Chapter 2

# The Theory of the Finite Element Method

## Introduction and some Basic Concepts

# 1- The Concept of an Element

## 1.1- The Finite Element Method

Physical visualization of a body or structure as an assemblage of building block-like elements, interconnected at the nodal points.

1) Majority of the problems in continuum mechanics are too complicated to handle exactly.
2) F.E. method is an approximate method to solve a continuum problem
3) F.E. method is subdivision of a continuum into a finite number of parts (called finite elements). The behavior of each finite part is specified by a finite number of parameters (also called generalized coordinates)
4) The solution of the complete system as an assembly of its elements follow precisely the same rules as those applicable to standard discrete problems e.g. Matrix structural analysis etc.
5) Question? Is the solution obtained near the exact solution of the problem? There is a lot more involved i.e. the mathematical theory behind the F.E. method, before we can answer this question.

Continuous → Discrete

Interpretation: Body is not subdivided into separate parts, instead the continuum is zoned into regions by imaginary lines (2D bodies) or imaginary planes (3D bodies) inscribed on the body. No physical separation is envisaged at these lines or planes.

We apply variational procedures to each element (region). We are interested in behavior of element. We need to define the element behavior in term of the elements geometry, material properties.
We then assemble each element into the assembled structure.

## 1.2- Boundary Value Problem

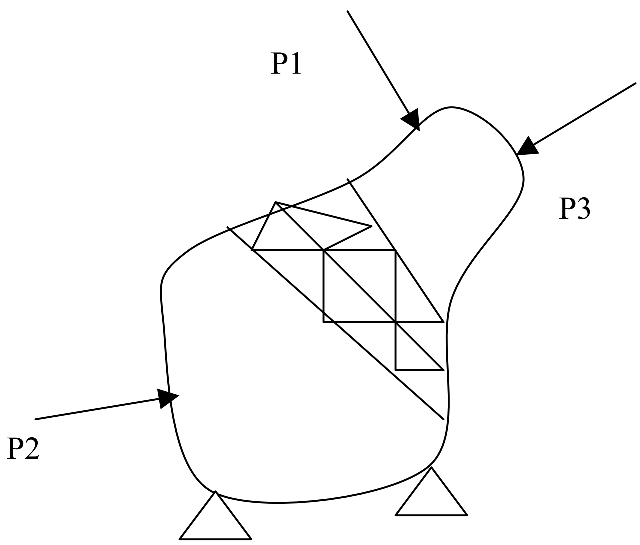
Problem governed by differential equation, in which values of state variables( or their normal derivatives)are given on the boundary.
Solution at a general interior point depends on the data at every point of the boundary. A change in only one boundary value affects the complete solution.
Initial Value Problem

Time as an independent variable, solution depends on initial conditions and boundary condition.

## 1.3- Schematic Picture of the Finite Element Method (Analysis of discrete systems)

Consider a complicated boundary value problem

1) In a continuum, we have an infinite number of unknown
   System Idealization
2) To get finite number of unknowns, we divide the body into a number of sub domains (elements) with nodes at corners or along the element edges with finite degrees of freedom.
3) Element equilibrium, the equilibrium requirement of each element is established in terms of state variables.
4) Element assemblage, the element interconnection requirements are invoked to establish a set of simultaneous equations for unknown state variable.
5) Solution of response, the simultaneous equations are solved.



Notes to be considered:

1) Selection of unknown state variables that characterize the response of the system
2) Identification of elements

There is some choice selection of state variables.

## 1.4- Various Element Shapes

Needs engineering judgment (geometry, no. of independent coordinates can be 2 oe 3 dimensional)
1-D ele. Idealized by line
2-D ele. Plane stress, plane strain and plate bending element can be triangular, rectangular, quadrilateral, axysymmetric
3-D ele. Tetrahedron, Rectangular prism, arbitrary hexahedron
Mixed assemblage e.g. beam elem. And plate bend.

## 2- Displacement Models

F.E. based on approximation of state variables if state variables is displacement, then the function that approximate displacement for each element is called displacement model or displacement function or displacement field.

Displacement functions are polynomials

Reasons: 1. Easy for mathematical applications (differentiation& integration)

2. Arbitrary order permits a recognizable approximation

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + ..... + \alpha_{n+1} x^n$$

$\alpha_i$'s *are generalized coordinates*

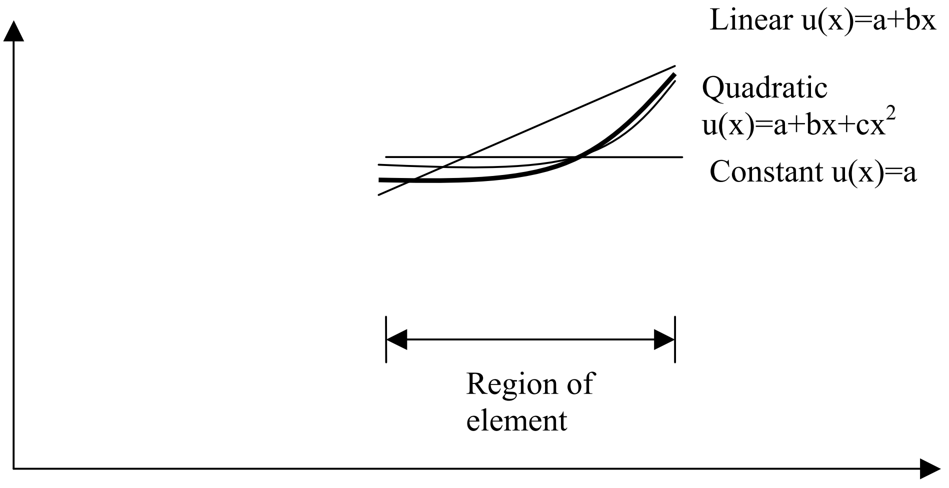
Number of terms in u(x) determines shape of displacement model

Magnitude of generalized coordinates governs the amplitude

$$u(x) = \{\phi\}^T \{x\}$$

$$\{\phi\}^T = [1 \quad x \quad x^2 \quad . \quad . \quad . \quad x^n]$$

$$\{x\}^T = [\alpha_1 \quad \alpha_2 \quad . \quad . \quad . \quad \alpha_{n+1}]$$




The greater no. of terms, then closer to exact solution
If exact solution is polynomial of order m, then terms in excess of m do not improve the representation.

Generalized coordinates displacement model is elementary form of models for the F.E. method. There is an alternative representation of polynomial displacement field that facilitates the formulation of the basic equations for the elements.

For 2-D displacement model

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \ldots\ldots + \alpha_m y^n$$

$$\upsilon(x,y) = \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \alpha_{m+4} x^2 + \alpha_{m+5} xy + \alpha_{m+6} y^2 \ldots\ldots + \alpha_{2m} y^n$$

$$m = \sum_{i=1}^{n+1} i$$

*u displacement in x and $\upsilon$ displacement in y directions*

$$\begin{Bmatrix} u \\ \upsilon \end{Bmatrix} = [\phi]\{\alpha\} = \begin{bmatrix} \{\phi\}^T & \{0\}^T \\ \{0\}^T & \{\phi\}^T \end{bmatrix} \{\alpha\}$$

$$\{\phi\}^T = [1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad . \quad . \quad .y^n]$$

$$\{\alpha\}^T = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad . \quad . \quad . \quad \alpha_{2m}]$$

Same can be done for 3-D (u,v,w)

## 2.1- Convergence Criteria

The numerical solution must converge or tend to the exact solution of the problem.

**Criteria**

Complete elements

a) Displacement models must include the rigid body displacements of the element (strain energy=0)
Beam element: w=constant (translation)
w=bx (rotation)

b) Displacement models must include the constant strain state
Beam element: $w=a+bx+cx^2$
(strain=curvature=$d^2w/dx^2=2c$=constant strain)
big element getting smaller and smaller till strains in each element approach constant values

Compatible (Conforming elements)

c) Certain displacement continuity (HISTORICALLY controversial)
Displacement continuity is sufficient conditions for monotonic convergence of the total potential energy
General complete elements have been successfully used.
Disadvantage of nonconforming element is that we no longer know in advance that the stiffness will be an upper bound (less stiff or more flexible)
In general continuity for the displacement and its derivatives to the order (m-1) where m is the highest derivatives in the potential energy functional J ($\pi$) is required for convergence.
For $C^1$continuity instead of $w_x$ and $w_y$ and $w_n$ only $w_n$ is sufficient.

D) spatial isotropy when dealing with 2D or 3D
Taking counterpart terms in Pascal triangle (Khayyam) For example $x^2 y$ and $y^2 x$ must be included
It helps fast convergence, without it convergence would / or might occur but slowly.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \text{Cons}\tan t\ \text{term} \\
 & & & & x & & x & & & & \text{Linear terms} \\
 & & & x & & xy & & y^2 & & & \text{quadratic terms} \\
 & & x^3 & & x^2y & & xy^2 & & y^3 & & \text{Cubic terms} \\
 & x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4 & \text{quartic terms} \\
x^5 & & x^4y & & x^3y^2 & & x^2y^3 & & xy^4 & y^5 & qu\,\text{int}\,ic\ \text{terms}
\end{array}
$$

Cubic displacement with 8 terms:

1) All constant + linear+quadratic + $x^3$ and $y^3$
2) All constant + linear+quadratic + $x^2y$ and $xy^2$

For the finite element method, the displacement formulation provides an upper bound to the true stiffness of the structure or the stiffness coefficients for a given displacement model have magnitudes higher than those for the exact solution (deforms less than the actual structure)
As F.E. division is made finer the approximate displacement solution will converge to the exact solution from below, we obtain lower bound to the solution
Total potential energy approaches from top to the actual TPE (also upper bound)

For selection of polynomial total number of generalized coordinates for an element must be equal to or greater than the number of joint or external degrees of freedom of the element (usually same number of generalized coordinates as the dof).
It is possible to utilize an excess of GC to improve the element stiffness matrix (less stiff or more flexible)
These excess coordinates associated with internal nodes and improve the approximation of equilibrium within the element. However, they do not improve interelement equilibrium.
More than a few extra coordinates are rarely justified.
The more additional DOF, more flexible becomes the element stiffness
But trade off, increasing complex formulation for individual element

## 2.2- Nodal Degrees of Freedom

Definition:
Nodal disp.s, rotations and /or strains necessary to specify completely the deformation of the finite element.
Min no. of DOF for a given problem is determined by the completeness requirements for convergence, the requirement of geometric isotropy, and necessity of an adequate representation of the terms in the potential energy function.

Additional DOF beyond minimum number may be included for any element by adding secondary external nodes or by specifying as DOF higher order derivatives of displacement at the primary nodes.
Element with additional DOF are called higher order elements.

Relation of DOF and Generalized coordinates

$$\{u\} = [\phi]\{A\}$$

$$\{\delta\}_{N\times 1} = \begin{Bmatrix} u(node1) \\ u_x(node1) \\ u(node2) \\ u_x(node2) \end{Bmatrix} = \begin{bmatrix} \phi(node1) \\ \phi_x(node1) \\ \phi(node2) \\ \phi_x(node2) \end{bmatrix}\{A\}$$

$$\{A\} = [T]^{-1}\{\delta\} \qquad [T] = transformation\ matrix$$

$$\{u\} = [\phi][T]^{-1}\{\delta\} = [N]\{\delta\}$$

"Express the displacement at any point within the element in terms of the element DOF vector {δ}"
N is total no. of DOF per element

Example 1.

Beam: transverse displacement w

$u(x,z) = -zw_x$

$\varepsilon(x,z) = u_x = -zw_{xx}$

need disp. Model for w

Strain is proportional to the second derivation of w

Minimum order of w is quadratic

$w = a + bx + cx^2$

$a = rigid \quad body\ translation$

$b = rigid \quad body\ rotation$

$c = cons\tan t\ bending\ strain(cons\tan t\ curvature)$

$\text{int}\,er\ element\ continuity\ is\ also\ satisfied$

completeness (no need for geometric isotropy)

DOF: $w_1$ and $w_2$ : we need one more DOF

Introduce two more DOF $w_{x1}$ ($\theta_1$ ) and $w_{x2}$ ($\theta_2$ )

Modify w

$w = a + bx + cx^2 + dx^3$

$$w = \{\phi\}^T\{A\} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

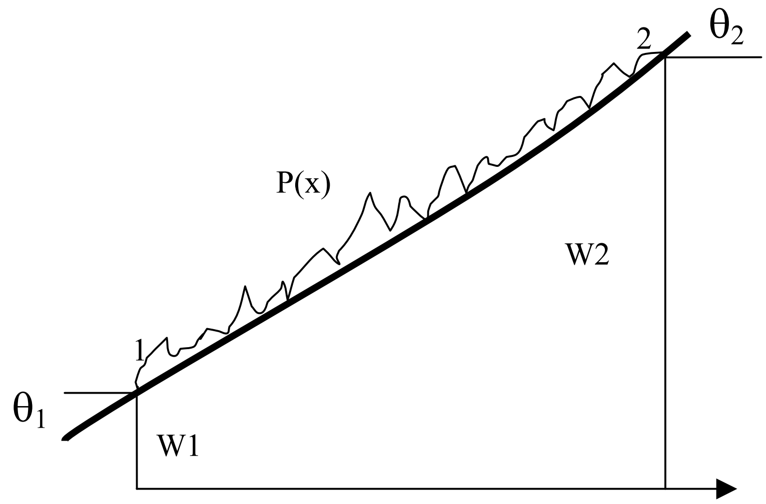
$$\theta = w_x = b + 2cx + 3dx^2 = \begin{bmatrix} 0 & 1 & 2x & 3x^2 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

$$\{\delta\} = \begin{Bmatrix} w(x=0) \\ \theta(x=0) \\ w(x=l) \\ \theta(x=l) \end{Bmatrix} = \begin{Bmatrix} w1 \\ \theta 1 \\ w2 \\ \theta 2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = [T]\{A\}$$

## 3- Beam Bending Finite Element

### 3.1- Derivation of Stiffness Matrix

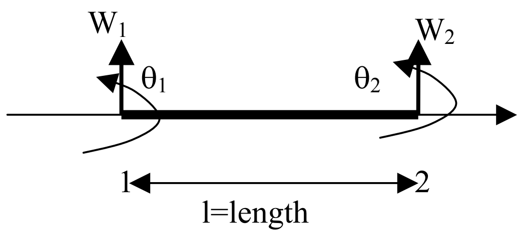
Consider an element of length l as shown,
Assume uniform EI and designate ends 1 and 2 as nodes.




Assume the displacement $w_1$ and $w_2$ and $\theta_1$ and $\theta_2$ as the generalized displacements i.e. 4 DOF.
To follow the displacement approach, assume an approximate displacement distribution within the element by a cubic polynomial in x:

$$w = a + bx + cx^2 + dx^3$$

The cubic has four constants and we have four degrees of freedom, i.e. four constants can be associated with four generalized coordinates. We know that any function can be approximately represented by a truncated Taylor's series.




Now find a, b, c and d in w in terms of $w_1$, $w_2$ , $\theta_1$ and $\theta_2$.

$$w(0) = a = w_1$$
$$\theta(0) = \frac{dw(0)}{dx} = b = \theta_1$$
$$w(l) = a + bl + cl^2 + dl^3 = w_2$$
$$\theta(l) = \frac{dw(l)}{dx} = b + 2cl + 3dl^2 = \theta_2$$

Define polynomial coefficient vector (also called generalized parameter) as:

$$\{A\} = [a \quad b \quad c \quad d]$$

and a generalized coordinate vector (or displacement vector for the beam problem)as:

$$\{\delta^e\}^T = [w_1 \quad \theta_1 \quad w_2 \quad \theta_2]$$

$$\{\delta^e\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

$$\{\delta^e\} = [T]\{A\}$$

[T]=Transformation matrix, is evaluated by simply substituting the co-ordinate values into w(x) or θ(x) that correspond to $w_1$, $w_2$ , $\theta_1$ and $\theta_2$.
We can numerically invert [T] to get $\{A\}=[T]^{-1}\{\delta^e\}$
This can be done because we know the numerical value of the coordinates.
Next step is to evaluate potential energy for w(x).
Note : in finite element method we are not interested in satisfying the differential equation exactly, only interested in energy or work done)

Rewrite displacement function as:

$$w(x) = a + bx + cx^2 + dx^3 = \sum_{i=1}^{4} a_i x^{m_i} \, where \quad m_i = i - 1$$

$$m_i \quad in \quad vector \quad form \quad \{M\}^T = [0 \quad 1 \quad 2 \quad 3]$$

$$\frac{dw}{dx} = \sum_{i=1}^{4} a_i m_i x^{m_i - 1}$$

$$\frac{d^2 w}{dx^2} = \sum_{i=1}^{4} a_i m_i (m_i - 1) x^{m_i - 2}$$

Now the strain energy within the element is given by:

$$U_e = \frac{EI}{2}\int_0^l (\frac{d^2w}{dx^2})^2 dx = \frac{EI}{2}\int_0^l (\sum_{i=1}^{4} a_i m_i (m_i - 1)x^{m_i - 2})(\sum_{j=1}^{4} a_j m_j (m_j - 1)x^{m_j - 2})dx$$

$$U_e = \frac{EI}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} a_i a_j m_i m_j (m_i - 1)(m_j - 1)\int_0^l x^{m_i + m_j - 4})dx$$

$$U_e = \frac{EI}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} a_i a_j m_i m_j (m_i - 1)(m_j - 1)\left[\frac{x^{m_i + m_j - 3}}{m_i + m_j - 3}\right]_0^l$$

$$U_e = \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \bar{k}_{ij}\, a_i a_j = \frac{1}{2}\{A\}^T \left[\bar{k}\right]\{A\}$$

$$where \quad \bar{k}_{ij} = m_i m_j (m_i - 1)(m_j - 1)\left[\frac{EI \quad l^{m_i + m_j - 3}}{m_i + m_j - 3}\right]$$

Note: Symmetry of $\bar{k}_{ij}$, interchanging i with j does not alter right hand side. $\bar{k}_{ij}$ are the components of the stiffness matrix $\left[\bar{k}\right]$ with respect to polynomial coefficient $a_i$ .

To get the final stiffness matrix with respect to the generalized displacement $\{\delta^e\}$,

$$U_e = \frac{1}{2}\{A\}^T \left[\bar{k}\right]\{A\} = \frac{1}{2}\left([T]^{-1}\{\delta^e\}\right)^T \left[\bar{k}\right]\left([T]^{-1}\{\delta^e\}\right)$$

$$U_e = \frac{1}{2}\{\delta^e\}^T \left(\left([T]^{-1}\right)^T \left[\bar{k}\right][T]^{-1}\right)\{\delta^e\}$$

$$U_e = \frac{1}{2}\{\delta^e\}^T \left[k^e\right]\{\delta^e\}$$

$$\left[k^e\right] = \left([T]^{-1}\right)^T \left[\bar{k}\right][T]^{-1}$$

$\left[k^e\right]$ is the required stiffness matrix with respect to the generalized displacement $\{\delta^e\}$.

Note: We don't know yet $\lfloor k^e \rfloor$ is stiffness matrix but from the form of the $U_e$ we can get a sense.

### 3.2- Consistent Load Vector

Loads are lumped at the discrete nodes of the element assemblage using virtual work principle.

Potential energy of the load:

$$W = \int_0^l p(x)w(x)dx$$

Let $p(x)=p_0$ a uniform load

$$W = \int_0^l p(x)w(x)dx = p_0 \int_0^l \sum_{i=1}^4 a_i x^{m_i} dx = p_0 \sum_{i=1}^4 a_i \frac{l^{m_i+1}}{m_i+1} = \{A\}^T \{\bar{p}\}$$

$$\bar{p}_i = p_0 \frac{l^{m_i+1}}{m_i+1}$$

Transformation to $\{\delta^e\}$.

$$W = \{A\}^T \left\{\bar{p}\right\} = \left([T]^{-1}\{\delta^e\}\right)^T \left\{\bar{p}\right\} = \{\delta^e\}^T \left(\left([T]^{-1}\right)^T \left\{\bar{p}\right\}\right)$$

$$W = \{\delta^e\}^T \{p^e\}$$

$\{p^e\}$ is the consistent load vector for the generalized displacement $\{\delta^e\}$.

Computing Steps:

1) program the transformation matrix [T] and invert numerically
2) Program the untransformed stiffness matrix $\left[\bar{k}\right]$ and the load vector $\left\{\bar{p}\right\}$
3) Transform (pre and post multiply with $\left([T]^{-1}\right)^T$ and $[T]^{-1}$ respectively) to $[k^e]$ and $[p^e]$

Comments: obviously the size of [T] depend s on the degree of polynomial. Therefore for higher degree polynomials, the larger [T] inverse process become very time consuming on computer.
There is another procedure which yields $[k^e]$ in the final form and hence saves a number of matrix multiplications.

### 3.3- Alternative Approach to Derive the Stiffness Matrix

$$w(0) = a = w_1$$

$$\theta(0) = \frac{dw(0)}{dx} = b = \theta_1$$

$$w(l) = a + bl + cl^2 + dl^3 = w_2$$

$$\theta(l) = \frac{dw(l)}{dx} = b + 2cl + 3dl^2 = \theta_2$$

four equations for four unknowns a, b, c and d, solving:

$$a = w_1$$

$$b = \theta_1$$

$$c = 3\frac{w_2 - w_1}{l^2} - \frac{2\theta_1 + \theta_2}{l}$$

$$d = \frac{\theta_1 + \theta_2}{l^2} + 2\frac{w_1 - w_2}{l^3}$$

substituting back into the cubic equation w yields:

$$w(x) = w_1 + \theta_1\, x + \left[\frac{w_2 - w_1}{l^2} - \frac{2\theta_1 + \theta_2}{l}\right] x^2 + \left[\frac{\theta_1 + \theta_2}{l^2} + 2\frac{w_1 - w_2}{l^3}\right] x^3$$

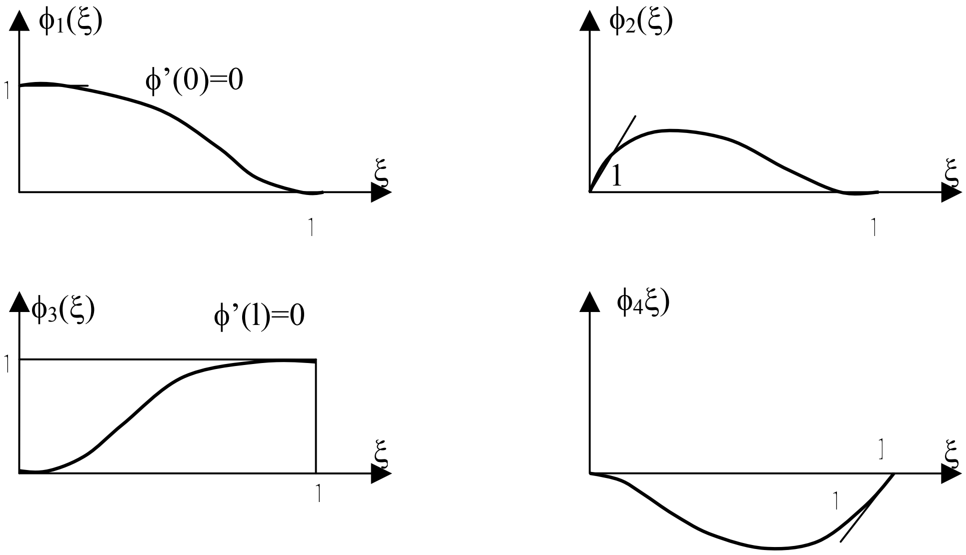
using ξ=x/l (nondimensional coordinate system), we can express the displacement function for the beam element as:

$$w(\xi) = (1 - 3\xi^2 + 2\xi^3)\, w_1 + (\xi - 2\xi^2 + \xi^3)\, l\theta_1 + (3\xi^2 - 2\xi^3)\, w_2 + (\xi^3 - \xi^2)\, l\theta_2$$

$$w(\xi) = \phi_1(\xi)\, w_1 + \phi_2(\xi)\, l\theta_1 + \phi_3(\xi)\, w_2 + \phi_4(\xi)\, l\theta_2$$

where $\phi_i$ 's are often called interpolation functions or shape functions.

Graphical Representation:




There are the possible configurations one wants to combine into the true shapes.

This method of solving for constants in the polynomial in terms of nodal degrees of freedom becomes difficult for higher degree polynomial approximations.

$$\pi = U - W$$

$$\frac{d^2w}{dx^2} = \frac{d^2w}{d\xi^2}\frac{1}{l^2} = \frac{1}{l^2}\left\{\frac{d^2\phi_1}{d\xi^2}w_1 + \frac{d^2\phi_2}{d\xi^2}l\theta_1 + \frac{d^2\phi_3}{d\xi^2}w_2 + \frac{d^2\phi_4}{d\xi^2}l\theta_2\right\}$$

$$U_e = \frac{EI}{2}\int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{EI}{2l^3}\left\{\begin{array}{l} 12w_1^2 + 4l^2\theta_1^2 + 12w_2^2 + 4l^2\theta_2^2 + 12lw_1\theta_1 \\ -24w_1w_2 + 12lw_1\theta_1 + 12lw_2\theta_1 + 4l^2\theta_1\theta_2 - 12lw_2\theta_2 \end{array}\right\}$$

Now the same procedure can be used as before.

Note: Stiffness coefficient can also be computed in a more convenient way(some times for lower order polynomials):

$$U_e = \frac{EI}{2}\int_0^l (\frac{d^2 w}{dx^2})^2 dx = \frac{1}{2}\{\delta^e\}^T [K^e]\{\delta^e\}$$

$$\{\delta^e\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \quad \{\phi\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \quad w = \sum w_i \phi_i$$

$$U_e = \frac{EI}{2}\int_0^l (\frac{d^2 w}{dx^2})^2 dx = \frac{EI}{2}\int_0^l \sum w_i \frac{d^2\phi_i}{dx^2} \sum w_j \frac{d^2\phi_j}{dx^2} dx = \sum\sum \frac{1}{2}\int_0^l EI w_i w_j \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2} dx$$

$$k_{ij} = \int_0^l EI \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2} dx = \frac{EI}{l^4}\int_0^l \frac{d^2\phi_i}{d\xi^2}\frac{d^2\phi_j}{d\xi^2} l d\xi$$

$$\frac{d^2}{dx^2} = \frac{1}{l^2}\frac{d^2}{d\xi^2} \qquad dx = l d\xi$$

$$k_{ij} = \frac{EI}{l^3}\int_0^l \phi_i'' \phi_j'' d\xi$$

*where the shape functions are given by* :

$$\phi_1(\xi) = (1 - 3\xi^2 + 2\xi^3)$$

$$\phi_2(\xi) = l(\xi - 2\xi^2 + \xi^3)$$

$$\phi_3(\xi) = (3\xi^2 - 2\xi^3)$$

$$\phi_4(\xi) = l(\xi^3 - \xi^2)$$

## 3.4- Potential Energy Theorem for Finite Element Discretization

For an element :

$$U^e = \frac{1}{2}\{\delta^e\}^T[K^e]\{\delta^e\}$$

$$W^e = \{p^e\}^T\{\delta^e\}$$

$$\pi_T = \sum_{n=1}^{nele} \left\{\frac{1}{2}\{\delta^e\}^T[K^e]\{\delta^e\} - \{p^e\}^T\{\delta^e\}\right\}$$

We know the stiffness matrix and load vector for each element.
Then when the elements are put together, we sum these energies to get totals of energy.
We define the nodal displacements vector for the entire assemblage, $\{X\}_{N\times 1}$ where N is structure total degrees of freedom and each element has n dof.

$$\pi_T = U - W$$

$$U = \frac{1}{2}\{X\}^T[K]\{X\}$$

$$W = \{p\}^T\{X\}$$

$$\pi = \pi_T = \left\{\frac{1}{2}\{X\}^T[K]\{X\} - \{p\}^T\{X\}\right\}$$

Where [k], {X} and {p} are stiffness matrix, displacement vector, and load vector for the global or total problem.
$\delta\pi=0$ (first variation set equal to zero)or extremum.

$$\delta\pi = \frac{1}{2}\left\{\{\delta X\}^T[K]\{X\} + \{X\}^T[K]\{\delta X\}\right\} - \{p\}^T\{\delta X\} = 0$$

$\{X\}^T[K]\{\delta X\}$ *is scalar then transpose it* :

$\{X\}^T[K]\{\delta X\} = \{\delta X\}^T[K]\{X\}$

$$\delta\pi = \left\{\{X\}^T[K] - \{p\}^T\right\}\{\delta X\} = 0$$

$\{\delta X\}$ *is arbitrary*

$\{X\}^T[K] - \{p\}^T = 0$

*or transposing* :

$$[K]\{X\} - \{p\} = 0$$

Take second variation to show that $\pi$, infact is minimum:

$$\delta^2\pi = \delta\left[\left\{\{X\}^T[K]\{\delta X\}\right\} - \{p\}^T\{\delta X\}\right]$$

$$\delta^2\pi = \{\delta X\}^T[K]\{\delta X\}$$

$\delta\left\{\{p\}^T\{\delta X\}\right\} = 0 \quad \{\delta X\}$ *is already varied and cannot vary again.*

But $\{\delta X\}$ is arbitrary variation of $\{X\}$ and if [k] is a positive definite matrix (which it is) then $\{\delta X\}^T[K]\{\delta X\}>0$, therefore $\pi$ is a minimum.

$$[K]\{X\} = \{p\}$$

$$\{p\}^T = \{X\}^T[K]$$

$$\pi = \frac{1}{2}\{X\}^T[K]\{X\} - \{X\}^T[K]\{X\}$$

$$\pi = -\frac{1}{2}\{X\}^T[K]\{X\}$$

$$\pi = -U_T \quad |\pi| = |U_T|$$

This is true when π is minimum in a discrete problem.

4- Stiffness Matrix and Load Vector Assembling

Nodal compatibility is used as the basis for the assemblage of the individual elements. Element adjacent to a particular node must have the same generalized displacement at the node such as displacement, translation and may be strain, curvature, and other derivatives of translations with respect to the spatial coordinates. The imposition of nodal compatibility represents the construction of the assemblage by rigidly joining the pieces of elements together at certain preselected joining points. Since the displacements are matched at the nodes, the loads and stiffness are added at these locations.

We applied the variational principle to the element. We will now apply it to the assemblage to obtain the assembly rules.

Assume element has n dof and structure has N dof. And we know the element stiffness matrix and load vector (including all the loadings on the body) for each elements.

$$\{p^e\} = \begin{Bmatrix} p_1^e \\ p_2^e \\ . \\ . \\ . \\ p_n^e \end{Bmatrix}_{n\times 1} \begin{matrix} 2 \\ k \\ \\ \\ \\ l \end{matrix} \qquad \{P_G{}^e\} = \begin{Bmatrix} 0 \\ p_1^e \\ \\ p_2^e \\ \\ p_n^e \\ \\ \end{Bmatrix}_{N\times 1} \begin{matrix} 1 \\ 2 \\ \\ k \\ \\ l \\ \\ N \end{matrix} \quad \textit{Numbers are global DOF}$$

$$\{p^e\}^T\{\delta^e\} = \{P_G{}^e\}^T\{X\}$$

With this concept the same can be done for Stiffness matrix

$$[k^e]_{n\times n} \Rightarrow \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{N\times N} \qquad \textit{such that} \quad \{X\}^T[K_G{}^e]\{X\} = \{\delta^e\}^T[k^e]\{\delta\}$$

$[K_G{}^e]$ *is the assembling of one element into global stiffness matrix*

$$\pi_T = \sum\nolimits_{n=1}^{nele} \left\{\frac{1}{2}\{\delta^e\}^T[k^e]\{\delta^e\} - \{p^e\}^T\{\delta^e\}\right\}$$

$$= \sum \frac{1}{2}\{X\}^T[K_G{}^e]\{X\} - \{P_G{}^e\}^T\{X\}$$

$$= \frac{1}{2}\{X\}^T\left(\sum_{n=1}^{nele}[K_G{}^e]\right)\{X\} - \left(\sum \{P_G{}^e\}^T\right)\{X\}$$

$$= \frac{1}{2}\{X\}^T[K_G]\{X\} - \{P_G\}^T\{X\}$$

$$\delta\pi = \{\delta X\}^T\left([K_G]\{X\} - \{P_G\}\right) = 0$$

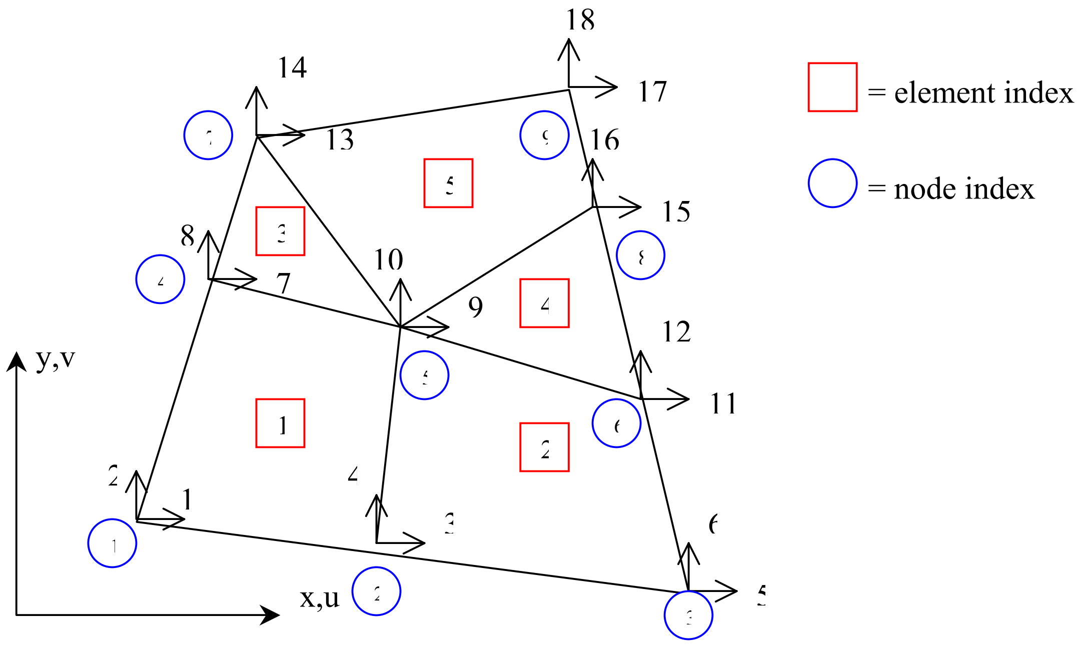
$$[K_G]\{X\} = \{P_G\}$$

Thus $\sum_{n=1}^{nele} [K_G{}^e]$ and $\sum \{P_G{}^e\}^T$ have the effects of each individual element stiffness matrix and consistent load vector in the global matrices. These are the equilibrium equations for the assemblage.

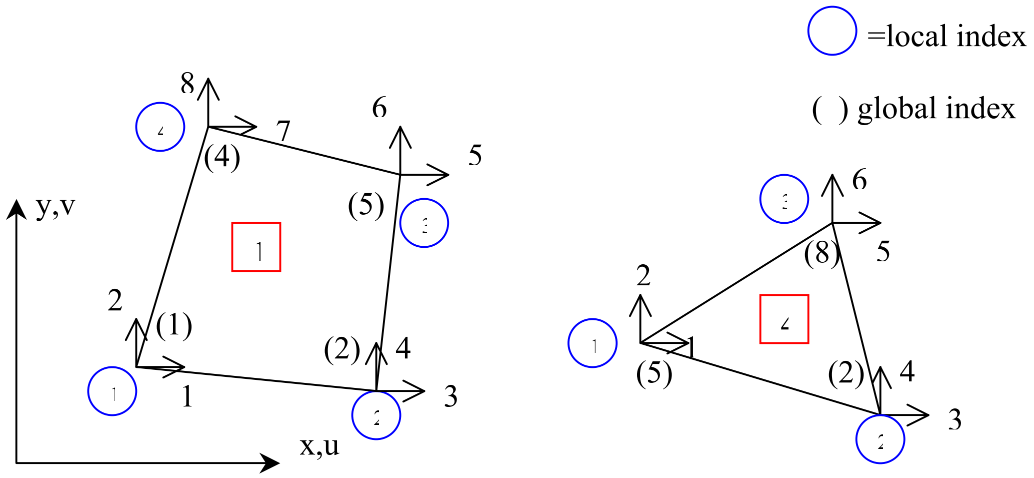
The equilibrium equation is results of joining of element together in which:
a)sum of the generalized forces at the nodes and equating to external loads (equilibrium)
b) equate of degrees of freedom at the nodes (compatibility)
It must be noted that equilibrium and compatibility are satisfied within the element and by doing it we satisfy them at the nodes. We must be worry only about the continuity between the elements.

Example: Direct Stiffness Method for the Assemblage

Two dimensional assemblage to illustrate direct stiffness method




Global indices for assemblage




Local indices for elements number 1 and 4

Typical element stiff nesses and loads with only selected nonzero entries shown

$$
\begin{array}{cc}
\begin{array}{cc} \textit{global} & \\ \textit{local} & \end{array} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 9 & 10 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 9 & 5 \\ 10 & 6 \\ 7 & 7 \\ 8 & 8 \end{array} &
\left[\begin{array}{cccccccc}
k_{11} & . & . & . & . & . & . & k_{18} \\
. & . & . & . & . & . & . & . \\
. & . & . & k_{35} & k_{36} & . & . & . \\
. & . & . & k_{45} & k_{46} & . & . & . \\
k_{51} & k_{52} & . & k_{55} & k_{56} & . & . & . \\
k_{61} & k_{62} & . & k_{65} & k_{66} & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & k_{88}
\end{array}\right]_{8\times 8}
\end{array}
\qquad
\left\{\begin{array}{c} . \\ . \\ . \\ . \\ q_5 \\ q_6 \\ . \\ . \end{array}\right\}_{8\times 1}
$$

*Stiffness matrix for element no.* 1 — *Load vector*

*Stiffness matrix for element no.* 4 — *Load vector*

$$
\begin{array}{cc}
\begin{array}{cc} \textit{global} & \\ \textit{local} & \end{array} & \begin{array}{cccccc} 9 & 10 & 11 & 12 & 15 & 16 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{cc} 9 & 1 \\ 10 & 2 \\ 11 & 3 \\ 12 & 4 \\ 15 & 5 \\ 16 & 6 \end{array} &
\left[\begin{array}{cccccc}
a_{11} & a_{12} & . & . & a_{15} & a_{16} \\
a_{21} & a_{22} & . & . & a_{25} & a_{26} \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
a_{11} & a_{11} & . & . & . & . \\
a_{11} & a_{11} & . & . & . & .
\end{array}\right]_{6\times 6}
\end{array}
\qquad
\left\{\begin{array}{c} b_1 \\ b_2 \\ . \\ . \\ . \\ . \end{array}\right\}_{6\times 1}
$$

Assemblage stiffness matrix and load vector showing the locations where entries are added from components of elements number 4 and 2

*Total stiffness matrix* *Total load*

$$\begin{array}{cc}
\textit{global} & \begin{array}{cccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \end{array} \\
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18 \end{array} &
\left[\begin{array}{cccccccccccccccccc}
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & k_{35} & k_{36} & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & k_{45} & k_{46} & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
k_{51} & k_{52} & . & . & . & . & . & . & . & k_{55}+a_{11} & k_{56}+a_{12} & . & . & . & a_{15} & a_{16} & . & . \\
k_{61} & k_{62} & . & . & . & . & . & . & . & k_{65}+a_{21} & k_{66}+a_{22} & . & . & . & a_{25} & a_{26} & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & a_{51} & a_{52} & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & a_{61} & a_{62} & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}\right]_{18\times 18}
\end{array}
\quad
\left\{\begin{array}{c} .\\.\\.\\.\\.\\.\\.\\.\\ q_5+b_1 \\ q_6+b_2 \\.\\.\\.\\.\\.\\.\\.\\. \end{array}\right\}_{18\times 1}$$

If we have external loads at nodes, we again add them to the load vector. We usually find the components on the global directions of the dof.
Subroutine for setup

*Element no. one*

$$\begin{aligned} LJ(I) &= \begin{bmatrix} 1 & 2 & 3 & 4 & 9 & 10 & 7 & 8 \end{bmatrix} \\ I &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix} \end{aligned}$$

*Element no. four*

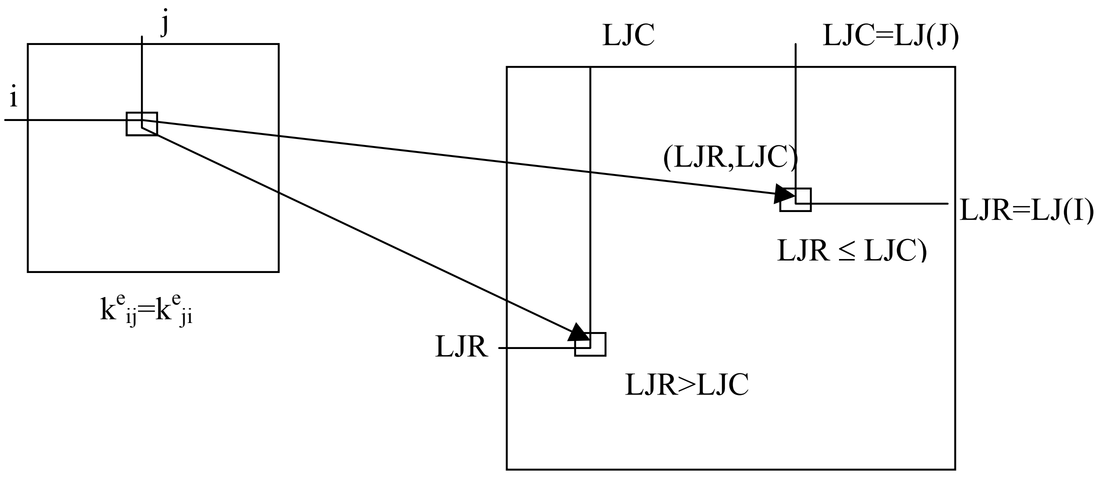
$$\begin{aligned} LJ(I) &= \begin{bmatrix} 9 & 10 & 11 & 12 & 15 & 16 \end{bmatrix} \\ I &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \end{aligned}$$

LJ is the address for the local stiffness matrix. Global stiffness matrix is 18×18 while the local stiffness matrix is 8×8 for the 4-node elements and 6×6 for 3-node element.

$$ele.\,no.\,1:$$
$$k^1(4,5) \rightarrow k(4,9)$$

$$ele.\,no.\,4:$$
$$k^4(3,4) \rightarrow k(11,12)$$

$$k^e(I,J) \rightarrow \qquad \begin{array}{l} LJR = LJ(I) = global\ row \\ LJC = LJ(J) = global\ column \end{array}$$




$k^e_{ij}=k^e_{ji}$ $\qquad$ $k_{ij}=k_{ji}$

## 5- Boundary conditions

So far we have not considered boundary conditions at all. A problem in solid mechanics is not completely specified unless boundary conditions are prescribed. Without imposition of boundary conditions, the element and total stiffness matrices are singular. It means that a loaded body or structure is free to experience unlimited rigid body motion unless some supports or kinematic constraints are imposed that will ensure the equilibrium of the loads. These constraints are the boundary conditions.

Types of Boundary Conditions
From the variational-method point of view, there are two basic types of boundary conditions

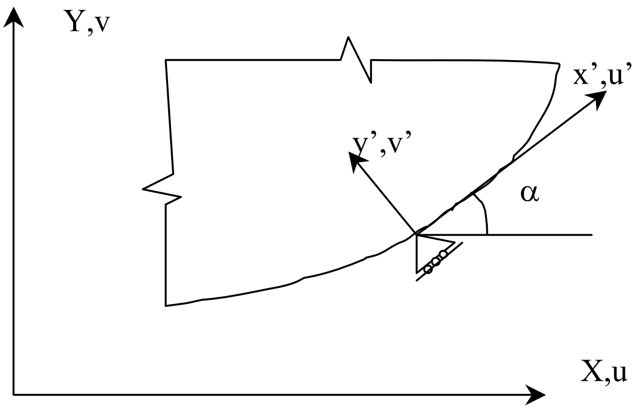
1. Geometric (essential)
2. Natural (force)

One of the principal advantages of the F.E. method is that we need specify only the geometric bc's; the natural bc's are implicitly satisfied in the solution procedures.
Traction boundary conditions are incorporated into the load vector.

In the displacement method of F.E. analysis, we can further categorize geometric boundary conditions as being

1. Homogeneous
2. Non-homogeneous((normal and skewed)

Homogeneous conditions occur at locations where completely constrained against movement (displacement=0).
Conversely, finite non zero values may be specified at some points; these are non-homogeneous bc's (e.g. support settlement). The distinction between normal and skewed conditions arises at locations on the boundary at which only some components of the displacement are restrained. If the restrained components are parallel to the global coordinates, the conditions are normal, otherwise they are skewed.



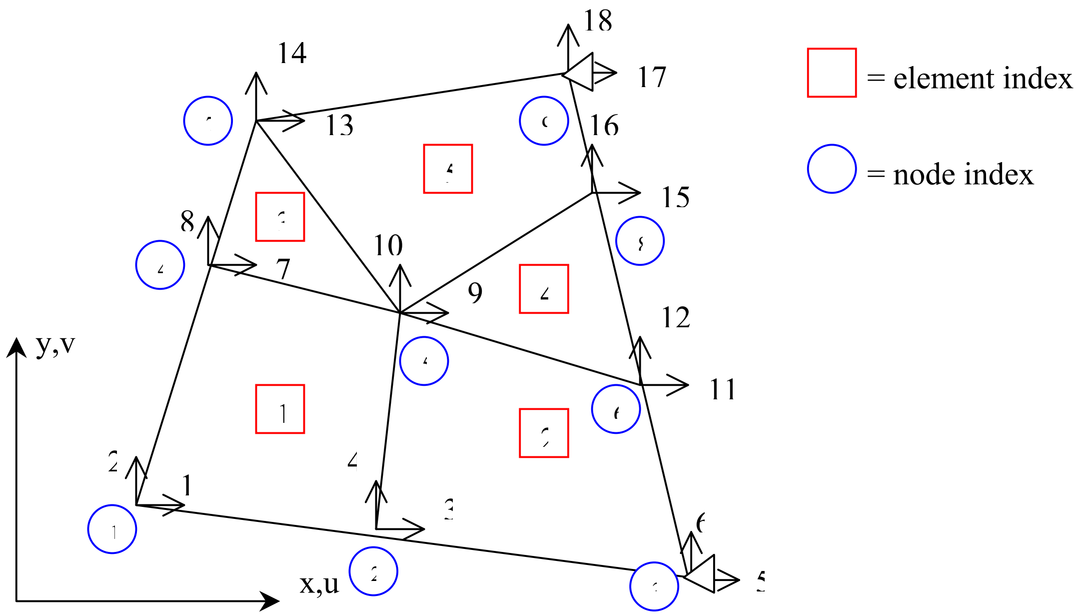

Skewed kinematic constraint

## 5.1- Essential Homogeneous Boundary Condition

### 5.1.1- First Approach

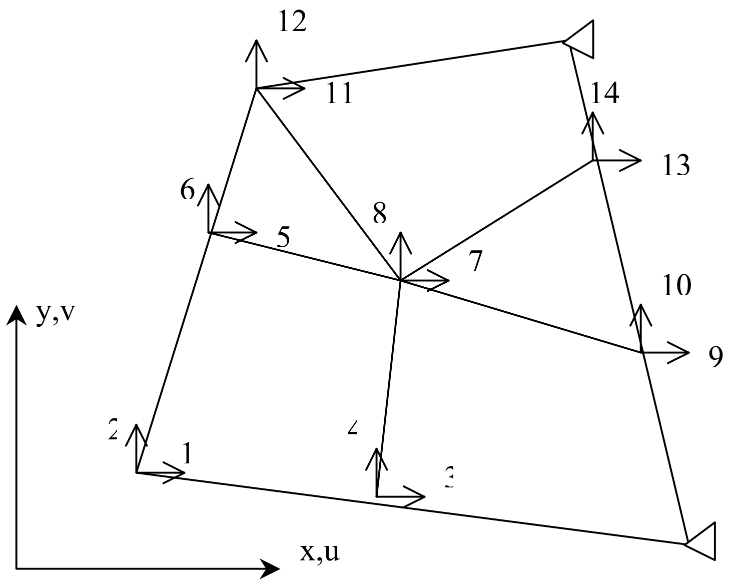
We delete all the rows and columns of the force deformation equations with the essential bc's.

$$
\begin{array}{c|cccccccccccccccccc|c}
\textit{Total stiffness matrix} & & & & & \Leftrightarrow & \Leftrightarrow & & & & & & & & & & & \Leftrightarrow & \Leftrightarrow & \textit{Total load} \\
\textit{global} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & \\
\hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{35} & k_{36} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{45} & k_{46} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Updownarrow 5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Updownarrow 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
9 & k_{51} & k_{52} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{55}+a_{11} & k_{56}+a_{12} & \cdot & \cdot & \cdot & a_{15} & a_{16} & \cdot & \cdot & q_5+b_1 \\
10 & k_{61} & k_{62} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{65}+a_{21} & k_{66}+a_{22} & \cdot & \cdot & \cdot & a_{25} & a_{26} & \cdot & \cdot & q_6+b_2 \\
11 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
13 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
14 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
15 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{51} & a_{52} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
16 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{61} & a_{62} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Updownarrow 17 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Updownarrow 18 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
 & \multicolumn{18}{r|}{18\times 18} & 18\times 1
\end{array}
$$




5.1.2- Second Approach

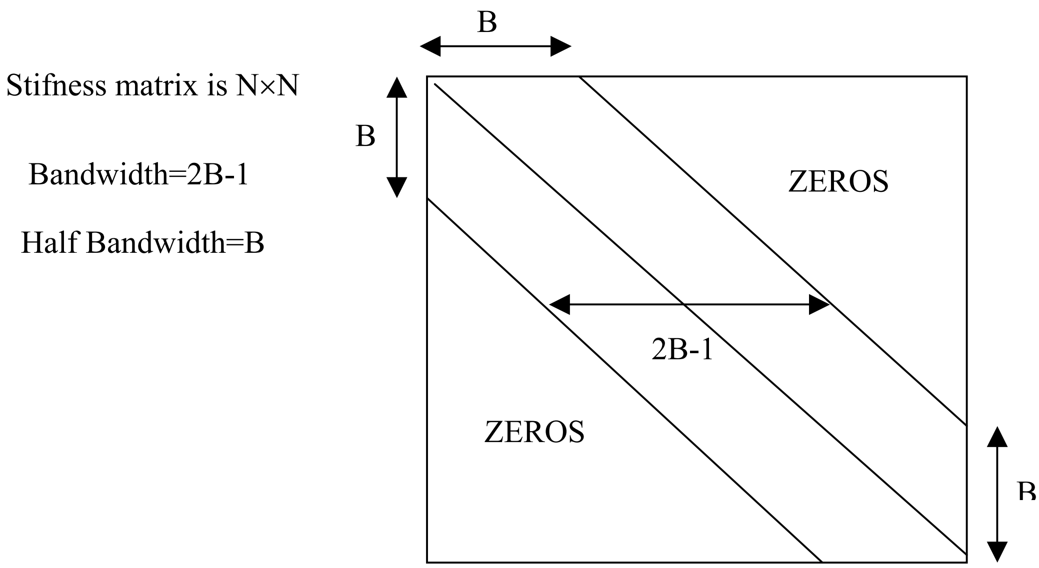
In the second approach, we do not number those essential bc's. Then where ever we have essential bc's we ignore then for the purpose of assembling. We give essential bc's 0 to show that they are not used in the assembled matrix.



## 6- Storage of the Total Stiffness Matrix

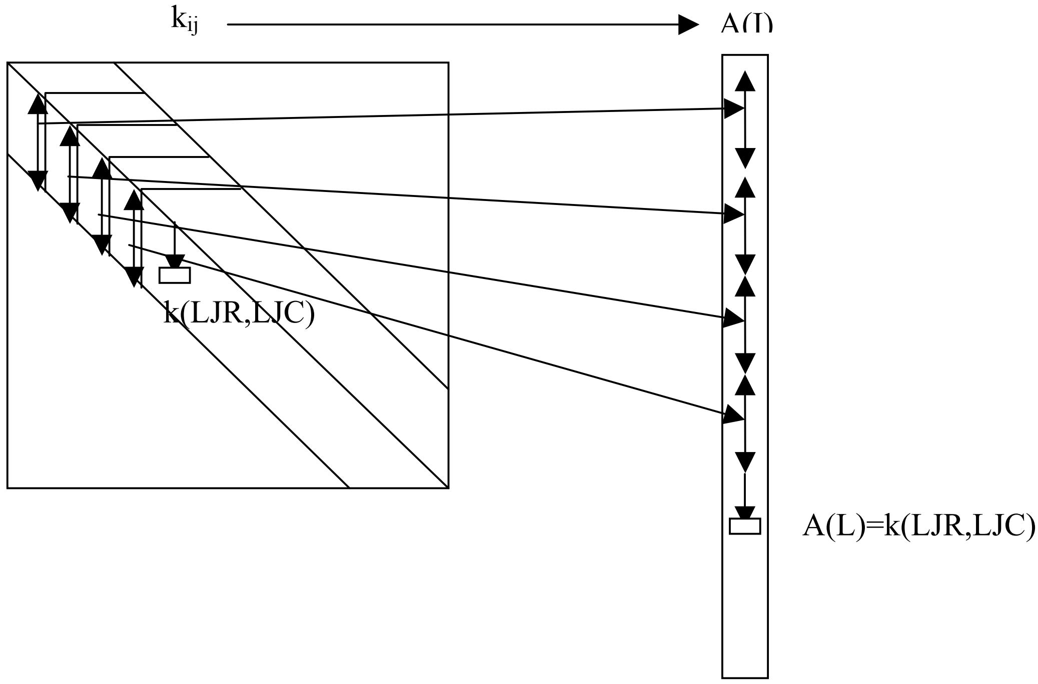
### 6.1- Bandwidth Method

Symmetric "usually thinly populated or sparse"
Bandwidth=2B-1
Half bandwidth=B




Economy of core storage is to only store N×B portion of the matrix (upper part)

B=LBAND+1
B=(D+1) f
D=maximum largest difference occurring for all elements of the assemblage
f=number of degree of freedom at each node

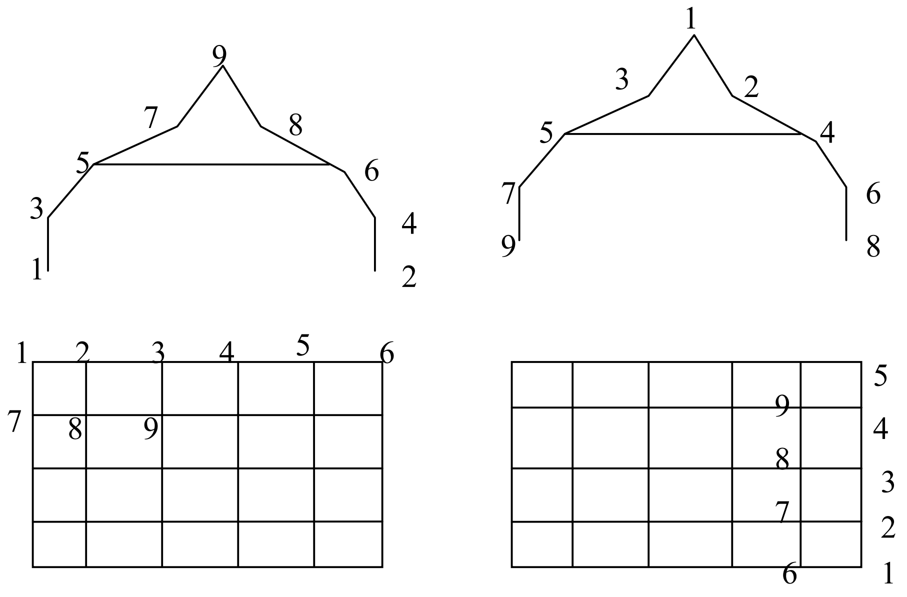



k(LJR,LJC) goes to A(L)

For LJR≥LJC $\quad A(L)=A(L)+k^e(I,J)$
$\quad L=B*(LJC-1)+(LJR-LJC)+1$

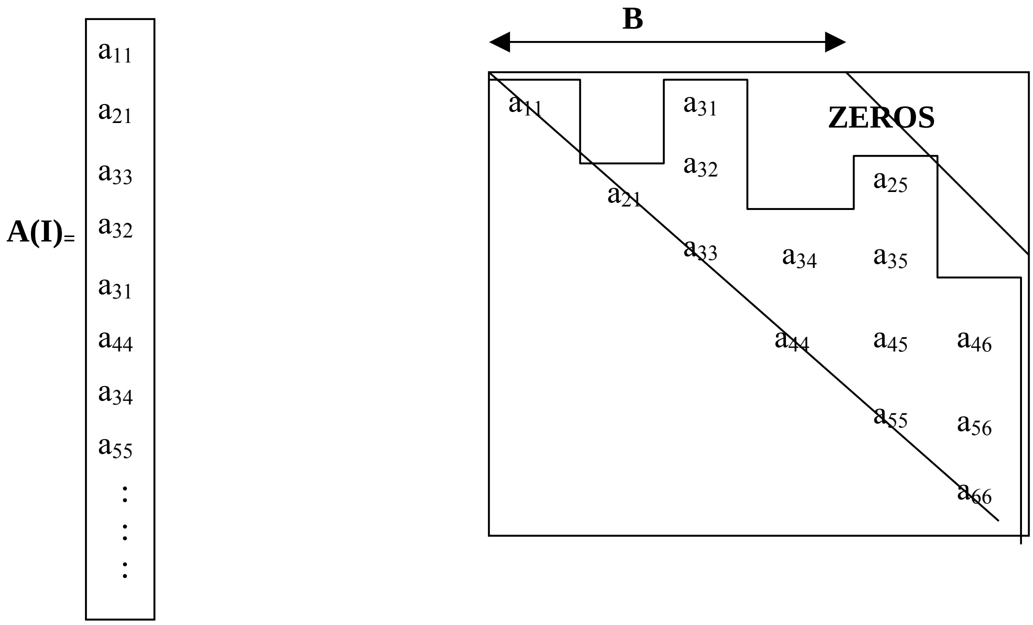
For LJR<LJC $\quad A(L)=A(L)+k^e(I,J)$
$\quad L=B*(LJR-1)+(LJC-LJR)+1$

Example: Is B different for this two numbering?



For extremely large systems of equations even this method of storage may be inadequate. Because B is a function of D which is the maximum of the largest difference in all elements of the assemblage.

## 6.2- Skyline Method



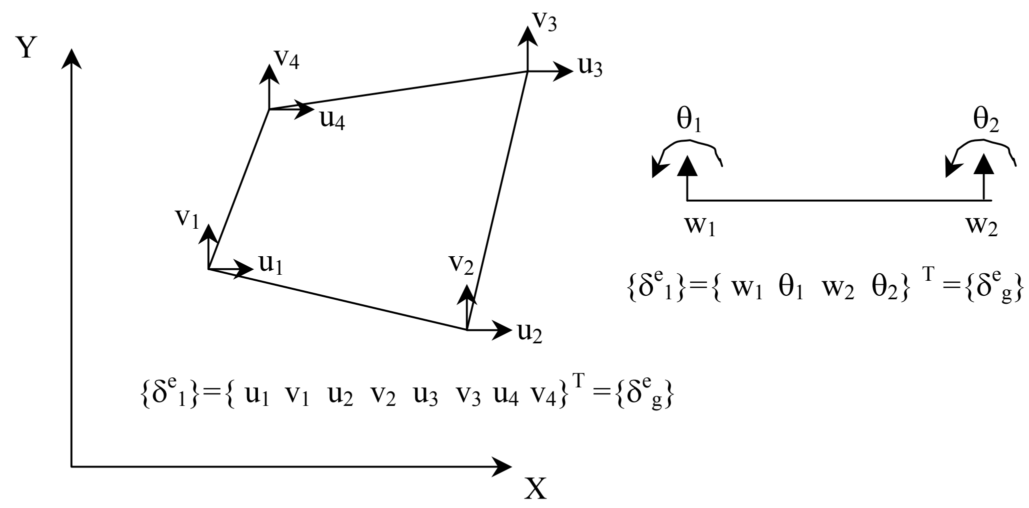

Sample problem

Write an algorithm for obtaining the address of each member by skyline method. (Use the algorithm from FEB.For).

## 7- Transformation to Global Coordinates

Local coordinate system is defined for particular element, where as global coordinate system refers to the entire assemblage.

It is usually possible to adopt local displacement directions that coincide with the global coordinates.




However for some types of problems, we may find it expedient to perform the analysis of individual elements with a local displacement system which has directions different from those in the global system. In such cases, before we can construct the equation for the assemblage, we must transform our element stiffness and load to a common frame of reference, the global coordinate system.

Say l refer to local and g to global. Relation between local and global element displacements at a **node** of an element:

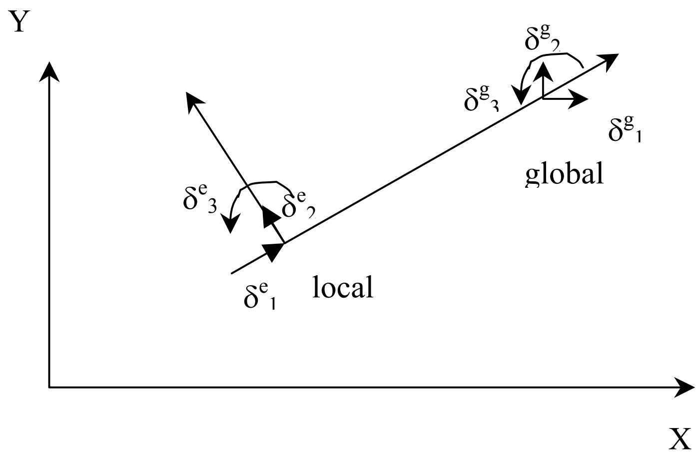
$$\{u_l\}=[t]\{u_g\}$$

we may construct the transformation for the nodal displacements:

$$\{\delta^{e}{}_{l}\} = [T]\{\delta_g\}$$

$$[T] = \begin{bmatrix} [t] & [0] & [0] & . & [0] \\ [0] & [t] & [0] & . & [0] \\ [0] & [0] & [t] & . & [0] \\ . & . & . & [t] & [0] \\ [0] & [0] & [0] & [0] & [t] \end{bmatrix}$$

Number of matrices [t] in [T] equals the number of element nodes.



$$\delta^e{}_1 = \delta^g{}_1 \cos\theta + \delta^g{}_2 \sin\theta$$

$$\delta^e{}_2 = -\delta^g{}_1 \sin\theta + \delta^g{}_2 \cos\theta$$

$$\delta^e{}_3 = \delta^g{}_3$$

$$[t] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\textit{before assembling}:$$

$$[k^e{}_g] = [T]^T [k^e{}_g][T]$$

$$\textit{note}: \quad [T]^{-1} = [T]^T \quad \textit{orthogonal matrix}$$

$$U^e = \{\delta^e{}_l\}^T [K^e{}_l]\{\delta^e{}_l\} = \{\delta^e{}_g\}^T [T]^T [K^e{}_l][T]\{\delta^e{}_g\}$$

$$W^e = \{p^e{}_l\}^T \{\delta^e{}_l\} = \{p^e{}_l\}^T [T]\{\delta^e{}_g\}$$

$$[k^e{}_g] = [T]^T [K^e{}_l][T]$$

$$\{p^e{}_g\} = \{p^e{}_l\}^T [T]$$

$$\pi_T = \sum_{n=1}^{nele} \left\{ \{\delta^e{}_g\}^T [k^e{}_g]\{\delta^e{}_g\} - \{p^e{}_g\}^T \{\delta^e{}_g\} \right\}$$

Notes: stiffness matrix and load vector are obtained using local coordinate system and then transformed to the global coordinate system.

After solution we may want calculate the local displacement or stresses in which the same transformation matrix can be used.

## 8- Modification of the Equilibrium Equations for Skewed Boundary Conditions

We need to transform the coordinates of the nodes where skewed bc's are specified into normal constraints. This is analogous to the transformation from local to global coordinates.
We can write the transformation for displacement at the i$^{th}$ node as:

$$\{r_i\} = [s_i]\{r'_i\}$$

$[s_i]$ is a simple point transformation involving the direction cosines which relate the global and skewed systems (same as [t]).
We can now write the transformation for the entire nodal displacement vector as:

$$\{r\} = [S]\{r'\}$$

$$[S] = \begin{bmatrix} [I] & [0] & [0] & . & [0] \\ [0] & [I] & [0] & . & [0] \\ [0] & [0] & [I] & . & [0] \\ . & . & . & [s_i] & [0] \\ [0] & [0] & [0] & [0] & [I] \end{bmatrix}$$

$$[I] = identity\ matrix\ of\ same\ order\ as\ [s_i]$$

Number of submatrices on diagonal of [S] is equal to the number of nodes in the assemblage. The order of [$s_i$] is equal to the number of displacement dof at each node.

The resulting transformation for the stiffness, loads are:

$$[k'] = [S]^T[K][S]$$

$$\{R'\} = [S]^T\{R\}$$

The procedure to transform skewed boundaries into normal boundaries, as outlined above, can be performed before the individual element stiffness and loads are assembled. In this case, the above equation apply for element

stiffness matrix and load vector rather than total stiffness matrix and load vector and the order of [S] is n rather than N.
Now, All the bc's are normal or they have been transformed to a skewed system in which they may be treated as normal. Tre results for displacement at the skewed bc's may want to be presented in the global direction in which can be transformed using the same transformation matrix.

**9- Prescribed Geometric Boundary Conditions**
Partitioning the global equilibrium equation results in:

$$\begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{12}]^T & [k_{22}] \end{bmatrix} \begin{Bmatrix} \{r_1\} \\ \{r_2\} \end{Bmatrix} = \begin{Bmatrix} \{R_1\} \\ \{R_2\} \end{Bmatrix}$$

where $\{r_1\}$ is the vector of unconstrained or free displacement, and $\{r_2\}$ is the vector of specified displacements.

$$[k_{11}]\{r_1\} = \{R_1\} - [k_{12}]\{r_2\}$$

Here $[k_{11}]$ is no longer singular. The reactions at the constraint displacement can be computed as:

$$\{R_2\} = [k_{12}]^T \{r_1\} + [k_{22}]\{r_2\}$$

In the case where homogeneous bc's, $\{r_2\}$ is null, the procedure is considerably simplified.

A more practical way of forming the modified equilibrium equation is to arrange the equations as below:

$$\begin{bmatrix} [k_{11}] & [0] \\ [o] & [I] \end{bmatrix} \begin{Bmatrix} \{r_1\} \\ \{r_2\} \end{Bmatrix} = \begin{Bmatrix} \{R_1\} - [k_{12}]\{r_2\} \\ \{r_2\} \end{Bmatrix}$$

The above can be done without reordering of the equations. The contribution to the subvector $\{R_1\} - [k_{12}]\{r_2\}$ are first constructed for each nonhomogeneous condition. Then the row and column of [k] corresponding to that condition are made null with the exception of the diagonal element, which is made unity. Finally, the prescribed value of the displacement is inserted in the load vector.
For a specified displacement $r_j$ occurring at the $j^{th}$ dof, the above process is summarized as:

$$R_i = R_i - k_{ij} r_j \quad for\ i=1,2,3,...,N\ \ if\ r_j \neq 0$$

$$k_{jm} = k_{mj} = 0 \quad for\ m = 1,2,3,..., N$$

$$k_{jj} = 1$$

$$R_j = r_j$$

These operations ensures that the equilibrium equations remain symmetrical.

**10- Accommodation of Elastic Supports in the Total Stiffness Matrix**

Elastic supports cab be readily accommodated by the FE method. They do not introduce a different type of bc's into the analysis. The deformation portion of such support is included as finite elements in the structure or body that discretized. The conventional geometric boundary conditions are then applied at the point where elastic supports are grounded. In practice, we do not add a new equation for these grounding points; rather the appropriate matrix element on the principal diagonal of the stiffness matrix is merely modified by adding the support stiffness to it.

**11- Solution of the Overall Problem**

Steps we have taken so far are as followings:

1. we have used potential energy theorem for each element
2. Obtained displacement field, calculated $\pi^e=U^e-W^e$
3. joined each element together, got an approximate of the total potential energy in a structure (is equivalent of satisfying equilibrium and compatibility at the nodes)
4. minimized the TPE to get an approximate solution of the problem
5. applied the boundary conditions; (only need to satisfy essential bc's; i.e. displacement and slopes and …. We do not have to satisfy the natural bc's for potential energy theorem)

We are ready to sole the equations for the unknown displacements. Once we obtained the displacements, we can proceed to evaluate whichever element stresses and or strains need to complete the analysis. It is important to know that the element stresses do not satisfy the equilibrium conditions for the individual element. In applying the principle of minimum potential energy, we approximate the overall equilibrium of the body, but do not provide for inter element equilibrium. Nevertheless, as our approximation to the total potential energy and to the displacement solution is improved either by using refined elements or by reducing the mesh, we also obtain improved results for the element stress components.

Because of the approximation involved, it is logical to use some average value of the stress(or strain) as representative for the element at the centroid of the element.