underline{E}$=E/(1-ν^2) $\underline{\nu}$=ν/(1-ν)

7- Curved, Isoparametric Elements

-Elements with curved boundaries are useful for problems with curved edges or surfaces for 3-dimensional problems.
-Unless edges of triangular elements are very small, we may introduce error due to approximation of curved boundaries by straight lines.
-We end up facing both the mathematical problem as well as shape problem

Elements of basic one-, two- or three dimensional types will be mapped into distorted shapes in the manner indicated in the following figures. In both figures s, t or L_1, L_2 and L_3 coordinates can be distorted into curvilinear set when plotted in Cartesian coordinates. Similarly, single straight line can be transformed into a curved line in Cartesian coordinate and flat sheet can be distorted into a three dimensional space. Figure 3 indicates two examples of two dimensional (s, t) element mapped into a three dimensional (x, y and z) space.

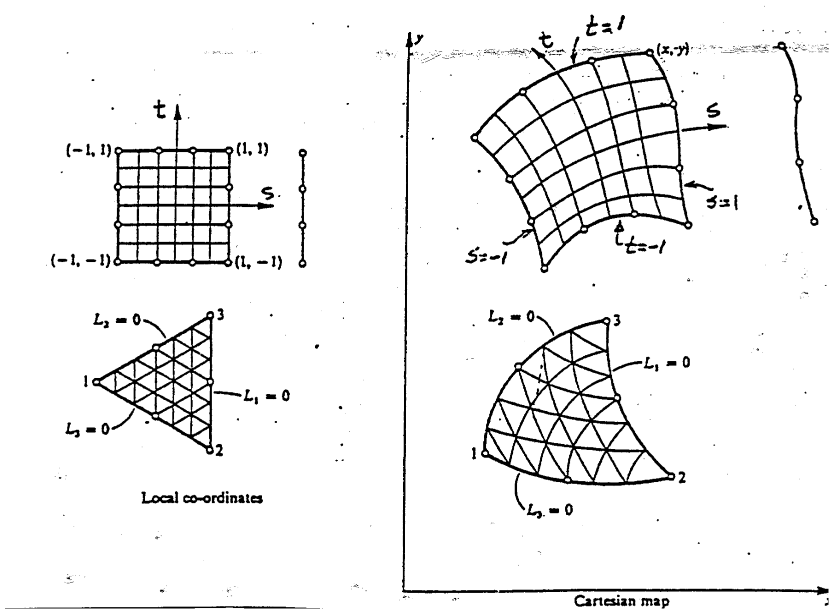


Figure 1. Two dimensional mapping of some elements

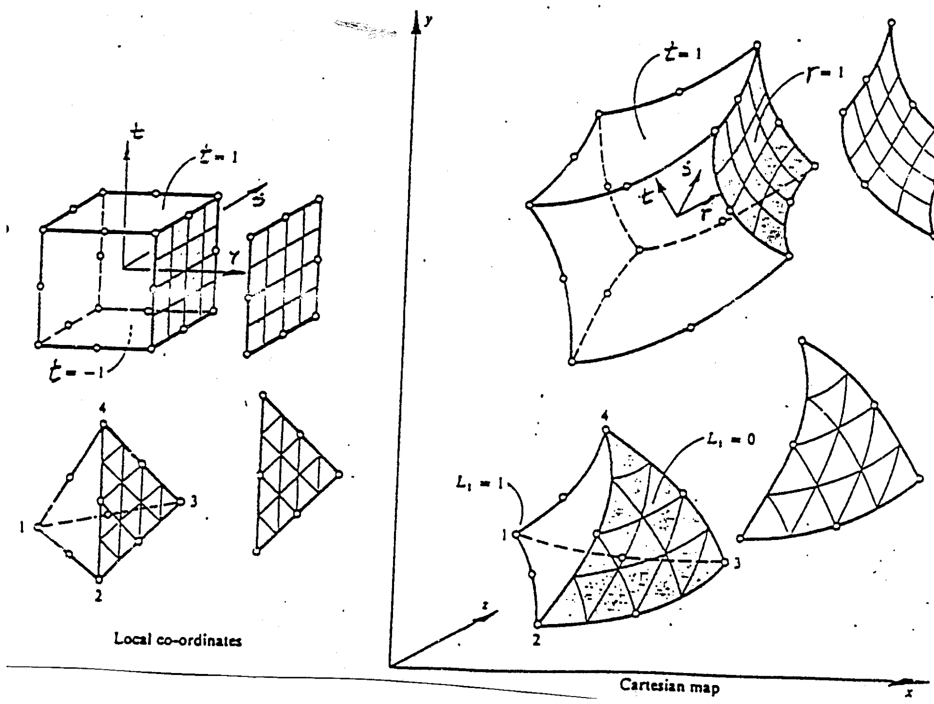


Figure 2. Thrre-dimensional mapping of some elemnts

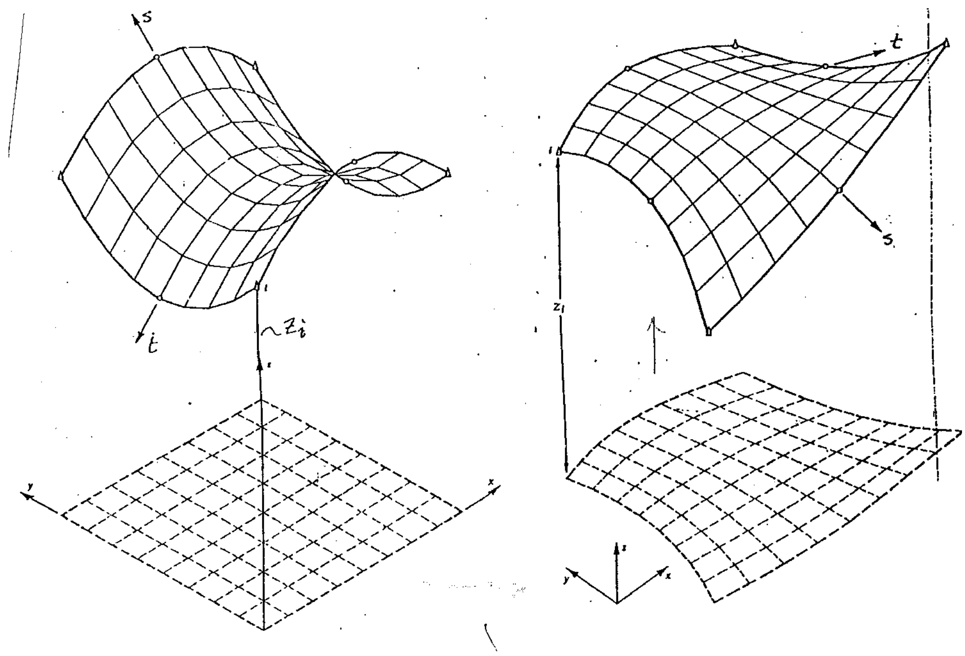


Figure 3. Flat elements (of parabolic type) mapped into thre dimensions

Coordinate transformations are required:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = f\begin{Bmatrix} s \\ t \end{Bmatrix} or\ f\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix}\ etc.$$

However, to apply the principle of transformations, there must exist a one to one correspondence between Cartesian and curvilinear coordinates i.e. no severe distortion- folding back or for a line element α i.e. no cross over etc.
A most convenient method of establishing co-ordinate transformations is to use shape functions discussed in the previous section and already used to represent the variation of unknown quantities or functions.

Express x and y for each element as:

$$x = N'_1\, x_1 + N'_2\, x_2 + ...$$
$$y = N'_1\, y_1 + N'_2\, y_2 + ...$$

when the shape functions of local coordinates s, t or L_1, L_2 and L_3 in two dimensional problems (fig. 1). Here the shape in x-y coordinate system is distorted and triangular or square shapes in local coordinates are called parent elements.
In above equations N'_i are the shape functions with unit values at the nodes. For curved shapes in x-y, these must be nonlinear functions of s, t or L_1, L_2 and L_3. x_1, x_2 … and y_1, y_2 … are nodal coordinates.

7.1- Geometric Conformability of elements

This requires that by the shape function transformation, the mapping of the parent element into real object should not leave any gaps or holes, figure4.

Theorem 1: If two adjacent elements are generated from parent elements in which the shape functions satisfy continuity requirements, then the distorted elements will be continuous.

The theorem above is obvious and follows from C_0 continuity implied by shape functions for any function that is approximated. Here, the functions are x and y and if the adjacent elements are given the same coordinates at the common nodes, continuity is implied.

Unknown Function within Distorted, Curvilinear Elements and Continuity Requirements

So far we have defined the shape of distorted elements by the shape functions N'_I for a parent element.
Suppose the unknown function to be determined in ϕ and can be approximated by:

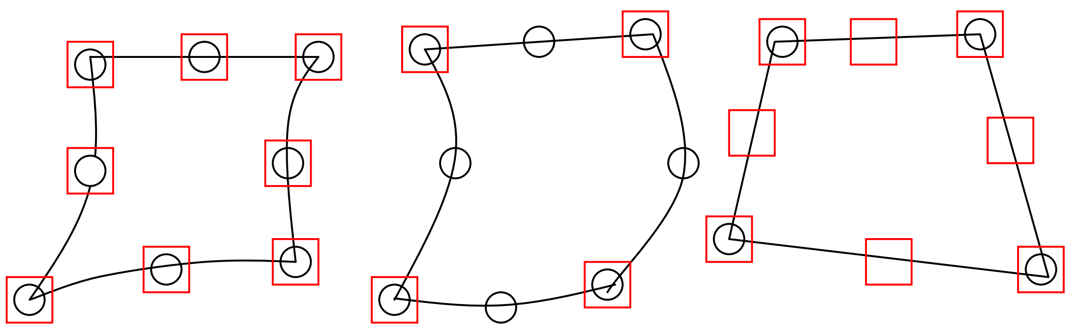
$$\phi = \sum_{i=1}^{m} N_i \phi_i \qquad m = number\ of\ dof\ per\ elements$$

where N_i are the usual shape functions and for now assume these are different from N'_i. ϕ_i are the nodal dof.

Theorem 2: If the shape functions N_i are such that continuity of ϕ is preserved in parent co-ordinates (or local coordinates)- then continuity requirements will be satisfied in distorted elements.

Proof is same as for theorem 1

The nodal values ϕ_I may or may not be associated with the same nodes as used to specify the element geometry.



Isoparametric Super parametric Subparametric

Points at which ϕ_I are specified

Nodes used for element geometry, coordinates are specified

Isoparametric	N_i=N'_i
Super-parametric	Variation of geometry is more general than the unknown ϕ
Sub-parametric	More nodes to define ϕ (i.e. more general) than to define geometry

Subparametric elements have been used more often in practice.

7.2- Constant Derivative Condition

This means that constant stress should be included in the approximation.

Consider ϕ in equation $\phi = \sum_{i=1}^{m_i} N_i \phi_i$. Constant derivative condition requires:

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y$$

Where α_1, α_2 and α_3 are generalized parameters.
At nodes (x_i, y_i):

$$\phi_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i$$

$$\phi = \sum_i N_i \phi_i = \sum_i N_i \left(\alpha_1 + \alpha_2 x_i + \alpha_3 y_i\right)$$

$$\phi = \alpha_1 \sum_i N_i + \alpha_2 \sum_i N_i x_i + \alpha_3 \sum_i N_i y_i$$

equating above equation yields :

$$\sum_i N_i = 1 \quad \sum_i N_i x_i = x \quad \sum_i N_i y_i = y \quad (ii)$$

Therefore, constant derivative condition will be obtained if equations ii are satisfied. Recall co-ordinate transformation in equation of:

$$x = \sum_i N'_i \, x_i$$

$$y = \sum_i N'_i \, y_i$$

where x_i and y_i are element nodal coordinates. For isoparametric elements N_i=N'_I therefore, equations of above are automatically satisfied.

Need $\sum_i N_i = 1$

Theorem 3: The constant derivative condition will be satisfied for all isoparametric elements providing $\sum_i N_i = 1$

The same requirement is necessary and theorem valid for sub parametric elements provided the shape function N'_I can be expressed as a linear combination of N_i i.e.:

$$N'_i = \sum_i C_{ij} N_i$$

where C_{ij} are constants. It is obvious that for the case of subparametric elements, N'_I are of lower order than N_i and above equation can be easily satisfied, however not so far superparametric elements. Some numerical tests may have to be performed in order to satisfy above equations. (Perhaps, a patch test should be performed, also an eigenvalyue analysis of the stiffness matrix may reveal the presence of constant stress nodes)

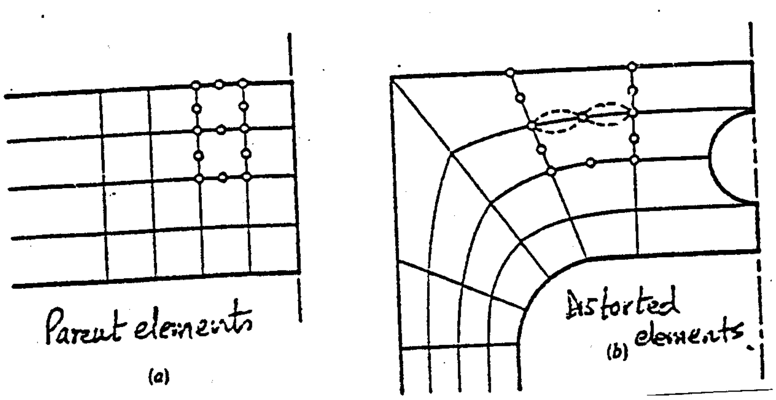
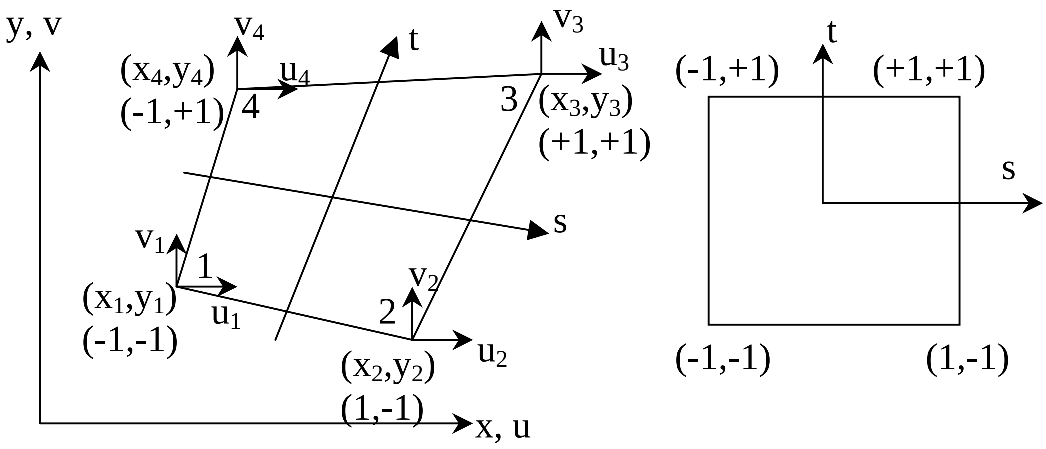


Figure 4. Compatibility requiremen in real subdivision for transformation

8- Quadrilateral Iso parametric Elements

8.1- Four node Quadrilateral Element



Distorted Element Parent Element

Local coordinates in s and t:

Shape functions

$N_1 = (1-s)(1-t)/4 \qquad -1 \le s \le +1 \quad -1 \le t \le +1$

$N_2 = (1+s)(1-t)/4$

$N_3 = (1+s)(1+t)/4$

$N_4 = (1-s)(1+t)/4$

coordinate transformation:

$$x = \sum_{i=1}^{4} N_i x_i$$

$$y = \sum_{i=1}^{4} N_i y_i$$

displacement approximations:

$$u = \sum_{i=1}^{4} N_i u_i$$

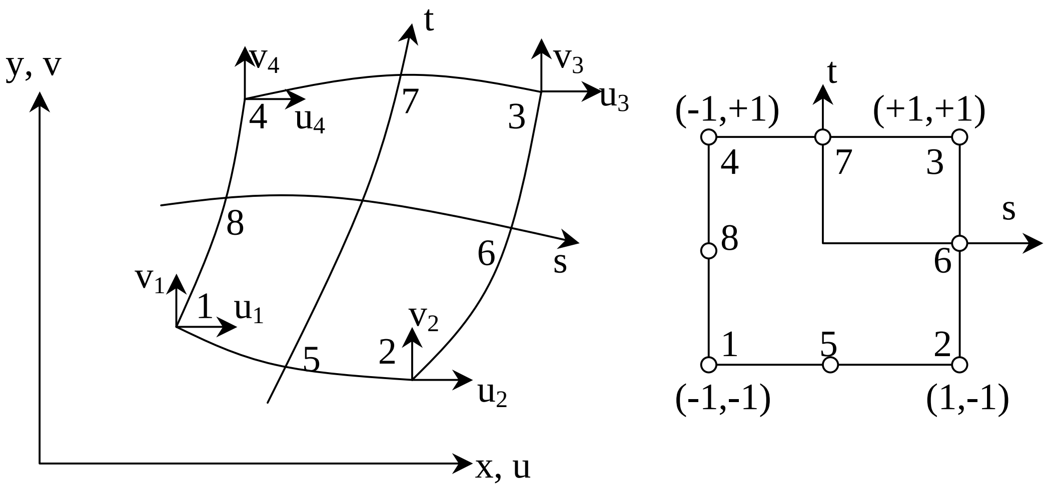
$$v = \sum_{i=1}^{4} N_i v_i$$

u_i and v_i are the degrees of freedom of node I in global coordinate system x,y.

note x and y are linear function of s and t , the local coordinates straight edges remain straight from parent to distorted element after transformation.

8.2- Eight Node Quadrilateral Element (Plane Elasticity)

A detailed derivation of isoparammetric elements is presented for this element.



Global system
S and t are curvilinear coordinates here

Local co-ordinates

Shape functions in local coordinates:

$$N_1 = -(1-s)(1-t)(1+s+t)/4 \qquad N_5 = (1-s^2)(1-t)/2$$

$$N_2 = -(1+s)(1-t)(1-s+t)/4 \qquad N_6 = (1-t^2)(1+s)/2$$

$$N_3 = -(1+s)(1+t)(1-s-t)/4 \qquad N_7 = (1-s^2)(1+t)/2$$

$$N_4 = -(1-s)(1+t)(1+s-t)/4 \qquad N_8 = (1-t^2)(1-s)/2$$

Try the following coordinate transformation:

$$x = \sum_{i=1}^{8} N_i x_i = f(s,t)$$

$$y = \sum_{i=1}^{8} N_i y_i = g(s,t)$$

Displacements are approximated by:

$$u = \sum_{i=1}^{8} N_i u_i$$

$$v = \sum_{i=1}^{8} N_i v_i$$

Both equation for displacement and coordinates satisfy the requirements of theorem 1, 2 and 3 in the previous section and hence the convergence criteria.

8.2.1- Element Properties (Evaluation of stiffness Matrix)

Displacement

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ . \\ . \\ u_8 \\ v_8 \end{Bmatrix}$$

$$\{u\} = [N]\{\delta\} \qquad \{u\}^T = \begin{bmatrix} u & v \end{bmatrix} \qquad \{\delta\}^T = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & . & . & u_8 & v_8 \end{bmatrix}$$

$$\{\varepsilon\} = [B]\{\delta\} \qquad [B] = [L][N]$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & . & . & . \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & . & . & . \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & . & . & . \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ . \\ . \\ u_8 \\ v_8 \end{Bmatrix}$$

$$\{\varepsilon\}_{3\times1} = [B]_{3\times16}\{\delta\}_{16\times1}$$

But N_i are functions of s and t and so are z, y functions of s and t. Therfore, using chain rule:

$$\frac{\partial N_i}{\partial s} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial N_i}{\partial t} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial s} \\ \frac{\partial N_i}{\partial t} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

The matrix [J] is called the Jacobian matrix and can be found explicitely in terms of local coordinates s and t, and x_i and y_i using equation 2. The left hand side of above eqn can also be evaluated using shape functions.

$$\frac{\partial x}{\partial s} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial s} x_i \qquad \frac{\partial x}{\partial t} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial t} x_i$$

$$\frac{\partial y}{\partial s} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial s} y_i \qquad \frac{\partial y}{\partial t} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial t} y_i$$

$$[J] = \begin{bmatrix} \frac{\partial N_i}{\partial s} x_i & \frac{\partial N_i}{\partial s} y_i \\ \frac{\partial N_i}{\partial t} x_i & \frac{\partial N_i}{\partial t} y_i \end{bmatrix}$$

$$[J]_{2\times 8} = \begin{bmatrix} \frac{\partial N_1}{\partial s} & \frac{\partial N_2}{\partial s} & \frac{\partial N_3}{\partial s} & . & . & . \\ \frac{\partial N_1}{\partial t} & \frac{\partial N_2}{\partial t} & \frac{\partial N_3}{\partial t} & . & . & . \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ . & . \\ . & . \\ . & . \\ x_8 & y_8 \end{bmatrix}$$

Now invert $[J]$ *numerically*

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial s} \\ \frac{\partial N_i}{\partial t} \end{Bmatrix}$$

$$[J]^{-1} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} = [I]$$

Next :

$$\frac{\partial N_1}{\partial s} = (1-t)(2s+t)/4 \qquad \frac{\partial N_1}{\partial t} = (1-s)(2t+s)/4$$

$$\frac{\partial N_2}{\partial s} = (1-t)(2s-t)/4 \qquad \frac{\partial N_2}{\partial t} = (1+s)(2t-s)/4$$

$$\frac{\partial N_3}{\partial s} = (1+t)(2s+t)/4 \qquad \frac{\partial N_3}{\partial t} = (1+s)(2t+s)/4$$

$$\frac{\partial N_4}{\partial s} = (1+t)(2s-t)/4 \qquad \frac{\partial N_4}{\partial t} = (1-s)(2t-s)/4$$

$$\frac{\partial N_5}{\partial s} = -s(1-t) \qquad \frac{\partial N_5}{\partial t} = -(1-s^2)$$

$$\frac{\partial N_6}{\partial s} = (1-t^2)/2 \qquad \frac{\partial N_6}{\partial t} = -t(1+s)$$

$$\frac{\partial N_7}{\partial s} = -s(1+t) \qquad \frac{\partial N_7}{\partial t} = (1-s^2)/2$$

$$\frac{\partial N_8}{\partial s} = -(1-t^2)/2 \qquad \frac{\partial N_8}{\partial t} = -t(1-s)$$

Also dxdy=det[J] dsdt. Note $\det[J]=\dfrac{\partial(x,y)}{\partial(s,t)}$ transform derivatives with respect to x and y to with respect to s and t.
Now back to strain $\{\varepsilon\}=[B]\{\delta\}$. Matrix [B] which consists of $N_{i,x}$ and $N_{i,y}$, can now be determined by using above equations:

$$\begin{Bmatrix} N_{i,x} \\ N_{i,y} \end{Bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \begin{Bmatrix} N_{i,s} \\ N_{i,t} \end{Bmatrix}$$

$$[B] = \begin{bmatrix} I_{11}N_{1,s}+I_{12}N_{1,t} & 0 & same & same & . & . & . \\ 0 & I_{21}N_{1,s}+I_{22}N_{1,t} & with & with & . & . & . \\ I_{21}N_{1,s}+I_{22}N_{1,t} & I_{11}N_{1,s}+I_{12}N_{1,t} & N_2 & N_2 & . & . & . \end{bmatrix}$$

stress

$$\{\tau\}=[D]\{\varepsilon\}$$

for plane stress

$$\{\tau\} = \begin{Bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}$$

$$\{\tau\}=[D]\{\varepsilon\}=[D][B]\{\delta\}$$

8.2.2- Stiffness Matrix

Stiffness matrix is then given by:

$$[k]_{16\times16} = t\iint_A [B]^T{}_{16\times3}[D]_{3\times3}[B]_{3\times16}\ dxdy$$

$$dxdy = \det[J]dsdt$$

and the limits of integration -1 to +1 for both s and t.

$$[k]_{16\times16} = t\int_{-1}^{1}\int_{-1}^{1} [B]^T{}_{16\times3}[D]_{3\times3}[B]_{3\times16}\ \det[J]dsdt$$

here the integration $[B]^T{}_{16\times3}[D]_{3\times3}[B]_{3\times16}\ \det[J]$, the entire expression is same complicated function of s and t. Although the limits of integration are simple when integration is carried out over the parent element in s and t coordinates, any typical term of the integrand matrix of above integration becomes very complicated algebraically and hence, we have to resort to numerical integration. However, for very simple elements, integration can be carried out exactly.

8.2.3- Comment on det[J]

A well known condition for one-to-one mapping desired for cured edge elements is that the sign of det[J] should remain unchanged at all points of the domain mapped i.e. s, t plane. Violent distortions may cause alteration of sign of det[J].

8.2.4- Body Forces

Suppose there are body forces p_x and p_y per unit volume involved in the problem being considered for stress analysis.

$$\{p\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

for plane elasticity, work done by {p} is:

$$W = t\iint_A \begin{bmatrix} u & v \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} dA = t\iint_A \{u\}^T \{p\} dA$$

$$\{u\} = [N]\{\delta\}$$

$$W = t\iint_A \{\delta\}^T [N]\{p\} dA = \{\delta\}^T \{f_b\}$$

$\{f_b\}$ *are body forces due to distributed body forces*

$$\{f_b\}^T = \begin{bmatrix} f_{b1} & f_{b2} & f_{b3} & . & . & . & f_{b16} \end{bmatrix}$$

$$\{\delta\}^T = \begin{bmatrix} u_1 & v_1 & u_2 & . & . & . & v_8 \end{bmatrix}$$

for gravity load p_x=0 and p_y=-γ (γ= weight per unit volume, constant). From above equations:

$$\{f_b\} = t\int_{-1}^{+1}\int_{-1}^{+1} [N]^T \{p\} \det[J] ds dt \quad (\textit{consistent load vector})$$

Again this can be integrated numerically. For gravity loading $\{p\} = \begin{Bmatrix} 0 \\ -\gamma \end{Bmatrix}$, further, for constant load,

$$\{f_b\} = t\{p\}^T \int_{-1}^{+1}\int_{-1}^{+1} [N] \det[J] ds dt$$

This integration can be done exactly i.e. by hand. If {p} is a function of x and y, then we may also interpolate {p} from the nodal values in terms of shape functions N_i i.e. $p_x = \sum_{i=1}^{8} N_i p_{xi}$ where p_{xi} are the values of $p_x(x,y)$ at the nodes, similarly for p_y.

8.2.5- Specific Boundary Stress (stress boundary condition)

Boundary work done by stresses is given by:

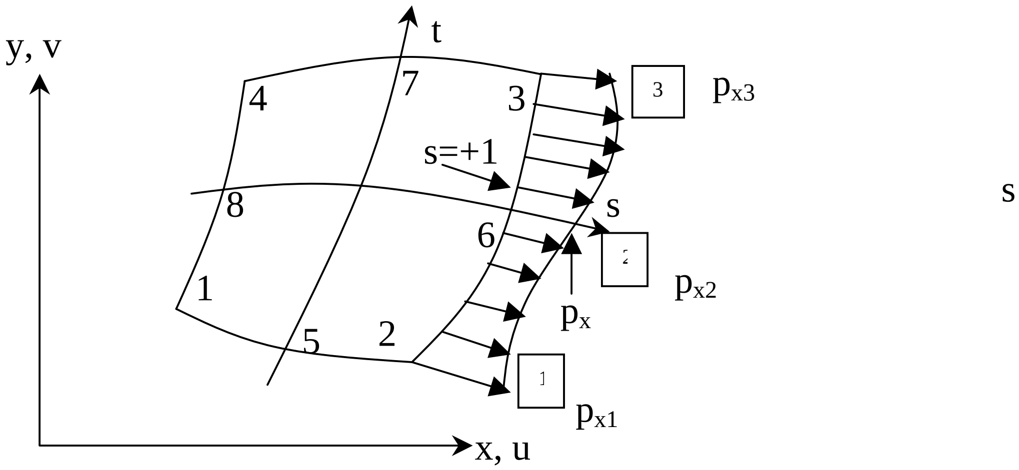
$$W_T = t\int_{S_T} \bar{T}_i \, u_i \, ds$$

$$W_T = t\int_{S_T} \{u\}^T \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} ds = \{\delta\}^T \underbrace{\{f_s\}}_{\text{Due to stress on } s_T}$$

where p_x and p_y are the stresses in x and y directions (((global system) on the boundary. S_T is part of boundary on which the stress are prescribed. {u} is now given by: $\{u\}=[N^s]\{\delta\}$ where $[N^s]$ is the same as [N] except its value on the boundary under consideration. Further, if p_x and p_y are functions of x and y on the boundary, again we can use the shape functions N_i to interpolate stress p_x and p_y but for N_i on the boundary. For example, consider edge 2-6-3 of the element shown.

Stress acting on this boundary is $p_x(x,y)$ as shown. For edge 2-6-3, s=+1 therefore, only nonzero N_i on s=+1 are:

$$N_2 = -\frac{t}{2}(1-t) \qquad N_6 = (1-t^2) \qquad N_3 = \frac{t}{2}(1+t)$$



$\boxed{1}$ are boundary nodes assigned temporarily

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_2 & 0 & N_3 & 0 & N_6 & 0 \\ 0 & N_2 & 0 & N_3 & 0 & N_6 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_6 \\ v_6 \end{Bmatrix}$$

$$\{u\} = [N^s]\{\delta^s\}$$

$$x = N_2 x_2 + N_3 x_3 + N_6 x_6 \quad Coordinate \quad Transforma \quad tion$$

$$y = N_2 y_2 + N_3 y_3 + N_6 y_6$$

$$approximat \quad ion \quad for \quad p_x \quad as \quad \bar{p}_x$$

$$\bar{p}_x = N_2 p_{x1} + N_3 p_{x3} + N_6 p_{x2}$$

Finally ,

$$dS = \sqrt{dx^2 + dy^2}$$

But x and y are given by above equations i.e. f(t) along 2-6-3. Therefore, dl, the infinitesimal element of length along 2-6-3 is given by:

$$dS = dl = \left[\left(\frac{\partial x}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial t} \right)^2 \right]^{\frac{1}{2}} dt \qquad note: \frac{\partial x}{\partial t} = \frac{dx}{dt} \quad along\ edge\ 2-6-3$$

$$similarly, if\ \ there\ is\ also\ a\ specific\ stress\ \ p_y(x, y)\ then:$$

$$\bar{p}_y = N_2 p_{y1} + N_3 p_{y3} + N_6 p_{y2}$$

$$\begin{Bmatrix} \bar{p}_x \\ \bar{p}_y \end{Bmatrix} = \begin{bmatrix} N_2 & 0 & N_3 & 0 & N_6 & 0 \\ 0 & N_2 & 0 & N_3 & 0 & N_6 \end{bmatrix} \begin{Bmatrix} p_{X1} \\ p_{y1} \\ p_{X3} \\ p_{y3} \\ p_{X2} \\ p_{y2} \end{Bmatrix}$$

$$\{\bar{p}\} = [N^S]\{p^i\}$$

$$W_T = t \int_{-1}^{+1} \{\delta^s\}^T [N^s]^T [N^s]\{p^i\} \underbrace{\left[\left(\frac{\partial x}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial t} \right)^2 \right]^{\frac{1}{2}}}_{at\ s=+1} dt$$

$$\{f_s\} = t \int_{-1}^{+1} [N^s]^T [N^s]\{p^i\} \underbrace{\left[\left(\frac{\partial x}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial t} \right)^2 \right]^{\frac{1}{2}}}_{at\ s=+1} dt$$

and :

$$\{f_s\}^T = \begin{bmatrix} f_{s3} & f_{s4} & f_{s5} & f_{s6} & f_{s11} & f_{s12} \end{bmatrix}$$

This procedure can be repeated for other edges as well. One have to be careful as to s or t =E etc.

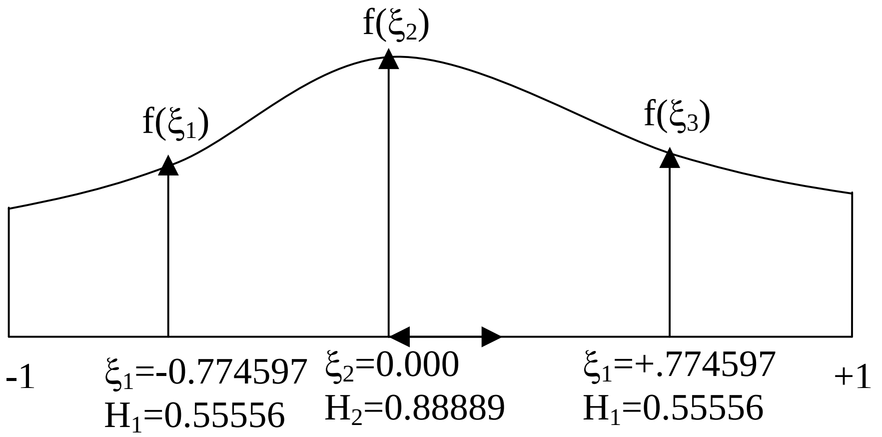
9- Numerical Integration (Gauss Quadrature)

Integration may be done analytically by using closed form formulas from a table of integrals. Alternatively, integration may be done numerically. Gauss quadrature is a commonly used form of numerical integration.
Instead of using Newton cotes quadrature method where point at which the function is to be found (i.e. numerical value) are determined a priori (usually equal intervals) we shall use Gauss Quadrature method. The coordinates of sampling points are determined for best accuracy.

9.1- Line Integration

$$I = \int_{-1}^{+1} f(s)ds = \sum_{i=1}^{n} H_i f(s_i)$$

If f(s) is a polynomial, n-point Gauss quadrature yields the exact integral if f(s) is of degree 2n-1 or less. Assume a polynomial expression to estimate I. For n sampling points we have 2n unknown (H_i and s_i) for which a polynomial of degree 2n-1 can be constructed and integrated exactly. This yields an error of order $O(\Delta^{2n})$. Thus, for n=3, a polynomial of degree 5 can be integrated exactly using Gaussian quadrature method.
Thus the form f(s)=a+bs is exactly integrated by a one-point rule. The form $f(s)=a+bs+cs^2$ is exactly integrated by a two-point rule, and so on. Use of an excessive number of points, for example, a two-point rule for f(s)=a+bs still yields the exact result.
In above equation,
n=number of sampling points often called Gauss points.
H_i=Weighting constants
s_i=Coordinates of sampling points
For any given n (by choice) H_i and s_i can be obtained from the tables.
n=3 Exact for 5^{th} degree polynomial 2n-1=5



$$I=0.55556f(-0.774597)+0.88889f(0.000)+0.55556f(+0.774597)$$

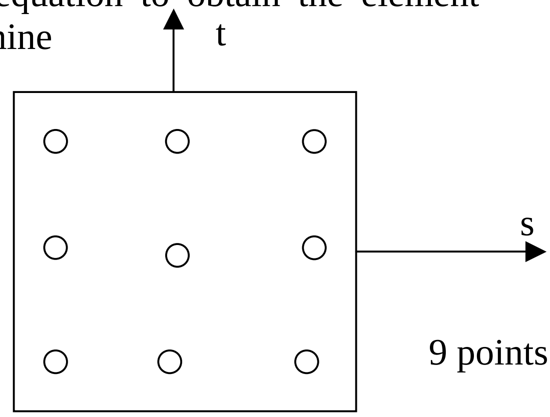
If f(s) is not a polynomial, but (say) the ratio of two polynomials, Gauss quadrature yields an approximate result. Accuracy improves as more Gauss points are used. Convergence toward exact result may not be monotonic.

9.2- Gauss Quadrature Numerical Integration over a Rectangle

$$I = \int_{-1}^{+1}\int_{-1}^{+1} f(s,t)\,ds\,dt = \int_{-1}^{+1} \sum_{i=1}^{n} H_i f(s_i, t)\,dt = \sum_{i=1}^{n} \sum_{j=1}^{n} H_i H_j f(s_i, t_i)$$

Example of n=3, we can use the same tables. Exact for polynomial of degree 5 in each direction, s and t.

Let us use n=3 for integration of above equation to obtain the element stiffness matrix [k]. Thus, we have to determine $[B]^T[D][B]\det[J]$ at each sample point (s_i, t_i)



In three dimensions, Gauss quadrature of order n over a hexahedron involves n^3 points. Three summations and the product of three weight factors.

Abscissae And Weight Coefficients Of The Gaussian Quadrature Formula

$$\int_{-1}^{1} f(s)\,ds = \sum_{j=1}^{n} H f(s_i)$$

n	±S	H
n = 1	0	2·00000 00000 00000
n = 2	0·57735 02691 89626	1·00000 00000 00000
n = 3	0·77459 66692 41483	0·55555 55555 55556
	0·00000 00000 00000	0·88888 88888 88889
n = 4	0·86113 63115 94053	0·34785 48451 37454
	0·33998 10435 84856	0·65214 51548 62546
n = 5	0·90617 98459 38664	0·23692 68850 56189
	0·53846 93101 05683	0·47862 86704 99366
	0·00000 00000 00000	0·56888 88888 88889
n = 6	0·93246 95142 03152	0·17132 44923 79170
	0·66120 93864 66265	0·36076 15730 48139
	0·23861 91860 83197	0·46791 39345 72691
n = 7	0·94910 79123 42759	0·12948 49661 68870
	0·74153 11855 99394	0·27970 53914 89277
	0·40584 51513 77397	0·38183 00505 05119
	0·00000 00000 00000	0·41795 91836 73469
n = 8	0·96028 98564 97536	0·10122 85362 90376
	0·79666 64774 13627	0·22238 10344 53374
	0·52553 24099 16329	0·31370 66458 77887
	0·18343 46424 95650	0·36268 37833 78362
n = 9	0·96816 02395 07626	0·08127 43883 61574
	0·83603 11073 26636	0·18064 81606 94857
	0·61337 14327 00590	0·26061 06964 02935
	0·32425 34234 03809	0·31234 70770 40003
	0·00000 00000 00000	0·33023 93550 01260
n = 10	0·97390 65285 17172	0·06667 13443 08688
	0·86506 33666 88985	0·14945 13491 50581
	0·67940 95682 99024	0·21908 63625 15982
	0·43339 53941 29247	0·26926 67193 09996
	0·14887 43389 81631	0·29552 42247 14753

9.3- Comment on Numerical Integration

Numerical integration, instead of exact may introduce additional errors and first attempt may be directed at reducing the error as much as possible. This may not be very economical. Therefore, the following should be determined:

a) The minimum integration requirement permitting convergence
b) The integration requirements necessary to preserve the rate of convergence which would results if exact integration were used.

Let p be the order of degree of complete polynomial and m be the order of differentials occurring in the strain energy expression. Providing the integration is exact to the order 2(p-m) or shows an error of O($h^{2(p-m)+1}$, or less, then no loss of convergence order will occur.

In curvilinear coordinates take a curvilinear dimension h of an element. For C_0 problem (m=1), the integration is of the following order:

p=1, Linear displacement O(h)
p=2, Quadratic displacement O(h^3)
p=3, Cubic displacement O(h^5)

With numerical integration , singular stiffness matrix may result for low integration orders making lower order integrations impractical. In general, there should be at least as many integration points as required to yield a number of independent relations equal or greater than the number of overall unknowns.

Consider again the plane eight-node isoparametric element discussed previously. Its stiffness matrix integrand $[B]^T[D][B]det[J]$ is an 16 by 16 matrix. Because it is a symmetric matrix, only 136 of 256 coefficienta sre different from one another. Each of these coefficients has the form f(s,t) and each must be integrated over the element area. In computer programming, a p-point integration rule requires p passes through a integration loop. Each pass requires ebaluation of [B] and det[J] at the coordinates of a Gauss point, computation of the product $[B]^T[D][B]det[J]$, and multiplication by weight factor. Each pass makes a contribution to [k] which is fully formed when all p passes have been completed. Clearly, there is considerable computation required in this process.

For an element of general shape, each coefficient in matrix $[B]^T[D][B]det[J]$ is the ration of two polynomials in s and t. The polynomial in the denominator comes from J^{-1} when [J] is inverted det[J] becomes the denominator of every coefficient in J^{-1} and hence appears in the denominator of every coefficient in [B]. Analytical integration of [k] would require the use of cumbersome formulas. Numerical intehration is simpler but in general it is not exact, so that [k] is only approximately integrated regardless of number of integration points. Should we use very few points for low computational expense or very many points to improve the accuracy of integration? The answer is neither, for reasons explained in the following.

9.4- Choice of Quadrature Rule, Instabilities

A FE model is usually inexact, and usually it errs by being too stiff. Overstiffness is usually made worse by using more Gauss points to integrate element stiffness matrix because additional points capture more higher-order terms in [k]. these terms resist some deformation modes that lower-order terms do not, and therefore act to stiffen an element. Accordingly, greater accuracy in the integration of [k] usually produce less accuracy in the FE solution, in addition to requiring more computation.

On the other hand, use of too few Gauss points produces an even worse situation known by various names, Instability, spurious singular mode, mechanism, kinematic mode, zero energy mode, and hourglass mode. Instability occurs if one or more deformation modes happen to display zero strain at all Gauss points. One must regard Gauss points as strain sensors. If Gauss points sense no strain under a certain deformation mode, the resulting [k] will have no resistance to that deformation mode.

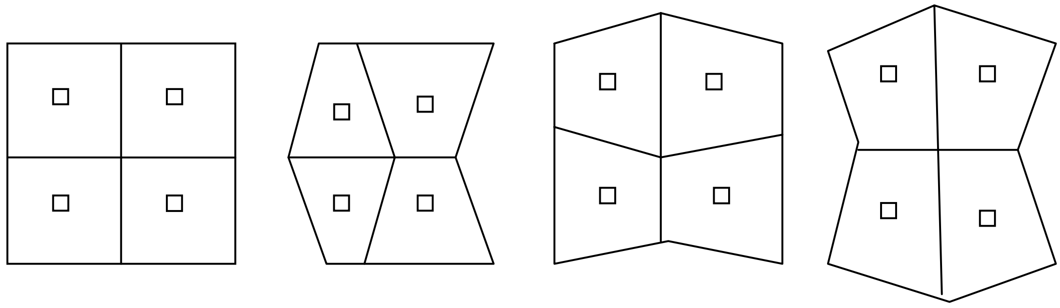
A simple illustration of instabilities are shown. Four-node plane elements are integrated by a one-point Gauss rule. In the lower left element with c a constant, the three instabilities shown have respective forms

b)u=cxy v=0

c)u=o v=-cxy

d)u=cy(1-x) v=cx(y-1)

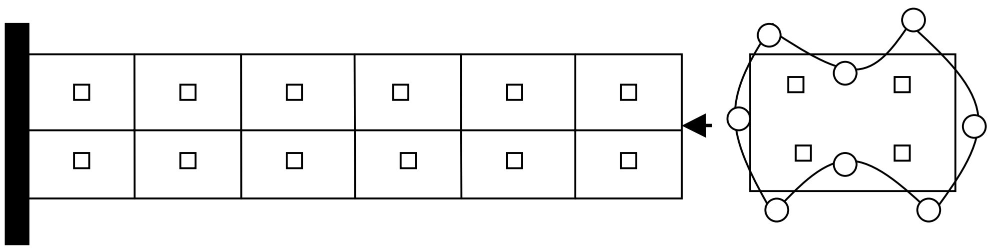
We easily check that each of these displacement fields produces strains $\varepsilon_{xx}=\varepsilon_{yy}=\gamma_{xy}=0$ at the Gauss point, x=y=0.



a) Undeformed plane 2 by 2 four-node square elements,
Gauss points are shown by square
B, c, d Instability displacement modes

Nonrectangular elements behave in the same way. Even if the mesh had just enough supports to prevent rigid body motion it could still display these

modes, without strain at the Gauss points, and hence without strain energy. The FE model would have no ressistance to loadings that would activate these modes. The global [k] would be singular regardless of how the structure is loaded.
When supports are adequate to make [k] nonsingular, there may yet be near-instability that is troublesome. In the figure, all dof are fixed at the support and each element is integrated with one point. Restraint provided by the support is felt less and less with increasing distance from it.
A plane eight-node element whose stiffness matrix is integrated with four Gauss points has the hourglass instability shown.



Mesh of four-node square elements with all nodes fixed at the support.
Gauss points are shown by square
Hourglass instability displacement mode in a single 8-node element integrated by Gauss points

There is no way that two adjacent elements can both display this mode while remaining connected.

9.5- Stress Calculation and Gauss Points

Calculated stress are often most accurate at Gauss points. It happens that the locations of greatest accuracy are apt to be the same Gauss points that were used for integration of the stiffness matrix.
In summary, it is common practice to use an order 2 Gauss rule (four points) to integrate [k] of four- and eight-node plane elements, and common practice to compute strains and stresses at these same points. Similarly, three-dimensional elements often use eight Gauss points for stiffness integration and stress calculation. Stresses at nodes or at other element locations are obtained by extrapolation or interpolation from Gauss point values.