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CONTROLLABILITY RESULTS FOR A CLASS OF ONE DIMENSIONAL DEGENERATE/SINGULAR PARABOLIC EQUATIONS

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Abstract. We study the null controllability properties of some degenerate/singular parabolic equations in a bounded interval of $\mathbb{R}$. For this reason we derive a new Carleman estimate whose proof is based on Hardy inequalities.

1. **Introduction.** In the recent years, there has been substantial progress in understanding the controllability properties of parabolic equations with variable coefficients. In particular, the null controllability for the following class of nondegenerate and nonsingular parabolic operators is well-known:

$$Pu = u_t - (a(x)u_x)_x, \qquad x \in (0,1),$$

where the coefficient $a(x)$ is a positive continuous function on $[0,1]$, see for instance [8, 9, 11, 15].

However, many problems that are relevant for applications are described by degenerate equations, with degeneracy occurring at the boundary of the space domain. For example, in [1], the reader will find a degenerate parabolic equation obtained as an example of a Crocco-type equation coming from the study of the velocity field of a laminar flow on a flat plate. Also, in [13] the authors consider the similar equation to study the control properties. Furthermore, degenerate parabolic operators naturally arise in various physical problems such as boundary layer models, gene frequency models for population genetics, see for instance, the Wright-Fischer model studied in [16], Bydyko-Sellers climate models [14].

Our main goal of this paper is to provide a full analysis of the null controllability problem for the following one dimensional equation that couples a degenerate diffusion coefficient with a singular potential:

$$u_t - (a(x)u_x)_x - \frac{\lambda}{x^\beta}u = h\chi_\omega, \qquad (t,x) \in (0,T) \times (0,1), \tag{1}$$

where the control h acts on a nonempty subinterval ω of $(0,1)$.

As one can see, the simplest example for the function $a(\cdot)$ is x^α which has been considered in [18] by Vancostenoble. For $\alpha \in [0,2)$ and under optimal conditions on the parameters $\beta, \lambda \in \mathbb{R}$, she deduced null controllability results for the corresponding evolution problem. One of the starting point of the study of purely singular

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equations is investigated in [20], when $\beta = 2$ and $a(x) \equiv 1$ is the nondegenerate diffusion coefficient. They showed controllability in this case for $\lambda \leq 1/4$. Also, the situation where there is no singular term has been investigated in [3, 4, 12] and null controllability properties were established for the operator

$$Pu = u_t - (a(x)u_x)_x, \qquad x \in (0,1),$$

with suitable boundary conditions. In [12], the authors proved the null controllabilty in the following assumption for the diffusion coefficient $a(\cdot)$

$$\lim_{x\to 0} \frac{xa'(x)}{a(x)} = \alpha < 2.$$

Besides, the same result has been deduced under the condition

$$xa'(x) \leq \alpha a(x), \qquad \forall x \in [0,1],$$

for some $\alpha \in (0,2)$ as shown in [3]. Furthermore, with the above condition, the controllability of the degenerate/singular equation (1) was established in [10] for $\beta \leq 2-\alpha$. We study the equation (1) to show the controllabilty results like [12, 18] under a weaker assumption. In fact, we admit the following assumption on the degeneracy coefficient $a(\cdot)$.

Hypothesis 1. We suppose that the degeneracy coefficient $a(\cdot)$ satisfies the following conditions:

(i) $a \in C([0,1]) \cap C^1((0,1])$, $a(x) > 0$ in $(0,1]$ and $a(0) = 0$.

(iii) If $\alpha \in [1,2)$, there exist $m > 0$ and $\delta_0 > 0$ such that for every $x \in [0, \delta_0]$, we have

$$a(x) \geq m \sup_{0\leq y\leq x} a(y).$$

Under these assumptions on the coefficient $a(\cdot)$, we prove that the related degenerate/singular parabolic equation is null controllable. The proof follows from a new Carleman estimate which is a consequence of Hardy inequalities (see section 2).

Remark 1. Note that part (ii) of Hypothesis 1 is weaker than the same condition on the coefficient $a(\cdot)$ in [10]. In fact, therein , instead of (ii), is assumed that

$(ii)^*$ there exists some $\alpha^* \in (0,2)$ such that $xa'(x) \leq \alpha^* a(x)$ for every $x \in [0,1]$.

Obviously, under this condition, the cofficient $a(\cdot)$ satisfies the assumption (ii) in Hypothesis 1 for some $\alpha \in [0,2]$. But (ii) and $(ii)^*$ are not equivalent. For example consider the function $a(x) = x^\alpha e^{2x}$, for which, one has

$$\limsup_{x\to 0} \frac{xa'(x)}{a(x)} = \limsup_{x\to 0} (\alpha + 2x) = \alpha,$$

and $a(\cdot)$ satisfies (iii) with $m = 1$, but $a(\cdot)$ fails $(ii)^*$: there exists no $\alpha^* < 2$ such that $xa'(x) \leq \alpha^* a(x)$ for any $x \in [0,1]$.

Remark 2. Clearly if $a(\cdot)$ is nondecreasing near zero, then it satisfies (iii), whereas there exist some nondereasing examples. For instance, consider $a(x) = x[1+\sin^2 \frac{1}{x}]$, and the function $a(x) = x^\alpha \exp(\sin \frac{1}{x})$ for $\alpha \geq 0$ is another example.

Remark 3. In fact, Hypothesis 1 is strictly weaker than the hypothesis on $a(\cdot)$ in the paper [3]. In that paper, instead of (ii) and (iii), they admit assumptoins $(ii)^*$ and the following, respectively:

$(iii)^*$ If $\alpha^* \in [1,2)$, then

$$\begin{cases} \exists \theta \in (1, \alpha^*] & \text{such that} \quad x \mapsto \frac{a(x)}{x^\theta} \quad \text{is nondecreasing near } 0, \quad \text{if} \quad \alpha^* > 1, \\ \exists \theta \in (0, 1) & \text{such that} \quad x \mapsto \frac{a(x)}{x^\theta} \quad \text{is nondecreasing near } 0, \quad \text{if} \quad \alpha^* = 1. \end{cases}$$

One can easily see that in this case, $a(\cdot)$ satisfies (iii) with $m = 1$ (in fact, $a(\cdot)$ is nondecreasing). But (iii) is strictly weaker than $(iii)^*$. For example, consider the function

$$a(x) = \sqrt{x} + \epsilon\sqrt{x^3}\sin^2\frac{1}{x},$$

which satisfies (iii) but there exists no $\theta > 1$ such that $x \mapsto \frac{a(x)}{x^\theta}$ is nondecreasing near 0. Note that, one has $\limsup_{x\to 0}\frac{xa'(x)}{a(x)} = \frac{1}{2} + 2\epsilon$, which for $\frac{1}{4} < \epsilon < \frac{3}{4}$, we have $\alpha^* = \frac{1}{2} + 2\epsilon \in (1,2)$.

Notice that the condition $\alpha < 2$ is necessary for controllability, because we know that the null controllability of (1) fails if $\frac{1}{\sqrt{a}} \notin L^1(0,1)$, [5]. And we can conclude that $\frac{1}{\sqrt{a}} \in L^1(0,1)$ from the assumption (ii).

The paper is organized as follows. In section 2, we derive an improved Hardy inequality which is a fundamental tool for the proof of the main results in our paper. Section 3 is devoted to well-posedness of the problem. In section 4, we state a Carleman estimate and an observability inequality, and as a consequence, a null controllability result is obtained for our problem.

2. **Hardy inequalities.** In this section, we prove some Hardy inequalities that will be useful for the rest of the paper. For this, we need some preliminaries. First, let us introduce the following weighted space.

$$H^1_a(0,1) := \{u \in L^2(0,1) \cap H^1_{loc}((0,1]) : \int_0^1 a(x)u_x^2 dx < \infty\},$$

with the norm

$$\|u\|^2_{H^1_a} := \int_0^1 u^2 + a(x)u_x^2 dx.$$

For the elements of $H^1_a(0,1)$ we have the following lemma (see [10]).

Lemma 2.1. *(i) For $\alpha \in [0,2)$ and every $u \in H^1_a(0,1)$, we have $\lim_{x\to 0} xu^2 = 0$ and $\lim_{x\to 0} xu = 0$.*
(ii) If $\alpha \in [0,1)$, then for every $u \in H^1_a(0,1)$ we have $u \in W^{1,1}(0,1)$.

Thus with respect to the part (ii) of the above Lemma, we can introduce the following space $H^1_{a,0}(0,1)$ depending on the values of α:

Definition 2.2. (i) For $\alpha \in (0,1)$, we define

$$H^1_{a,0}(0,1) := \{u \in H^1_a(0,1) : u(0) = u(1) = 0\},$$

(ii) For $\alpha \in [1,2)$, we let

$$H^1_{a,0}(0,1) := \{u \in H^1_a(0,1) : u(1) = 0\}.$$

There exists an important density result for the elements of $H^1_{a,0}$ (see [10]).

Proposition 1. *(i) For $\alpha \in [0,1)$, the space $C_c^\infty(0,1)$ is dense in $H^1_{a,0}(0,1)$.*
(ii) In the case $\alpha \in [1,2)$, the subset of functions of $C^\infty([0,1])$ which vanish at $x = 1$ is dense in $H^1_{a,0}(0,1)$.

For the proof of well-posedness, we need to use some "Hardy-type" inequalities. One of them is the following.

Lemma 2.3. *Suppose $\alpha \in [0,2)$ and $\beta < 2-\alpha$. There exists an optimal constant $\lambda^*(a,\beta) > 0$ such that for every $u \in H^1_{a,0}(0,1)$, we have*

$$\int_0^1 a(x)u_x^2 dx \geq \lambda^*(a,\beta)\int_0^1 \frac{u^2}{x^\beta}dx. \tag{2}$$

Proof. Note that by Proposition 1 it is enough to prove (2) for $u \in C_c^\infty(0,1)$ in the case $\alpha \in (0,1)$ and for $u \in C^\infty([0,1])$ such that $u(1) = 0$ in the case $\alpha \in [1,2)$. First, choose $\eta > 0$ such that $\beta < 2-\alpha-\eta$, then we can write

$$\int_0^1 (x^{\frac{\alpha+\eta}{2}}u_x - \frac{1-\alpha-\eta}{2}\frac{u}{x^{\frac{2-\alpha-\eta}{2}}})^2 dx \geq 0.$$

Thus we get

$$\int_0^1 x^{\alpha+\eta}u_x^2 dx + \frac{(1-\alpha-\eta)^2}{4}\int_0^1 \frac{u^2}{x^{2-\alpha-\eta}}dx - \frac{(1-\alpha-\eta)}{2}\int_0^1 \frac{1}{x^{1-\alpha-\eta}}(u^2)_x dx \geq 0.$$

Now, if we use integration by parts, in the case $\alpha \in (0,1)$ there exists no boundary term, also in the case $\alpha \in [1,2)$, since $u \in C^\infty([0,1])$ the term $\frac{u^2}{x^{1-\alpha-\eta}}$ tends to zero as $x \to 0$. Therefore, we obtain

$$\int_0^1 x^{\alpha+\eta}u_x^2 dx - \frac{(1-\alpha-\eta)^2}{4}\int_0^1 \frac{u^2}{x^{2-\alpha-\eta}}dx \geq 0.$$

On the other hand, since $\limsup_{x\to 0}\frac{xa'(x)}{a(x)} = \alpha$, there exists $\delta > 0$ so that $\frac{xa'(x)}{a(x)} \leq \alpha+\eta$ for all $x \in (0,\delta)$, which means that the function $x \mapsto \frac{a(x)}{x^{\alpha+\eta}}$ is decreasing on the interval $(0,\delta)$. Thus

$$a(x) \geq x^{\alpha+\eta}a(\delta)\delta^{-\alpha-\eta}, \qquad \forall x \in [0,\delta].$$

Besides, $a(x) > 0$ for all $x \in [\delta,1]$, so one can choose $M = M(a,\eta) > 0$ such that $a(x) \geq Mx^{\alpha+\eta}$, for every $x \in [0,1]$, and deducely we obtain

$$\int_0^1 a(x)u_x^2 dx \geq M\frac{(1-\alpha-\eta)^2}{4}\int_0^1 \frac{u^2}{x^\beta}dx$$

$\square$

Remark 4. For $\alpha \in [0,2)$, by (2) we have

$$\int_0^1 a(x)u_x^2 dx \geq C\int_0^1 u^2 dx, \qquad \forall u \in H^1_{a,0}(0,1).$$

Then we can consider the new equivalent norm $\|u\|_{H^1_{a,0}(0,1)} := (\int_0^1 a(x)u_x^2 dx)^{\frac{1}{2}}$ on $H^1_{a,0}(0,1)$.

Another useful inequality to achieve the desired result is the following improved Hardy inequality.

Theorem 2.4. *Suppose that the function $a(\cdot)$ satisfies Hypothesis 1 and $\beta < 2-\alpha$. Then for all $n > 0$ and $\gamma < 2-\alpha$, there exist some positive constants $C_0 = C_0(a,\beta,\gamma,n)$ and $\mu = \mu(a,\beta)$ such that, for all $u \in H^1_{a,0}(0,1)$, the following inequality holds:*

$$\int_0^1 a(x)u_x^2 dx + C_0\int_0^1 u^2 dx \geq \mu\int_0^1 \frac{u^2}{x^\beta}dx + n\int_0^1 \frac{u^2}{x^\gamma}dx. \tag{3}$$

Proof. First, choose $\eta > 0$ such that

$$\beta < 2 - \alpha - \eta, \qquad \gamma < 2 - \alpha - \eta. \tag{4}$$

On the other hand, by (ii) of Hypothesis 1, there exists $M > 0$ so that

$$a(x) \geq M x^{\alpha+\eta}, \qquad \forall x \in [0,1].$$

Now, we can use the Theorem proved by Vancostenoble in [18]: there exists some positive constant $C_0' = C_0'(\alpha+\eta, \gamma, n, M) > 0$ such that, for all $u \in C_c^\infty(0,1)$, the following inequality holds:

$$\int_0^1 x^{\alpha+\eta} u_x^2 dx + C_0' \int_0^1 u^2 dx \geq \frac{(1-\alpha-\eta)^2}{4} \int_0^1 \frac{u^2}{x^{2-\alpha-\eta}} dx + \frac{n}{M} \int_0^1 \frac{u^2}{x^\gamma} dx. \tag{5}$$

Also $C_0'(\alpha+\eta, \gamma, n, M)$ is explicitly given by

$$C_0'(\alpha+\eta, \gamma, n, M) = (\frac{n}{M} + 1)^{\frac{2-\alpha-\eta+\gamma}{2-\alpha-\eta-\gamma}} \frac{2-\alpha-\eta-\gamma}{2-\alpha-\eta+\gamma} (\frac{4\gamma}{(2-\alpha-\eta)^2 - \gamma^2})^{\frac{2\gamma}{2-\alpha-\eta-\gamma}}.$$

Now, considering Proposition 1, we deduce that the inequality (5) is true for every $u \in H^1_{a,0}(0,1)$ in the case $\alpha \in (0,1)$. For $\alpha \in [1,2)$, we can prove the inequality (5) for every $u \in C^\infty([0,1])$ such that $u(1) = 0$, by the similar method used in [18]. Thus the inequality will be true for every $u \in H^1_{a,0}(0,1)$ and $\alpha \in (0,2)$.
Hence we deduce that for all $n > 0$ and $\gamma < 2 - \alpha - \eta$, there exists some positive constant $C_0 = C_0(a, \beta, \gamma, n) > 0$ such that for all $u \in H^1_{a,0}(0,1)$, the inequality (3) holds. □

3. **Well-posedness of the problem.**

Consider the operator

$$Au := (a(x)u_x)_x + \frac{\lambda}{x^\beta} u \tag{6}$$

where the coefficient $a(\cdot)$ satisfies Hypothesis 1. Let ω be a nonempty subinterval of $(0,1)$ and consider the following initial-boundary value problem in the domain $Q_T = (0,T) \times (0,1)$.

$$\begin{cases} u_t - Au = h\chi_\omega, & & (t,x) \in Q_T, \\ u(t,0) = u(t,1) = 0, & \text{in the case } \alpha \in (0,1), & t \in (0,T), \\ (a(x)u_x)(t,0) = u(t,1) = 0, & \text{in the case } \alpha \in [1,2), & t \in (0,T), \\ u(0,x) = u_0(x), & & x \in (0,1). \end{cases} \tag{7}$$

where the initial condition u_0 is given in $L^2(0,1)$ and $h \in L^2(Q_T)$. We are interested in the null controllability of (7) in time $T > 0$ with a distributed control supported in ω, i.e. for all $u_0 \in L^2(0,1)$, does there exist $h \in L^2(Q_T)$ such that $u(T,x) = 0$ for every $x \in [0,1]$? In this purpose, we derive Carleman estimate for the operator A in Section 4. But before going any further, we state some conditions on the parameters β, λ for which problem (7) is well-posed.

First note that when $\alpha \geq 2$, the null controllability might be false. For example consider $a(x) = x^\alpha$ for $\alpha \geq 2$. In this case, the necessary condition for null controllability which was proved in [5] does not hold, i.e. $\frac{1}{\sqrt{a(x)}}$ does not belong to $L^1(0,1)$. But for $\alpha < 2$, if we choose $\eta > 0$ so that $\alpha + \eta < 2$, as proved before, there exists some $M = M(a, \eta) > 0$ such that $a(x) \geq M x^{\alpha+\eta}$, for every $x \in [0,1]$. Therefore $\frac{1}{\sqrt{a}} \in L^1(0,1)$.

For a given singular potential, Cabré and Martel in [2] proved that existence versus blow-up of positive solutions is connected to the existence of some Hardy inequality involving the considered potential, like as the following:

$$\int_0^1 a(x)u_x^2 dx \geq \lambda \int_0^1 \frac{u^2}{x^\beta} dx, \tag{8}$$

for any $u \in H^1_{a,0}(0,1)$. The condition $\beta < 2-\alpha$ insures the validity of this inequality. The following proposition shows that the Hardy inequality (8) is false for β large enough.

Proposition 2. *Suppose that the function $a(\cdot)$ satisfies the conditions (i) and (ii) of Hypothesis 1 and set*

$$\alpha_* := \liminf_{x\to 0} \frac{xa'(x)}{a(x)}.$$

Then, the Hardy inequality (8) is false for any $\beta > 2 - \alpha_*$.

Proof. Let $\epsilon > 0$ such that $\beta + \alpha_* - \epsilon > 2$. Then, on a interval $(0, \delta]$, one has $\frac{xa'(x)}{a(x)} \geq \alpha_* - \epsilon$, which implies the function $x \mapsto \frac{a(x)}{x^{\alpha_*-\epsilon}}$ is increasing on $(0, \delta]$. So, there exists $M > 0$ such that $a(x) \leq Mx^{\alpha_*-\epsilon}$ for every $x \in [0, 1]$. Therefore,

$$\int_0^1 a(x)u_x^2 dx \leq M \int_0^1 x^{\alpha_*-\epsilon} u_x^2 dx.$$

If (8) is true for β, setting $u(x) := x^{r+\frac{\beta}{2}}(1 - x)$, and let $r \to 0^+$, one can obtain a contradiction. (Note that $u \in H^1_{a,0}$.) □

Note that if two functions $a(\cdot)$ and $b(\cdot)$ are equivalent near zero; i.e. there exist two positive constants c_1 and c_2 such that $c_1b(x) \leq a(x) \leq c_2b(x)$ in some neighbourhood of zero, then two spaces $H^1_a(0,1)$ and $H^1_b(0,1)$ are the same with equivalent norms. Therefore one can easily obtain the following Lemma.

Lemma 3.1. *Let $a, b : [0, 1] \to \mathbb{R}$ be in $C([0, 1]) \cap C^1((0, 1])$, $a(0) = b(0) = 0$, $a > 0$ and $b > 0$ on $(0, 1]$. Moreover, assume that there exist two positive constants c_1 and c_2 such that*

$$c_1b \leq a \leq c_2b,$$

in a neighborhood of zero. Then, for any $\beta > 0$, Hardy inequality (8) holds for the function $a(\cdot)$ if and only if it holds for $b(\cdot)$.

Following examples show that for $\beta \in [2 - \alpha, 2 - \alpha_*]$, the inequality (8) may be true or false.

Example 1. For $\alpha \in (0, 2), \gamma \in (0, \alpha)$ such that $\alpha - \gamma \neq 1$, define

$$a(x) = x^{\alpha-\gamma} \exp\,[-\gamma \int_x^1 \frac{\sin\frac{1}{t}}{t} dt], \qquad\qquad b(x) = x^{\alpha-\gamma}.$$

Note that $xb'(x) \leq (\alpha - \gamma)b(x)$ for all $x \in [0, 1]$, so by the Lemma proved in [10], for the function $b(\cdot)$, (8) is true for any $\beta \leq 2 - \alpha + \gamma$ and it fails for $\beta > 2 - \alpha + \gamma$ respect to Proposition 2 and the fact that

$$\liminf_{x\to 0} \frac{xb'(x)}{b(x)} = \alpha - \gamma.$$

Now, there exist positive constants c_1 and c_2 such $c_1b(x) \leq a(x) \leq c_2b(x)$ for every $x \in [0, 1]$. Thus, by Lemma 3.1 and the preceding paragraph, for the function $a(x)$, Hardy inequality (8) is true for $\beta \leq 2-\alpha+\gamma$ and it is false for $\beta > 2-\alpha+\gamma$. But one has

$$\limsup_{x\to 0} \frac{xa'(x)}{a(x)} = \alpha, \qquad \liminf_{x\to 0} \frac{xa'(x)}{a(x)} = \alpha - 2\gamma.$$

Example 2. Now, consider the function $a(x) = \frac{x^\alpha}{(\ln \frac{x}{2})^2}$. In this case, one has $\lim_{x\to 0} \frac{xa'(x)}{a(x)} = \alpha$, and for $\beta = 2-\alpha$, the function $u(x) = x^{\frac{1-\alpha}{2}}(1-x)$ contradicts Hardy inequality (8). So, (8) is false for $\beta \geq 2-\alpha$.

The above examples, lead us to assume $\beta < 2-\alpha$. Here, we show well-posedness and controllability of (7) for

$$\alpha \in [0,2), \qquad 0 < \beta < 2-\alpha, \qquad \lambda \in \mathbb{R}. \tag{9}$$

Remark 5. In [18], the author study problem (7) in the special case $a(x) = x^\alpha$ under the following assumptions

$$\begin{cases} \alpha \in [0,2), & 0<\beta<2-\alpha, \quad \lambda \in \mathbb{R}, \\ \alpha \in (0,2)\setminus\{1\}, & \beta = 2-\alpha, \quad \lambda \leq (1-\alpha)^2/4, \end{cases}$$

Note that $\frac{(1-\alpha)^2}{4}$ is the optimal constant in the following Hardy inequality:

$$\int_0^1 x^\alpha u_x^2 dx \geq \frac{(1-\alpha)^2}{4} \int_0^1 \frac{u^2}{x^{2-\alpha}} dx. \tag{10}$$

Taking $a(x) = x^\alpha$ one retrives exactly the results of [18] in subcritical case $\beta < 2-\alpha$. In fact, the possibility of considering critical case $\beta = 2-\alpha$ in [18] is a consequence of holding Hardy inequality (10) and also improved Hardy (3) for $\beta = 2-\alpha$, which is false in our case because of "limit" assumption (ii) in Hypothesis 1 on α. (Also, see Example 3.4). However, in critical case $\beta = 2-\alpha$ in [18], one needs to consider only the values of λ with $\lambda \leq \frac{(1-\alpha)^2}{4}$. Hence the case of critical value $\beta = 2-\alpha$ has still to be considered. Yet, this question leads to the problem of finding the optimal costants in Hardy and Improved Hardy inequalities, which seems to be not easy for general $a(\cdot)$.

Before going to define $D(A)$, we state the following lemma which results that the boundary condition makes sense in the case $\alpha \in [1,2)$ (see [10]).

Lemma 3.2. *Assume that $\alpha \in [1,2)$ and the condition (9) holds. Then for all $u \in H^1_a(0,1)$ such that $(au_x)_x + \frac{\lambda}{x^\beta}u \in L^2(0,1)$, we have $au_x \in W^{1,1}(0,1)$.*

Definition 3.3. (i) For $\alpha \in (0,1)$, we define

$$D(A) := \{u \in H^1_{a,0}(0,1) \cap H^2_{loc}((0,1]) : (au_x)_x + \frac{\lambda}{x^\beta} u \in L^2(0,1)\},$$

(ii) For $\alpha \in [1,2)$, we change the definition of $D(A)$ in the following way

$$D(A) := \{u \in H^1_{a,0}(0,1) \cap H^2_{loc}((0,1]) : (au_x)_x + \frac{\lambda}{x^\beta} u \in L^2(0,1), (au_x)(0) = 0\}.$$

Thus, in the case $\alpha \in (0,1)$, if $u \in D(A)$, then u satisfies the Dirichlet boundary conditions $u(0) = u(1) = 0$ and in the case $\alpha \in [1,2)$, every $u \in D(A)$ satisfies the Neumann boundary condition $(au_x)(0) = 0$ and the Dirichlet boundary condition $u(1) = 0$.

For the operator $(A, D(A))$ the following proposition holds (see [10]).

Proposition 3. *Assume that the condition (9) holds, then there exists a constant $k \geq 0$ such that the operator $(-(A - kI), D(A))$ is a self-adjoint and positive operator.*

Consequently, we have the following well-posedness result (see e.g. [7]).

Theorem 3.4. *Assume that the condition (9) holds and consider the problem (7) with $h \equiv 0$. Then, for all initial condition $u_0 \in L^2(0,1)$, the problem (7) has a unique solution*

$$u \in C^0([0,T], L^2(0,1)) \cap C^0((0,T], D(A)) \cap C^1((0,T], L^2(0,1)). \tag{11}$$

Moreover, if $u_0 \in D(A)$, then

$$u \in C^0([0,T], D(A)) \cap C^1([0,T], L^2(0,1)). \tag{12}$$

In addition, the inhomogeneous problem (7) with $h \in L^2(Q_T)$, has a unique solution $u \in C^0([0,T], L^2(0,1))$ for all initial condition $u_0 \in L^2(0,1)$.

4. **Carleman estimates and applications to controllability.** As it is well-known, very useful tools to study controllability are provided by observability inequalities for the adjoint problem

$$\begin{cases} v_t + (a(x)v_x)_x + \frac{\lambda}{x^\beta} v = 0, & & (t,x) \in Q_T, \\ v(t,1) = 0, & & t \in (0,T), \\ v(t,0) = 0, & \text{in the case } \alpha \in (0,1), & t \in (0,T), \\ (av_x)(t,0) = 0, & \text{in the case } \alpha \in [1,2), & t \in (0,T), \\ v(T,x) = v_T(x), & & x \in (0,1). \end{cases} \tag{13}$$

For this problem, we prove the following observability inequalities.

Proposition 4. *Assume that (9) holds and the coefficient $a(\cdot)$ satisfies Hypothesis 1. Let $T > 0$ be given and ω be a nonempty subinterval of $(0,1)$. Then there exists a positive constant $C = C(T, a, \beta, \omega, \lambda)$ such that every solution v of (13) satisfies*

$$\int_0^1 v^2(0,x)dx \leq C \int_0^T \int_\omega v^2(t,x)dxdt. \tag{14}$$

Now, by standard arguments (see [11]), a null controllability result follows.

Theorem 4.1. *Assume that (9) holds and the coefficient $a(\cdot)$ satisfies Hypothesis 1. Let $T > 0$ be given, and let ω be a nonempty subinterval of $(0,1)$. Then, for all $u_0 \in L^2(0,1)$, there exists $h \in L^2((0,T) \times \omega)$ such that the solution of (7) satisfies $u(T) \equiv 0$ in $(0,1)$. Furthermore, we have the estimate*

$$\|h\|_{L^2((0,T)\times\omega)} \leq C' \|u_0\|_{L^2(0,1)},$$

for some $C' = C'(T, a, \beta, \omega, \lambda) > 0$.

For the proof of the observability inequality (14), we need Carleman estimates for the degenerate and singular problem

$$\begin{cases} v_t + (a(x)v_x)_x + \frac{\lambda}{x^\beta} v - rv = h, & & (t,x) \in Q_T, \\ v(t,1) = 0, & & t \in (0,T), \\ v(t,0) = 0, & \text{in the case } \alpha \in (0,1), & t \in (0,T), \\ (av_x)(t,0) = 0, & \text{in the case } \alpha \in [1,2), & t \in (0,T), \\ v(T,x) = v_T(x), & & x \in (0,1), \end{cases} \tag{15}$$

where r is a nonnegative fixed constant. Our main result is the following.

Theorem 4.2. *Suppose that the function $a(\cdot)$ satisfies Hypothesis 1 and assumption (9) holds. Also let $T > 0$, then for every $\gamma < 2-\alpha$ there exists $\sigma : (0,T)\times[0,T] \to \mathbb{R}$ of the form $\sigma(t,x) = \theta(t)p(x)$ with*

$$p(x) < 0 \quad \forall x \in [0,1] \quad \text{and} \quad \theta(t) \to +\infty \quad \text{as} \quad t \to 0^+, T^-,$$

and two positive constants C and R_0, such that for all $v_T \in L^2(0,1)$ and $h \in L^2(Q_T)$, the solution v of (15) satisfies, for all $R \geq R_0$,

$$\begin{aligned} &R^3 \int_0^T \int_0^1 \theta^3 \frac{x^2}{a(x)} e^{2R\sigma(t,x)} v^2 dxdt + R \int_0^T \int_0^1 \theta a(x) e^{2R\sigma(t,x)} v_x^2 dxdt \\ &+ R \int_0^T \int_0^1 \theta \frac{v^2}{x^\beta} e^{2R\sigma(t,x)} dxdt + R \int_0^T \int_0^1 \theta \frac{v^2}{x^\gamma} e^{2R\sigma(t,x)} dxdt \\ \leq &C(\int_0^T \int_0^1 |h|^2 e^{2R\sigma(t,x)} dxdt + R \int_0^T \theta(t) v_x^2(t,1) e^{2R\sigma(t,1)} dt). \end{aligned} \tag{16}$$

Remark 6. This Theorem extends the results obtained in [18] in the case $\beta < 2-\alpha$, where the author considers only the case $a(x) = x^\alpha$. Also, in [12], in which the problem is purely degenerate and the same inequality exposes weaker than (16).

Remark 7. In [17], the authors treat an inverse problem for the one dimensional Sellers climate model and get an unconditional global Lipschitz stability of an unknown coefficient in a nonlinear term in this model. In fact, the main tool for this purpose is the Carleman estimate. So, we think that the results of the present paper could be used to treat some inverse problems following the method of Imanuvilov-Yamamoto in the spirit of the recent works by [6, 17, 19].

For the proof of Theorem 4.2, consider $0 < \gamma < 2-\alpha$ and $\sigma(t,x) := \theta(t)p(x)$ where

$$\theta(t) := \frac{1}{[t(T-t)]^k}, \qquad k := 1 + \frac{2}{\gamma} > 1, \tag{17}$$

$$p(x) := c_1(\int_0^x \frac{y}{a(y)} e^{\xi y^2} dy - c_2). \tag{18}$$

Observe that there exists some $c > 0$ such that for all $t \in (0,T)$

$$|\theta_t(t)| \leq c\theta^{1+\frac{1}{k}}(t), \qquad |\theta_{tt}(t)| \leq c\theta^{1+\frac{2}{k}}(t) \tag{19}$$

Two positive constants c_1 and ξ will be defined later. Also, one can choose c_2 so that $p(x) < 0$ for all $x \in [0,1]$. In fact, choose $\eta > 0$ such that $\alpha + \eta < 2$. Then , there exists some $M > 0$ so that $a(x) \geq Mx^{\alpha+\eta}$ for every $x \in [0,1]$. Now, for any $c_2 > \frac{e^\eta}{M(1-\alpha-\xi)}$, we have the desired result.

Next, for $R > 0$, we define the function $w(t,x) := e^{R\sigma(t,x)}v(t,x)$, where v is the solution of (15). Notice that $w|_{t=0} = w|_{t=T} \equiv 0$ and w satisfies

$$\begin{cases} (e^{-R\sigma}w)_t + (a(x)(e^{-R\sigma}w)_x)_x + \dfrac{\lambda}{x^\beta}(e^{-R\sigma}w) - re^{-R\sigma}w = h, & (t,x) \in Q_T, \\ w(t,1) = 0, & t \in (0,T), \\ w(t,0) = 0, \quad \text{for } \alpha \in (0,1), & t \in (0,T), \\ (aw_x)(t,0) = R(\sigma_x aw)(t,0), \quad \text{for } \alpha \in [1,2), & t \in (0,T), \\ w(T,x) = w(0,x) = 0, & x \in (0,1). \end{cases} \tag{20}$$

Thanks to the definitions of p and σ, we have $(\sigma_x aw)(t,x) = c_1 x e^{\xi x^2}\theta(t)w(t,x)$. Also, for $t \in [0,T]$, the function $w(t,\cdot)$ is in $H_a^1(0,1)$, therefore $xw(t,x)|_{x=0} = 0$ for

$t \in [0,T]$ using Lemma 2.1. Thus $(\sigma_x a w)(t,x)|_{x=0} = 0$ and the previous problem can be recast as follows. Set

$$Lv := v_t + (a(x)v_x)_x + \frac{\lambda}{x^\beta} v - rv, \qquad L_R w := e^{R\sigma} L(e^{-R\sigma} w).$$

Then (20) becomes

$$\begin{cases} L_R w = h e^{R\sigma}, & & (t,x) \in Q_T, \\ w(t,1) = 0, & & t \in (0,T), \\ w(t,0) = 0, & \text{in the case } \alpha \in (0,1), & t \in (0,T), \\ (a w_x)(t,0) = 0, & \text{in the case } \alpha \in [1,2), & t \in (0,T), \\ w(T,x) = w(0,x) = 0, & & x \in (0,1). \end{cases} \tag{21}$$

We have the following proposition which implies the Carleman estimates.

Proposition 5. *Suppose that $\alpha \in [0,2)$, $\beta < 2-\alpha$, $\lambda \in \mathbb{R}$ and $T > 0$, also for every $\gamma < 2-\alpha$, consider η as introduced in (4). Then, there exist constants $M > 0$ and $R_0 = R_0(a, \eta, \gamma, \lambda) > 0$ such that, for all $R \geq R_0$ and all solutions w of (21), we have*

$$\begin{aligned} &3R^3 c_1^2 \int_0^T \int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt + 2R \int_0^T \int_0^1 \theta a(x) w_x^2 dxdt \\ &+ \frac{M(1-\alpha-\eta)^2}{4} R \int_0^T \int_0^1 \theta \frac{w^2}{x^\beta} dxdt + R \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} dxdt \\ &\leq \frac{1}{2} \int_0^T \int_0^1 |h|^2 e^{2R\sigma(t,x)} dxdt + R c_1 a(1) \int_0^T \theta(t) w_x^2(t,1) dt, \end{aligned} \tag{22}$$

where $c_1 = \frac{4}{2-\alpha-\eta}$.

For the proof of this proposition, we introduce the following self-adjoint and skew-adjoint operators

$$\begin{aligned} L_R^+ w &:= (a(x) w_x)_x - R\sigma_t w + R^2 a(x) \sigma_x^2 w + \frac{\lambda}{x^\beta} w - rw, \\ L_R^- w &:= w_t - 2R a(x) \sigma_x w_x - R(a(x)\sigma_x)_x w. \end{aligned}$$

Then one has

$$L_R^+ w + L_R^- w = h e^{R\sigma},$$

and therefore

$$\|h e^{R\sigma}\|^2 = \|L_R^+ w\|^2 + \|L_R^- w\|^2 + 2\langle L_R^+ w, L_R^- w\rangle \geq 2\langle L_R^+ w, L_R^- w\rangle, \tag{23}$$

where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the usual norm and scalar product in $L^2(Q_T)$, respectively. The proof of Proposition 5, is based on the computation of the scalar product $\langle L_R^+ w, L_R^- w\rangle$, which comes in the following lemma.

Lemma 4.3. *The scalar product $\langle L_R^+ w, L_R^- w\rangle$ may be written as a sum of a distributed term A and a boundary term B, $\langle L_R^+ w, L_R^- w\rangle = A + B$, where the distributed term A is given by*

$$\begin{aligned} A = &- 2R^2 \int_0^T \int_0^1 \theta \theta_t p_x^2 a(x) w^2 dxdt + \frac{R}{2} \int_0^T \int_0^1 \theta_{tt} p w^2 dxdt \\ &+ R \int_0^T \int_0^1 \theta (2a^2(x) p_{xx} + a(x) a'(x) p_x) w_x^2 dxdt + R \int_0^T \int_0^1 \theta a (a p_x)_{xx} w w_x dxdt \\ &+ R^3 \int_0^T \int_0^1 \theta^3 (2a^2(x) p_{xx} + a(x) a'(x) p_x) p_x^2 w^2 dxdt - \beta \lambda R \int_0^T \int_0^1 \theta p_x \frac{a(x)}{x^{\beta+1}} w^2, \end{aligned}$$

whereas the boundary term B is given by

$$B = -R\int_0^T \theta(t)a^2(1)p_x(1)w_x^2(t,1)dt,$$

where $\theta(t)$ and $p(x)$ defined by (17) and (18).

Proof. The proof is similar to one stated in [3, 18]. However, in these refrences the term $-rw$ has not appeared, this term is neutral in the scalar product $\langle L_R^+ w, L_R^- w\rangle$, since

$$\begin{aligned}\langle -rw, L_R^+ w\rangle =& \frac{r}{2}\int_0^1\int_0^T \frac{d}{dt}w^2 dtdx + rR\int_0^1\int_0^T (a(x)\sigma_x w)_x dtdx \\ =& \frac{r}{2}\int_0^1 w^2(t,x)|_{t=0}^{t=T}dx + rR\int_0^T (a(x)\sigma_x w^2)(t,x)|_{x=0}^{x=1}dt. \qquad (24)\end{aligned}$$

The first sentence is zero because of the boundary conditions on w. Also,

$$a(x)\sigma_x w^2 = c_1\theta e^{\xi x^2} x w^2,$$

so the second sentence in the above is zero according to Lemma 2.1 and the boundary conditions on w. $\square$

Now we estimate the distributed term A in the following lemma.

Lemma 4.4. *Suppose that $\alpha \in [0,2)$, $\beta < 2-\alpha$, $\lambda \in \mathbb{R}$ and $T > 0$, also for every $\gamma < 2-\alpha$, consider η as introduced in (4). Then, there exists a constant $R_0 = R_0(a,\eta,\gamma,\lambda) > 0$ such that, for all $R \geq R_0$ we have*

$$\begin{aligned}A \geq& 3R^3c_1^2\int_0^T\int_0^1 \theta^3\frac{x^2}{a(x)}w^2 dxdt + 2R\int_0^T\int_0^1 \theta a(x)w_x^2 dxdt \\ &+ \frac{M(1-\alpha-\eta)^2}{4}R\int_0^T\int_0^1 \theta\frac{w^2}{x^\beta}dxdt + R\int_0^T\int_0^1 \theta\frac{w^2}{x^\gamma}dxdt,\end{aligned}$$

where $c_1 = \frac{4}{2-\alpha-\eta}$.

Proof. By Lemma 4.3 and the fact that

$$p_x = c_1\frac{x}{a(x)}e^{\xi x^2}, \qquad p_{xx} = c_1\Big(\frac{a(x)-xa'(x)}{a(x)^2} + \frac{2\xi x^2}{a(x)}\Big)e^{\xi x^2},$$

we obtain

$$\begin{aligned}A =& -2R^2c_1^2\int_0^T\int_0^1 \theta\theta_t\frac{x^2}{a(x)}e^{\xi x^2}w^2 dxdt + \frac{R}{2}\int_0^T\int_0^1 \theta_{tt}pw^2 dxdt \\ &+ Rc_1\int_0^T\int_0^1 \theta(2a(x) - xa'(x) + 4\xi x^2 a(x))e^{\xi x^2}w_x^2 dxdt \\ &+ R^3c_1^3\int_0^T\int_0^1 \theta^3(2a(x) - xa'(x) + 4\xi x^2 a(x))\frac{x^2}{a(x)^2}e^{\xi x^2}w^2 dxdt \\ &- \beta\lambda Rc_1\int_0^T\int_0^1 \theta\frac{e^{\xi x^2}}{x^\beta}w^2 dxdt + R\int_0^T\int_0^1 \theta a(ap_x)_{xx}ww_x dxdt.\end{aligned}$$

Now, take $\eta > 0$ such that $\alpha + \eta < 2$. Since $\limsup_{x\to 0}\frac{xa'(x)}{a(x)} = \alpha$, one has $2a(x) - xa'(x) \geq (2-\alpha-\eta)a(x)$ near zero. On the other hand, $a(x) > 0$ for every $x \in (0,1)$, so for $\xi > 0$ large enough, one has $2a(x) - xa'(x) + 4\xi x^2 a(x) \geq$

$(2-\alpha-\eta)a(x)$ for every $x \in [0,1]$. For the simplicity, we set $c := 2-\alpha-\eta$. Therefore one can estimate A in the following way

$$\begin{aligned}
A \geq & -2R^2c_1^2\int_0^T\int_0^1 \theta\theta_t \frac{x^2}{a(x)}e^{\xi x^2}w^2dxdt + \frac{R}{2}\int_0^T\int_0^1 \theta_{tt}pw^2dxdt \\
& + Rcc_1\int_0^T\int_0^1 \theta e^{\xi x^2}a(x)w_x^2dxdt + R^3cc_1^3\int_0^T\int_0^1 \theta^3 e^{\xi x^2}\frac{x^2}{a(x)}w^2dxdt \\
& - \beta\lambda Rc_1\int_0^T\int_0^1 \theta\frac{e^{\xi x^2}}{x^\beta}w^2dxdt + \int_0^T\int_0^1 \theta a(ap_x)_{xx}ww_xdxdt.
\end{aligned}$$

Let

$$\begin{aligned}
A_1 &:= -2R^2c_1^2\int_0^T\int_0^1 \theta\theta_t\frac{x^2}{a(x)}e^{\xi x^2}w^2dxdt, \\
A_2 &:= \frac{R}{2}\int_0^T\int_0^1 \theta_{tt}pw^2dxdt, \\
A_3 &:= Rcc_1\int_0^T\int_0^1 \theta e^{\xi x^2}a(x)w_x^2dxdt + R^3cc_1^3\int_0^T\int_0^1 \theta^3e^{\xi x^2}\frac{x^2}{a(x)}w^2dxdt \\
A_4 &:= -\beta\lambda c_1R\int_0^T\int_0^1 \theta\frac{e^{\xi x^2}w^2}{x^\beta}dxdt, \\
A_5 &:= R\int_0^T\int_0^1 \theta a(ap_x)_{xx}ww_xdxdt.
\end{aligned}$$

In estimating the above terms, we use the fact that $1 \leq e^{\xi x^2} \leq e^\xi$ for every $x \in [0,1]$, to get rid of the term $e^{\xi x^2}$. First, we estimate the term A_1. According to the relation (19), we know that $|\theta\theta_t| \leq c\theta^{2+\frac{1}{k}} \leq \tilde{c}\theta^3$, and obtain

$$|A_1| \leq 2R^2\tilde{c}c_1^2e^\xi\int_0^T\int_0^1 \theta^3\frac{x^2}{a(x)}w^2dxdt.$$

Also, for every $\epsilon > 0$ we can write

$$\begin{aligned}
A_5 &\leq \epsilon R\int_0^T\int_0^1 \theta a(x)w_x^2dxdt + \frac{4R}{\epsilon}\int_0^T\int_0^1 \theta a((ap_x)_{xx})^2w^2dxdt \\
&= \epsilon R\int_0^T\int_0^1 \theta a(x)w_x^2dxdt + \frac{4Rc_1^2}{\epsilon}\int_0^T\int_0^1 \xi^2\theta x^2[6+4\xi x^2]^2a(x)w^2dxdt \\
&\leq \epsilon R\int_0^T\int_0^1 \theta a(x)w_x^2dxdt + \frac{40N\xi^2c_1^2\hat{c}R}{\epsilon}\int_0^T\int_0^1 \theta^3\frac{x^2}{a(x)}w^2dxdt,
\end{aligned}$$

where the positive constants $\hat{c}$ and N are such that

$$\begin{aligned}
&\theta(t) \leq \hat{c}\theta^3(t), && \forall t \in (0,T), \\
&a(x)^3 \leq N, && \forall x \in (0,1).
\end{aligned}$$

Therefore

$$\begin{aligned}
A \geq & (R^3cc_1^3 - 2R^2c_1^2\tilde{c}e^\xi - \frac{40N\xi^2c_1^2\hat{c}R}{\epsilon})\int_0^T\int_0^1 \theta^3\frac{x^2}{a(x)}w^2dxdt \\
& + R(cc_1-\epsilon)\int_0^T\int_0^1 \theta a(x)w_x^2dxdt + A_2 + A_4. \qquad (25)
\end{aligned}$$

In the following, we produce estimates of the last two terms A_2 and A_4 to complete the proof. We want to prove the result for all γ satisfying $0 < \gamma < 2 - \alpha$. However, if the result holds for any γ such that $\beta \leq \gamma < 2 - \alpha$, then it obviously also holds for all γ such that $0 < \gamma < 2 - \alpha$. Therefore, we consider here γ, such that $\beta \leq \gamma < 2-\alpha$ and we study the term A_4. In the case $\lambda > 0$, we apply the improved Hardy inequality (3) with $n = \lambda c_1 e^{\xi}(2-\alpha) + 3 - \alpha \geq \beta\lambda c_1 e^{\xi} + 3 - \alpha$, which gives:

$$\int_0^1 a(x) w_x^2 dx + C_0 \int_0^1 w^2 dx \geq \frac{M(1-\alpha-\eta)^2}{4} \int_0^1 \frac{w^2}{x^\beta} dx + (\beta\lambda c_1 e^{\xi} + 3 - \alpha) \int_0^1 \frac{w^2}{x^\gamma} dx,$$

for a suitable $C_0 = C_0(a, \beta, n, \gamma) = C_0(a, \lambda, \gamma)$. Therefore we can write

$$\begin{aligned} A_4 =& -\beta\lambda c_1 R \int_0^T \int_0^1 \theta \frac{e^{\xi x^2} w^2}{x^\beta} dxdt \geq -\beta\lambda c_1 e^{\xi} R \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} dxdt \\ \geq& \frac{M(1-\alpha-\eta)^2}{4} R \int_0^T \int_0^1 \theta \frac{w^2}{x^\beta} dxdt - R \int_0^T \int_0^1 \theta a(x) w_x^2 \\ &+ R(3-\alpha) \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} dxdt - C_0 R \int_0^T \int_0^1 \theta e^{\xi x^2} w^2 dxdt. \end{aligned}$$

For $\lambda \leq 0$, we have

$$A_4 = -\beta\lambda c_1 R \int_0^T \int_0^1 \theta \frac{e^{\xi x^2} w^2}{x^\beta} dxdt \geq 0.$$

Applying (3) with $n = 3 - \alpha$, that is

$$\int_0^1 a(x) w_x^2 + C_0 \int_0^1 w^2 \geq \frac{M(1-\alpha-\eta)^2}{4} \int_0^1 \frac{w^2}{x^\beta} + (3-\alpha) \int_0^1 \frac{w^2}{x^\gamma},$$

we obtain the same estimate as in the case $\lambda > 0$. It follows that

$$\begin{aligned} A \geq& (R^3 c c_1^3 - 2R^2 c_1^2 \tilde{c} e^{\xi} - \frac{40 N \xi^2 c_1^2 \hat{c} R}{\epsilon}) \int_0^T \int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt \\ &+ R(cc_1 - \epsilon - 1) \int_0^T \int_0^1 \theta a(x) w_x^2 dxdt + \frac{M(1-\alpha-\eta)^2}{4} R \int_0^T \int_0^1 \theta \frac{w^2}{x^\beta} dxdt \\ &+ R(3-\alpha) \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} dxdt + A_2 - C_0 R \int_0^T \int_0^1 \theta w^2 dxdt. \end{aligned} \tag{26}$$

Finally, we need to estimate the two last terms in the above inequality. By (19), we have

$$|\theta_{tt}| \|p\|_\infty \leq K^* \theta^{1+\frac{2}{k}},$$

for some $K^* > 0$. It follows that

$$|A_2 - C_0 R \int_0^T \int_0^1 \theta w^2 dxdt| \leq R K' \int_0^T \int_0^1 \theta^{1+\frac{2}{k}} w^2 dxdt, \tag{27}$$

for some $K' = K'(a, \lambda, \gamma, c_1, \xi) > 0$. At this stage, we use the special choice of k, that is

$$k = 1 + \frac{2}{\gamma},$$

and consider $q=\frac{k}{k-1}$ and $q'=k$, so that $\frac{1}{q}+\frac{1}{q'}=1$. Then, for all $\epsilon>0$, we have

$$\begin{aligned}\int_0^T\int_0^1 \theta^{1+\frac{2}{k}}w^2dxdt =& \int_0^T\int_0^1 (\theta^{1+\frac{2}{k}-\frac{3}{q'}}a^{\frac{1}{q'}}x^{\frac{-2}{q'}}w^{\frac{2}{q}})(\theta^{\frac{3}{q'}}x^{\frac{2}{q'}}a^{\frac{-1}{q'}}w^{\frac{2}{q'}})dxdt\\ \le& \epsilon\int_0^T\int_0^1 \theta^{(1+\frac{2}{k}-\frac{3}{q'})q}a^{\frac{q}{q'}}x^{\frac{-2q}{q'}}w^2dxdt + C(\epsilon)\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt,\end{aligned}$$

where $C(\epsilon)=(\epsilon q)^{\frac{-q'}{q}}q'^{-1}$. Note that

$$q(1+\frac{2}{k}-\frac{3}{q'})=1, \qquad \frac{2q}{q'}=\gamma.$$

Now if $K''>0$ be such that $a(x)^{\frac{q}{q'}}\le K''$ for every $x\in[0,1]$, then we obtain

$$R\int_0^T\int_0^1\theta^{1+\frac{2}{k}}w^2dxdt \le \epsilon K''R\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dxdt + C(\epsilon)R\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt. \tag{28}$$

Putting the estimate (28) in (27) and using (26), we obtain

$$\begin{aligned}A \ge& (R^3cc_1^3-2R^2c_1^2\tilde{c}e^\xi-\frac{40N\xi^2c_1^2\hat{c}R}{\epsilon}-C(\epsilon)K'R)\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt\\ &+R(cc_1-\epsilon-1)\int_0^T\int_0^1\theta a(x)w_x^2dxdt+\frac{M(1-\alpha-\eta)^2}{4}R\int_0^T\int_0^1\theta\frac{w^2}{x^\beta}dxdt\\ &+R(3-\alpha-\epsilon K''K')\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dxdt.\end{aligned}$$

Now, let $c_1=\frac{4}{c}$. Also let

$$\epsilon=\min\{cc_1-3,\frac{2-\alpha}{K''K'}\}.$$

Thus there exists $R_0=R_0(a,\eta,\lambda,\gamma,c_1,\xi)>0$ such that for all $R\ge R_0$

$$\begin{aligned}A \ge& 3R^3c_1^2\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt+2R\int_0^T\int_0^1\theta a(x)w_x^2dxdt\\ &+\frac{M(1-\alpha-\eta)^2}{4}R\int_0^T\int_0^1\theta\frac{w^2}{x^\beta}dxdt+R\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dxdt.\end{aligned}$$

□

Note that $B=-Rc_1\int_0^T a(1)e^\xi\theta(t)w_x^2(t,1)dt$, therefore from Lemma 4.3, Lemma 4.4 and the inequality (23) we can easily imply that Proposition 5. On the other hand, since $v=e^{-R\sigma}w$, one has

$$v_x^2\le 2e^{-2R\sigma}(w_x^2+R^2c_1^2\theta^2\frac{x^2}{a(x)^2}w^2).$$

So, the left hand of (16) is smaller than

$$\begin{aligned}&(R^3+2R^3c_1^2)\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt+2R\int_0^T\int_0^1\theta a(x)w_x^2dxdt\\ &+\frac{M(1-\alpha-\eta)^2}{8}R\int_0^T\int_0^1\theta\frac{w^2}{x^\beta}dxdt+R\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dxdt.\end{aligned}$$

which is smaller than the left hand of (22), since $c_1\ge 1$. Also, note that $w_x(t,1)=v_x(t,1)e^{R\sigma(t,1)}$, since $v(t,1)=0$. Now, by Proposition 5 we can easily complete the proof of Theorem 4.2. □

Proof of Proposition 4. First, define $\tilde{v}(t,x) := e^{rt}v(t,x)$, where v is the solution of (15). Then $\tilde{v}$ satisfies

$$\begin{cases} \tilde{v}_t + (a(x)\tilde{v}_x)_x + \dfrac{\lambda}{x^\beta}\tilde{v} - r\tilde{v} = 0, & & (t,x) \in Q_T, \\ \tilde{v}(t,1) = 0, & & t \in (0,T), \\ \tilde{v}(t,0) = 0, & \text{in the case } \alpha \in (0,1), & t \in (0,T) \\ (a\tilde{v}_x)(t,0) = 0 & \text{in the case } \alpha \in [1,2), & t \in (0,T), \\ \tilde{v}(T,x) = e^{rT}v_T(x), & & x \in (0,1). \end{cases}$$

Obviously, if the observability inequality (14) is true for $\tilde{v}$, then it is also true for v. Now, observability inequality for $\tilde{v}$ is a consequence of the Carleman estimate (16). For the complete proof, we refer the reader to [10] since the argument here is similar. The parameter r will be defined so that the map $t \mapsto \int_0^1 \tilde{v}^2(t,x)dx$ is nondecreasing. (for detail see [10]). $\square$

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