PATTERN FORMATION

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Workshop on Biomathematics
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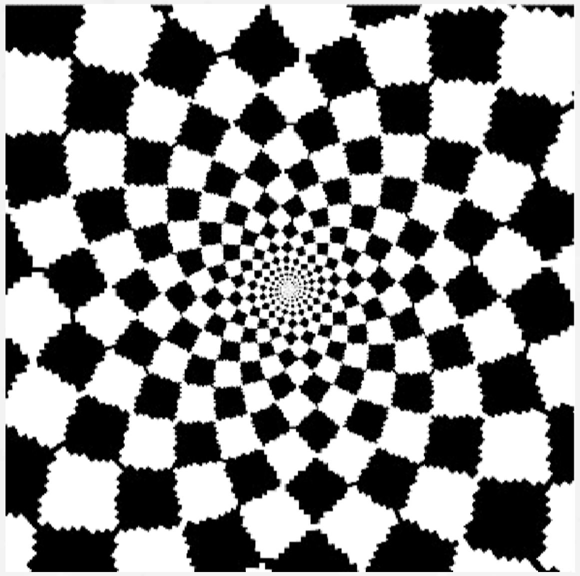
SOME FEATURES

HALLUCINATION

(*a*)



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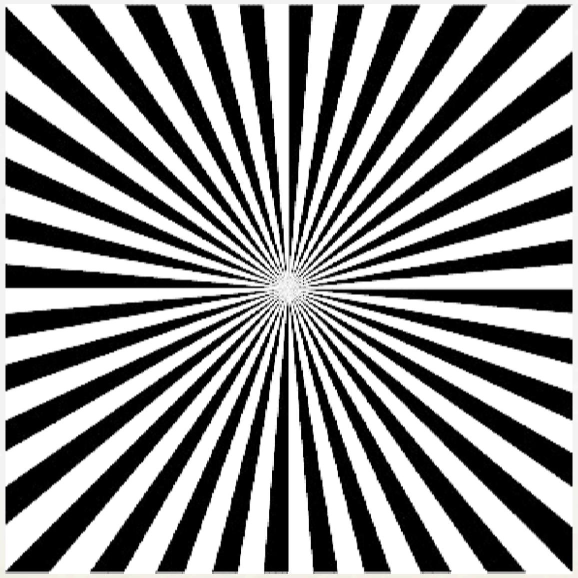

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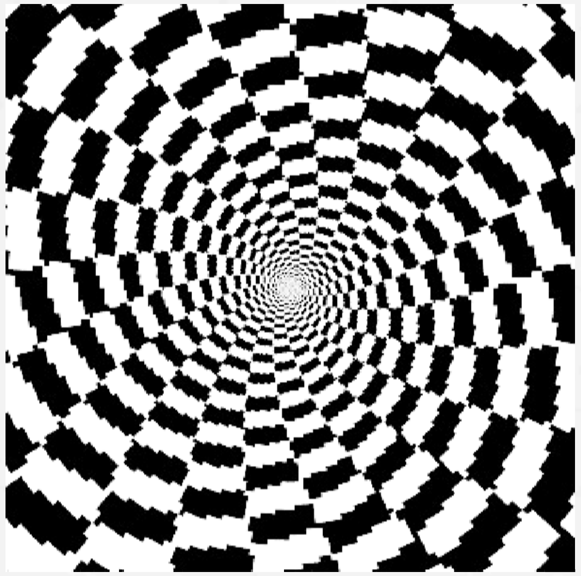

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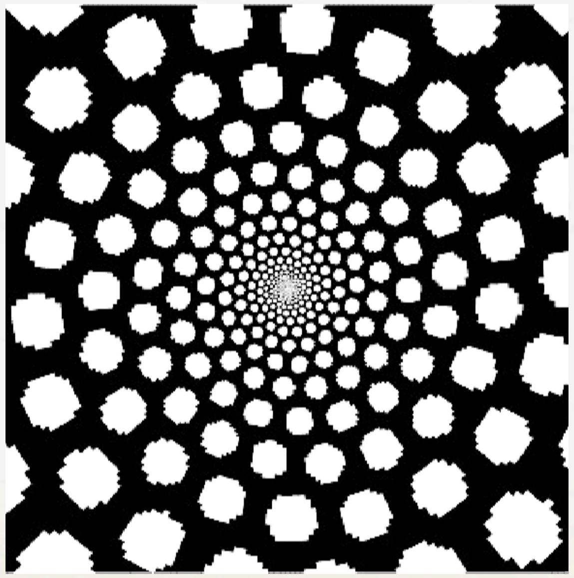
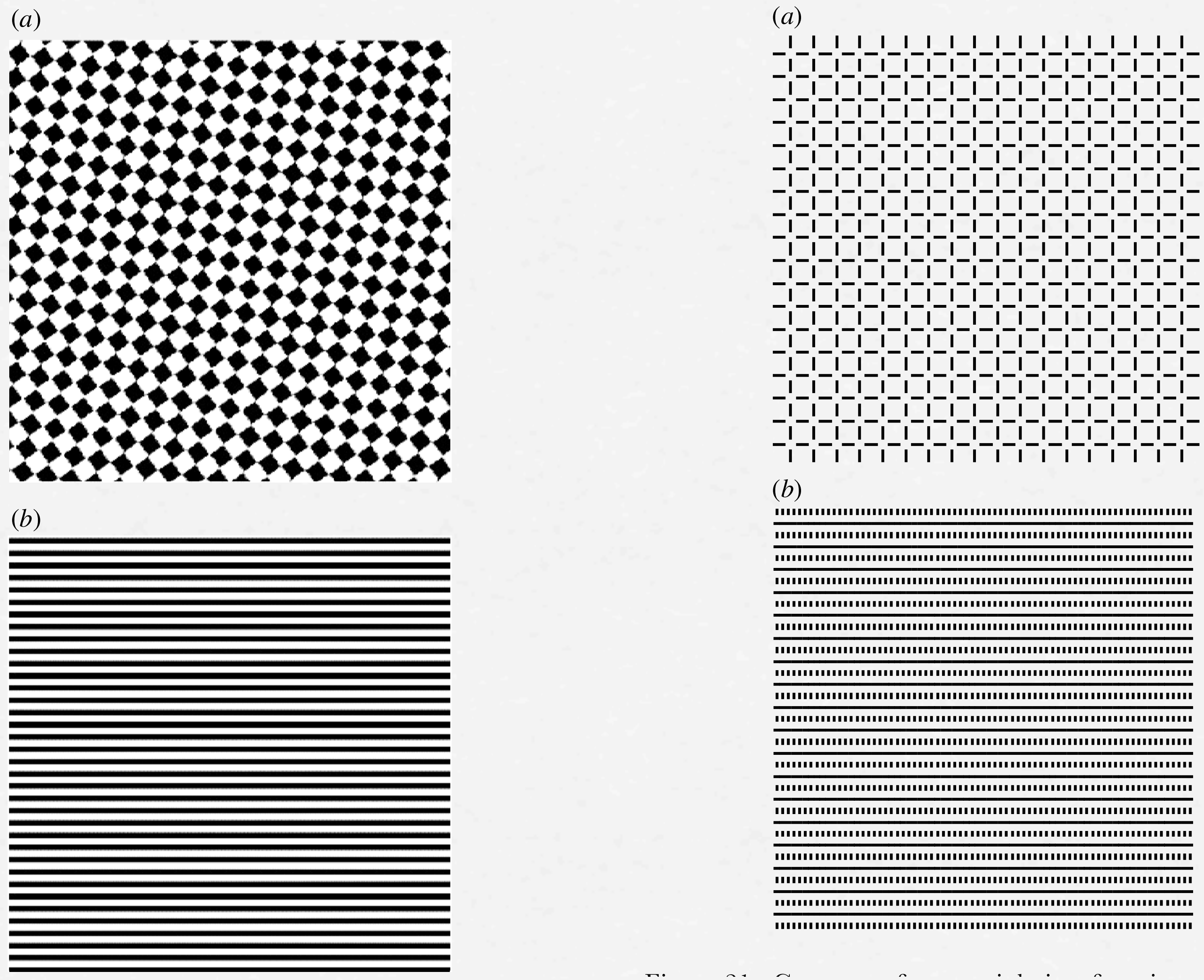


PATTERNS IN NEURAL FIELDS

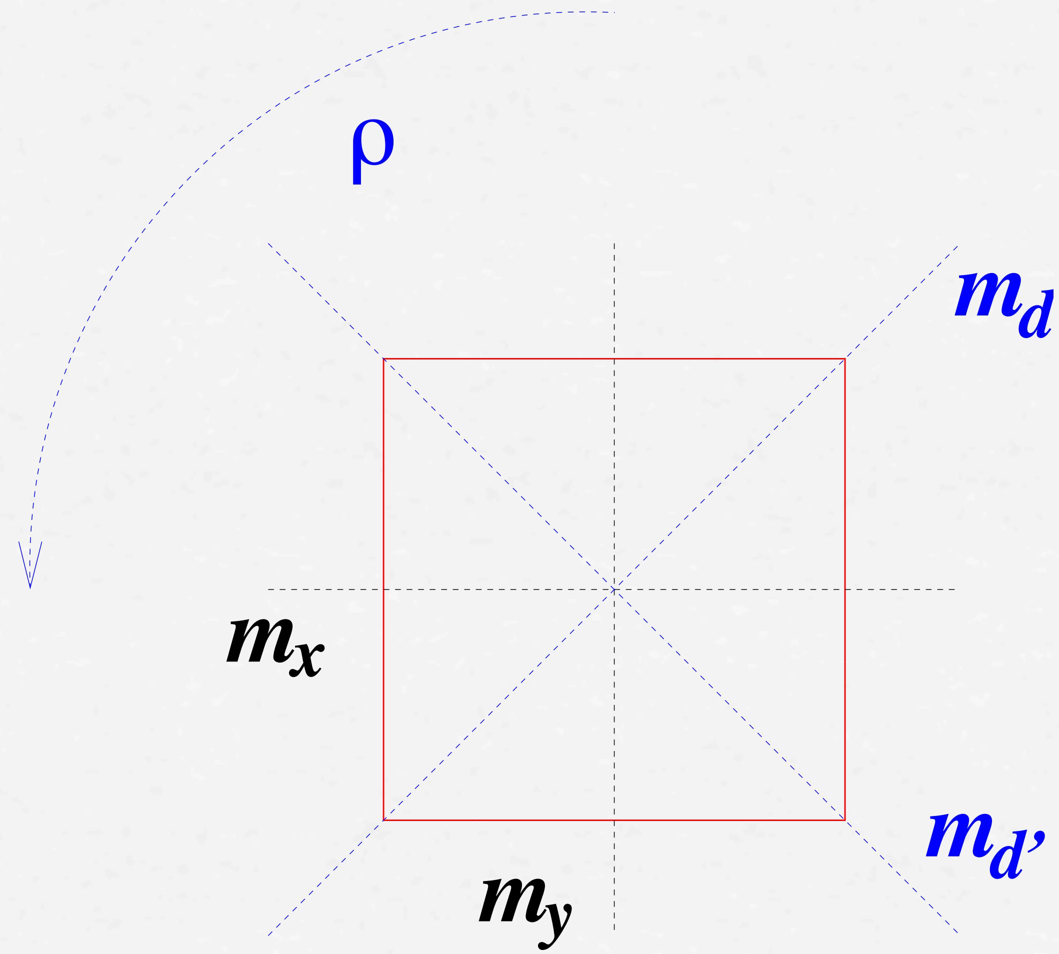


$$u_t = -u + \int w(x, y) f(u(y, t)) dy$$

SYMMETRIC REPRESENTATION

$$\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$m = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$D_4 \quad = \quad \{I, m_x, m_y, m_d, m_{d'}, \rho, \rho^2, \rho^3\}$$

SYMMETRIC EQUATIONS

- Vector Field $\frac{dx}{dt} = f(x, \mu)$
- Group Action $\Gamma \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$
- Vector field $f(x, \mu)$ has the symmetry Γ if for every solution $x(t)$, the trajectory $\gamma.x(t)$ is also a solution for every $\gamma \in \Gamma$
- $$\gamma.f(x, \mu) = f(\gamma.x, \mu)$$

An Example of D_4 Symmetry

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, x_2, \mu) \\ \frac{dx_2}{dt} = f_2(x_1, x_2, \mu) \end{cases}$$

$$m.f(x) = f(m.x) \Longrightarrow \begin{cases} -f_1(x_1, x_2) = f_1(-x_1, x_2) \\ f_2(x_1, x_2) = f_2(-x_1, x_2) \end{cases}$$

$$f(x) = \mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + a_1 \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} + a_2 \begin{pmatrix} x_1 x_2^2 \\ x_1^2 x_2 \end{pmatrix} + \cdots$$

SYMMETRIC BIFURCATION

- Definition1: (Isotropy Subgroup)

$$\Sigma_x = \{\sigma \in \Gamma : \sigma.x = x\}$$

- Isotropic subgroups are constant along solution curves.

- Definition2: (Fixed Point Invariant Subspace)

$$\mathrm{Fix}(\Sigma) = \{x \in \mathbb{R}^n : \sigma.x = x, \forall \sigma \in \Sigma\}$$

- One possible line of finding the Σ symmetry solution is to restrict the dynamics to Fix(Σ).

EQUIVARIANT BRANCHING LEMMA

Let Γ be a finite group acting on $\mathbb{R}^n$ with Fix $(\Gamma) = \{0\}$.

Let $\frac{dx}{dt} = f(x, \mu)$ be a Γ symmetry with $f(0, \mu) = 0$,

$D_x f|_{(0,0)} = 0\,, D_x f_\mu|_{(0,0)} v \neq 0$ for a $0 \neq v \in \mathrm{Fix}(\Sigma)$

Σ is an isotropy subgroup of Γ where $\dim\mathrm{Fix}(\Sigma) = 1$.

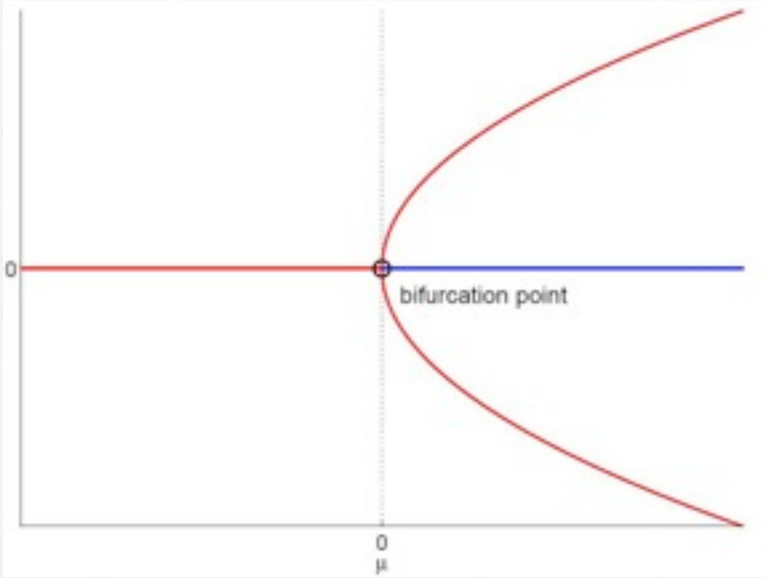
Then there is a curve $x = sv$, $\mu = \mu(s)$ of critical points.

$$f(sv, \mu(s)) = 0$$

EXAMPLE

$$\Sigma = \{e, m\} \Longrightarrow \mathrm{Fix}(\Sigma) = \{(0, y) : y \in \mathbb{R}\}$$

$$f(0, y, \mu(y)) = 0, \quad \mu \approx -a_1 y^2$$



Pitchfork Bifurcation

LATTICE PATTERNS

$$u_t = F(u, \mu)$$

$$u(x + \overrightarrow{l}, t) = u(x, t)$$

$$\mathcal{L} = \{n_1 \overrightarrow{l_1} + n_2 \overrightarrow{l_2} : n_1, n_2 \in \mathbb{Z}\}$$

$$u(x, t) = \sum_{k \in \mathcal{L}^*} z_k(t) \exp(ik.x) + c.c.$$

$$\frac{dz}{dt} = g(z, \mu)$$

LATTICE SYMMETRIES GROUP

Rotation:

$$u(x_1, x_2, t) = z_1 \exp(ix_1) + z_2 \exp(ix_2) + c.c.$$
$$u(-x_2, x_1, t) = z_1 \exp(-ix_2) + z_2 \exp(ix_1) + c.c.$$

$$\Longrightarrow \rho.(z_1, z_2) = (z_2, \overline{z_1})$$

Reflection: $$m.(z_1, z_2) = (\overline{z_1}, z_2)$$

Translation: $$p.(z_1, z_2) = (e^{-ip_1} z_1, e^{-ip_2} z_2)$$

PDE TO ODE

Example: (Square Lattice) $\dot{z} = g(z, \mu)$

symmetry group: $\Gamma = D_4 \times T^2$

$$\gamma . g(z) = g(\gamma . z) \qquad \forall \gamma \in \Gamma$$

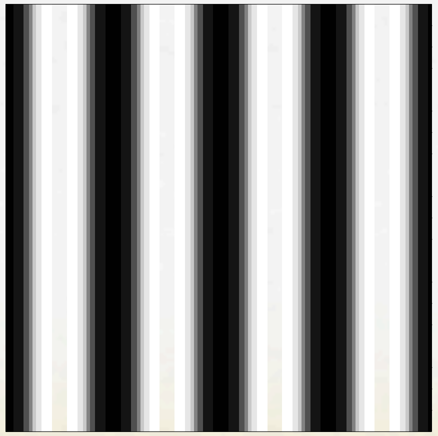
$$\begin{cases} \frac{dz_1}{dt} = \mu z_1 - \alpha |z_1|^2 z_1 - \beta |z_2|^2 z_1 + \dots \\ \frac{dz_2}{dt} = \mu z_2 - \beta |z_1|^2 z_2 - \alpha |z_2|^2 z_2 + \dots \end{cases}$$

APPLYING BRANCHING LEMMA

$$\Sigma = D_2 \times S^1 = \{e, m\} \times (p_1, 0)$$

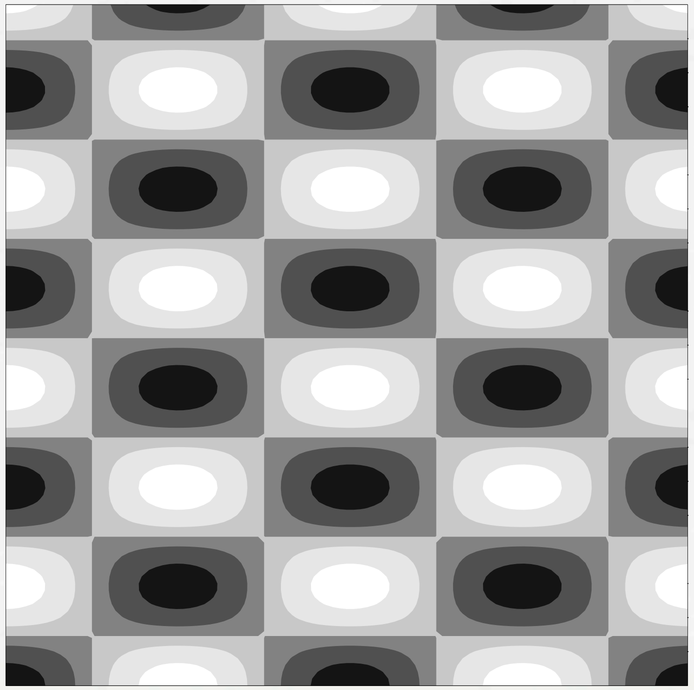
$$\mathrm{Fix}(\Sigma) = \{(z_1, 0) : z_1 \in \mathbb{R}\}$$

$$u(x, t, \mu) = \sqrt{\frac{\mu}{\alpha}} \exp(ix_1) + c.c. = 2\sqrt{\frac{\mu}{\alpha}} \cos(x_1)$$



$$\Sigma = D_4 \qquad \text{Fix}(\Sigma) = \{(z_1, z_1) : z_1 \in \mathbb{R}\}$$

$$u(x,t,\mu) \approx 2\sqrt{\frac{\mu}{\alpha+\beta}}(\cos(x_1) + \cos(x_2))$$



REFERENCE

Marty Golubitsky

Rebecca Hoyle