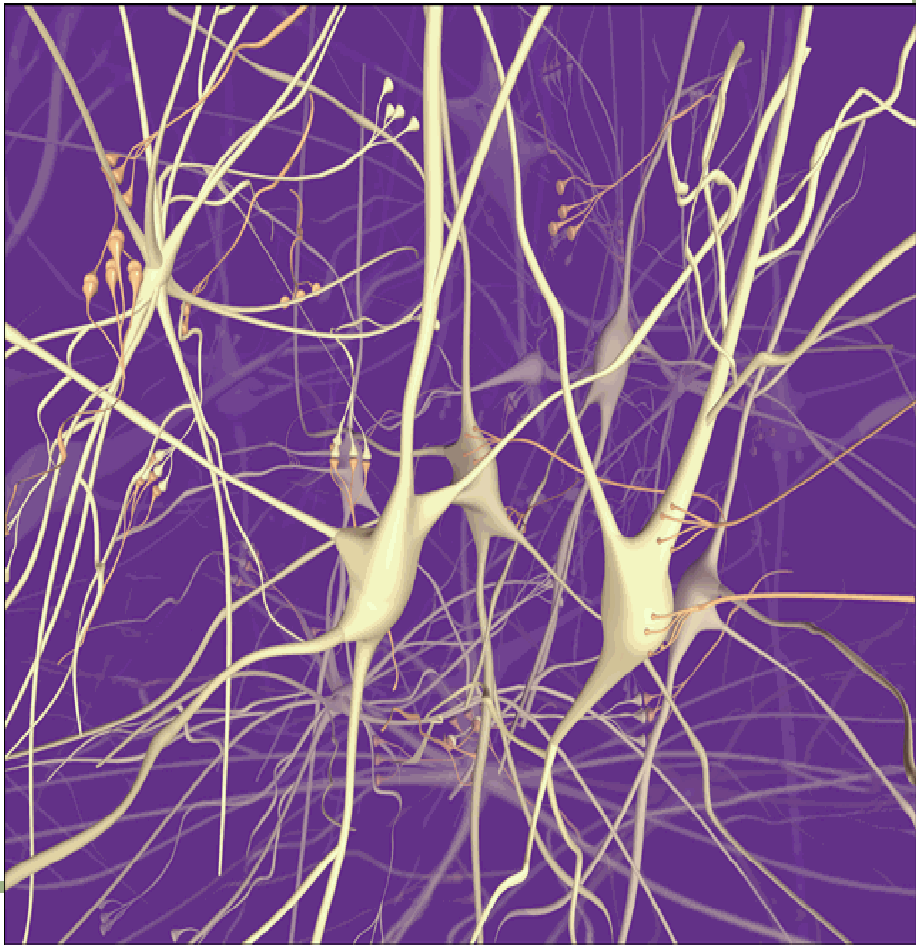
Neural Fields Models

Morteza Fotouhi

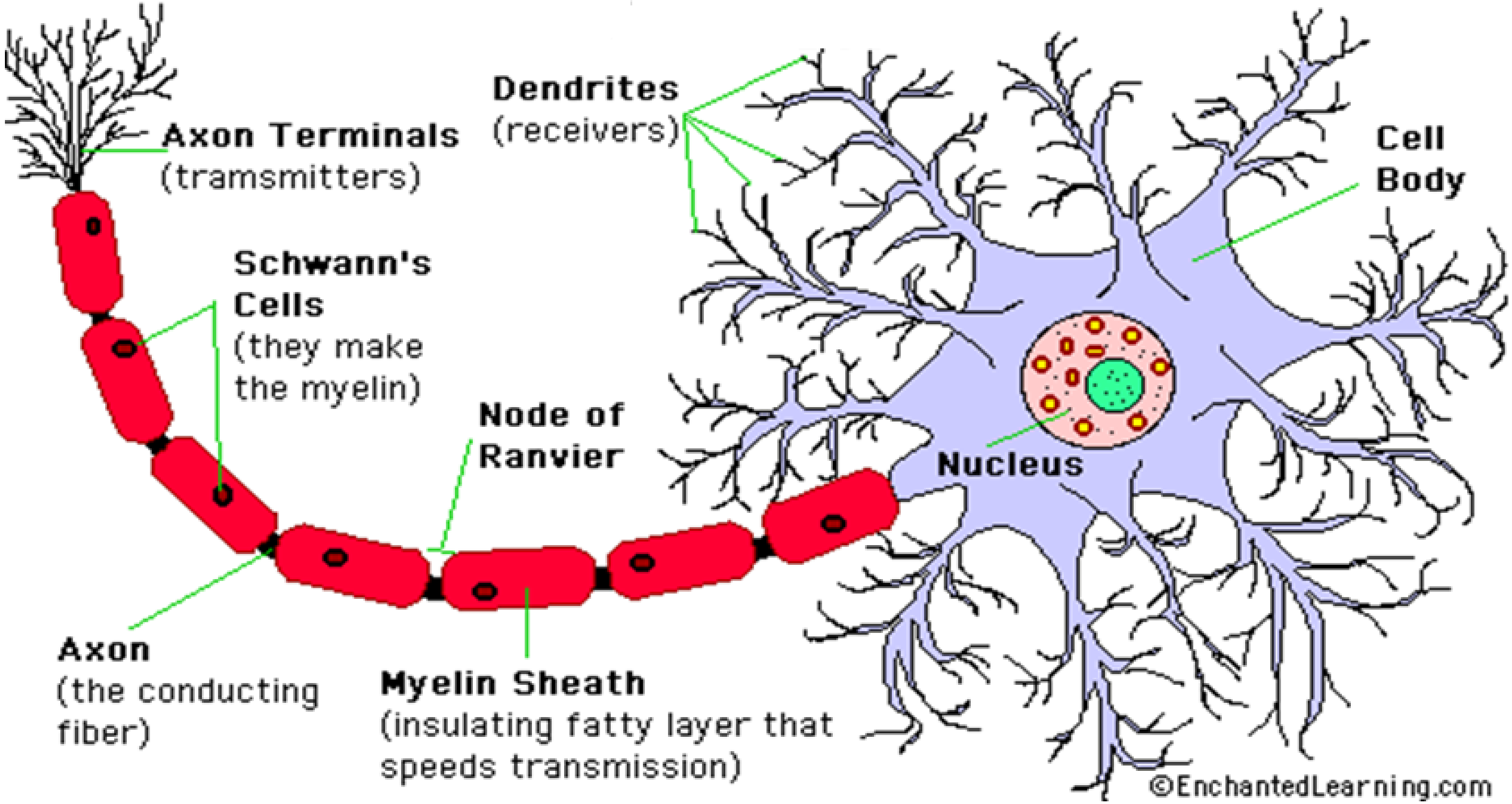
December 2011

Mathematical Sciences Department,
Sharif Univ. of Technology

BioMath group,
School of Mathematics, IPM



Neuron



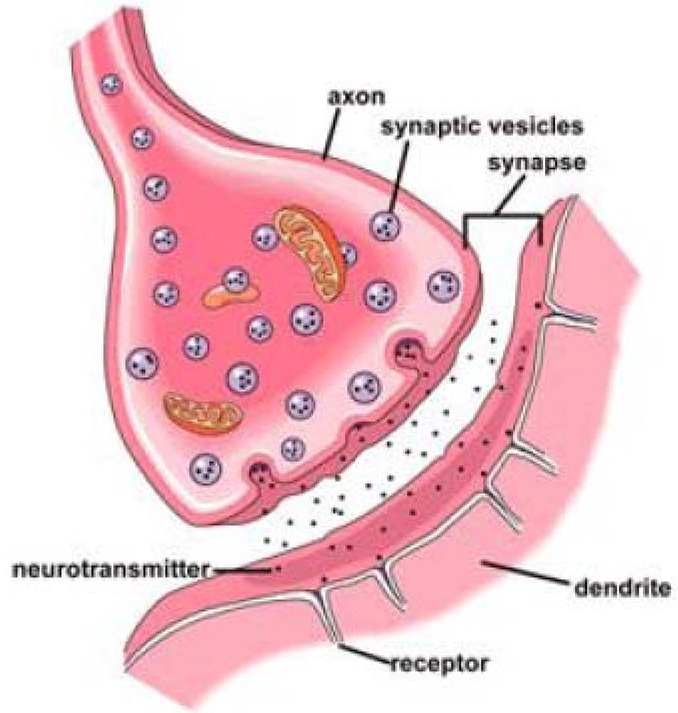
$$C\frac{dV}{dt} = -I_{con} + I_{syn} + I_{ext}$$

Synaptic Processing

$$I_{syn}(t) = g_{syn}(t)(V_{syn} - V(t))$$

Excitatory

$$V_{syn} > V_{rest} \approx -65mV$$



Synaptic Conductance

$$g_{syn}(t) = \overline{g}\eta(t - T), \qquad t \geq T$$

Model 1:

$$\eta(t) = (\frac{1}{\alpha} - \frac{1}{\beta})^{-1}\Big[\exp(-\alpha t) - \exp(-\beta t)\Big]H(t)$$

$$\mathcal{L}\eta = \delta, \qquad \mathcal{L} = (1 + \frac{1}{\alpha}\frac{d}{dt})(1 + \frac{1}{\beta}\frac{d}{dt})$$

Synaptic Conductance

$$g_{syn}(t) = \overline{g}\eta(t-T), \qquad t \geq T$$

Model 1:

$$\eta(t) = (\frac{1}{\alpha} - \frac{1}{\beta})^{-1}\left[\exp(-\alpha t) - \exp(-\beta t)\right]H(t)$$

$$\mathcal{L}\eta = \delta, \qquad \mathcal{L} = (1 + \frac{1}{\alpha}\frac{d}{dt})(1 + \frac{1}{\beta}\frac{d}{dt})$$

Model 2:

$$\eta(t) = \alpha^2 t \exp(-\alpha t)H(t)$$

Synaptic Conductance

$$g_{syn}(t) = \overline{g}\eta(t-T), \qquad t \geq T$$

Model 1:

$$\eta(t) = (\frac{1}{\alpha} - \frac{1}{\beta})^{-1}\left[\exp(-\alpha t) - \exp(-\beta t)\right]H(t)$$

$$\mathcal{L}\eta = \delta, \qquad \mathcal{L} = (1 + \frac{1}{\alpha}\frac{d}{dt})(1 + \frac{1}{\beta}\frac{d}{dt})$$

Model 2:

$$\eta(t) = \alpha^2 t \exp(-\alpha t)H(t)$$

Model 3:

$$\eta(t) = \alpha \exp(-\alpha t)H(t)$$

Synaptic current from a train of spikes

$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest})\sum_m \eta(t - T^m)$$

Synaptic current from a train of spikes

$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest})\sum_m \eta(t - T^m)$$

$$\mathcal{L}I = \overline{g}(V_{syn} - V_{rest})\sum_m \delta(t - T^m)$$

Synaptic current from a train of spikes

$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest})\sum_m \eta(t - T^m)$$

$$\mathcal{L}I = \overline{g}(V_{syn} - V_{rest})\sum_m \delta(t - T^m)$$

$$r(t) = \frac{1}{\Delta}\int_{t-\Delta}^{t} \sum_m \delta(s - T^m) ds$$

Rate Based Model

$$r(t) = \frac{1}{\Delta}\int_{t-\Delta}^{t}\sum_{m}\delta(s-T^m)ds$$

$$\frac{1}{\Delta}\int_{t-\Delta}^{t}\sum_{m}\eta(s-T^m)ds = \frac{1}{\Delta}\int_{t-\Delta}^{t}\sum_{m}\int_{-\infty}^{+\infty}\eta(\tau)\delta(s-T^m-\tau)d\tau ds$$

$$= \int_{-\infty}^{+\infty}\eta(\tau)r(t-\tau)d\tau$$

$$= \int_{-\infty}^{t}\eta(t-\tau)r(\tau)d\tau$$

Rate Based Model

$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest})\int_{-\infty}^{t} \eta(t-\tau)r(\tau)d\tau$$

Rate Based Model

$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest})\int_{-\infty}^{t} \eta(t-\tau)r(\tau)d\tau$$

$$\mathcal{L}I = \overline{g}(V_{syn} - V_{rest})r(t) = w_0 r(t)$$

Rate Based Model

$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest})\int_{-\infty}^{t} \eta(t-\tau)r(\tau)d\tau$$

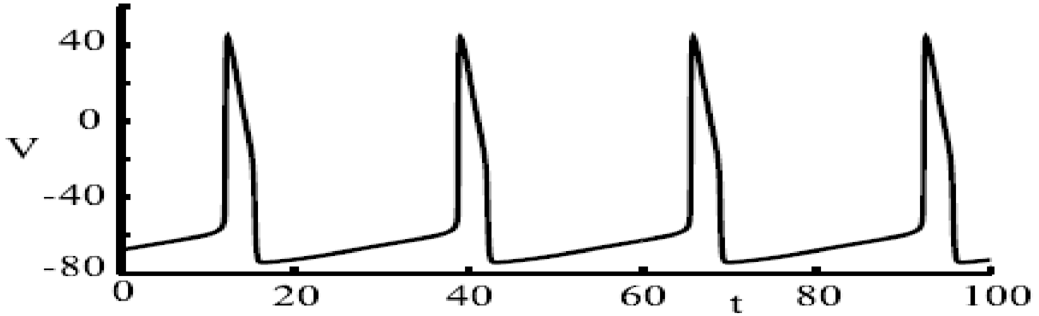
$$\mathcal{L}I = \overline{g}(V_{syn} - V_{rest})r(t) = w_0 r(t)$$

Excitatory: $w_0 > 0$

Inhibitory: $w_0 < 0$

Firing rate function

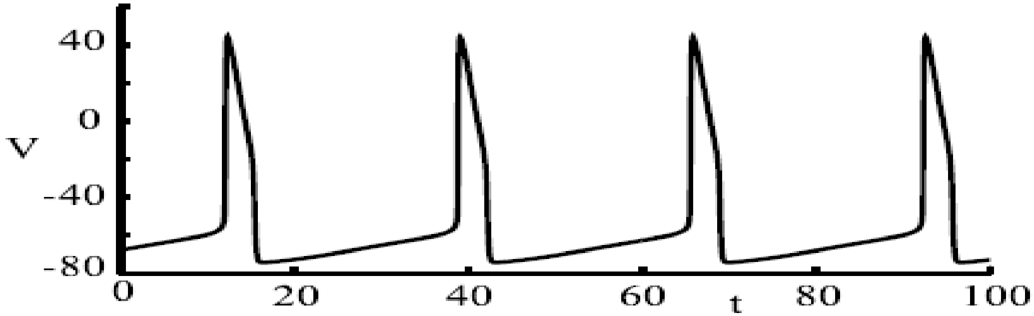
$$T^m = \inf\{t > T^{m-1} : V(t) = h\}$$



Firing rate function

$$T^m = \inf\{t > T^{m-1} : V(t) = h\}$$

$$C\frac{dV}{dt} = -V + I_{syn} \qquad V(T^{m-1}) = V_{rest}$$

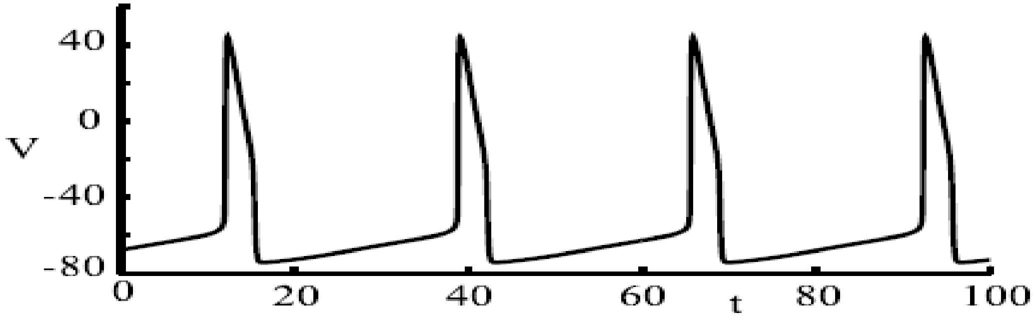


Firing rate function

$$T^m = \inf\{t > T^{m-1} : V(t) = h\}$$

$$C\frac{dV}{dt} = -V + I_{syn} \qquad V(T^{m-1}) = V_{rest}$$

$$T^m - T^{m-1} = C \ln \frac{I_{syn} - V_{rest}}{I_{syn} - h}$$



Firing rate function

$$r(t) = F(I)$$

$$F(I) = \frac{F_0 H(I-h)}{\ln(\frac{I-V_{rest}}{I-h})} \approx (I-h)F_0 H(I-h)$$

Firing rate function

$$r(t) = F(I)$$

$$F(I) = \frac{F_0 H(I-h)}{\ln\left(\frac{I - V_{rest}}{I-h}\right)} \approx (I-h) F_0 H(I-h)$$

$$F(I) = \frac{F_0}{1 + \exp(-\beta(I-h))}$$

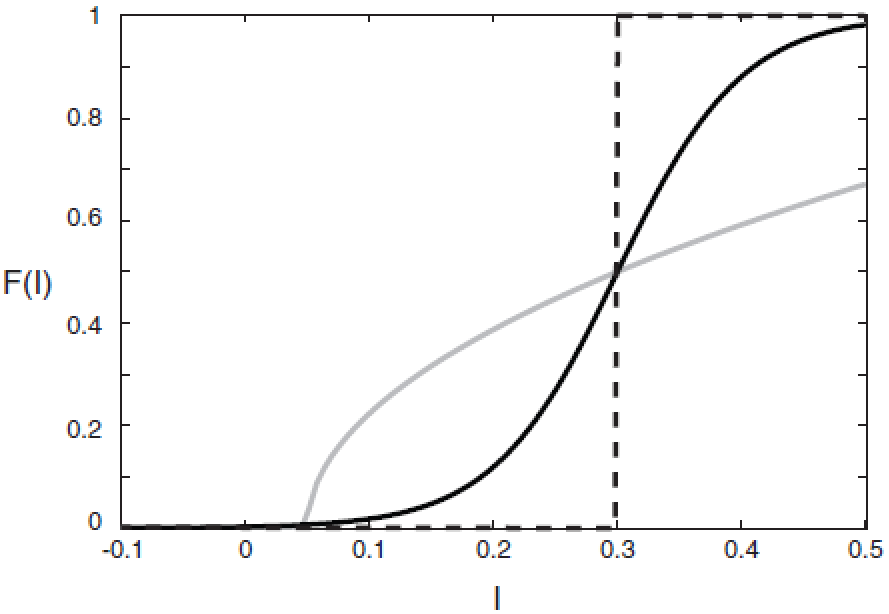
Firing rate function

$$r(t) = F(I)$$

$$F(I) = \frac{F_0 H(I-h)}{\ln(\frac{I - V_{rest}}{I-h})} \approx (I-h)F_0 H(I-h)$$

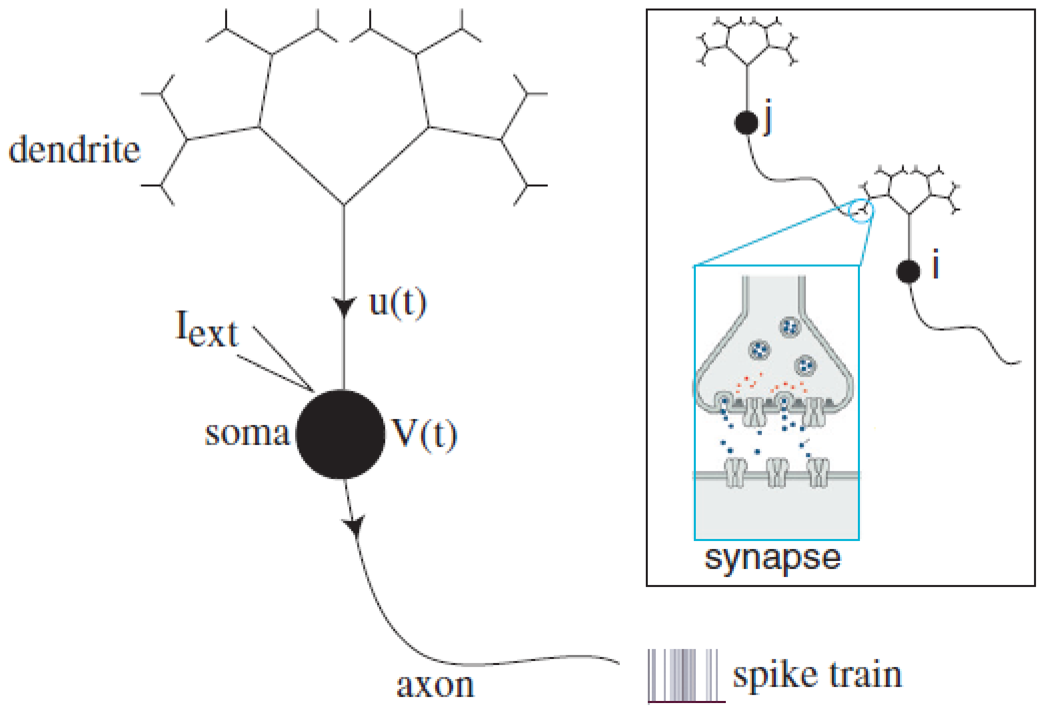
$$F(I) = \frac{F_0}{1 + \exp(-\beta(I-h))}$$

$$F(I) = F_0 H(I-h)$$



Firing Rate Model

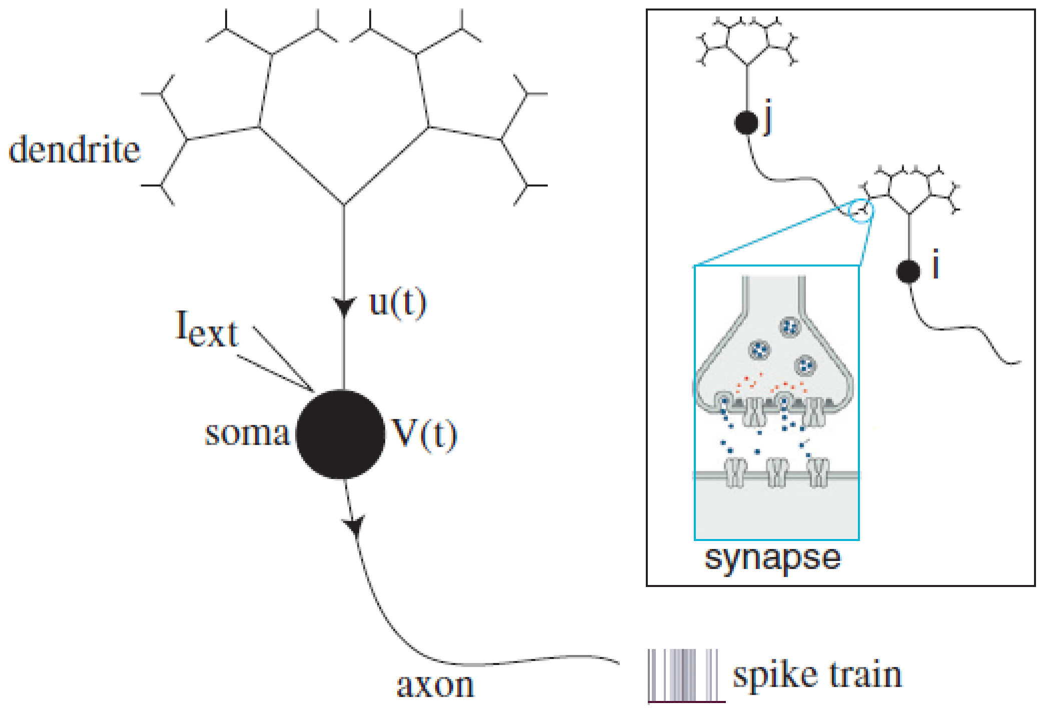
$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest})\int_{-\infty}^{t} \eta(t-\tau)r(\tau)d\tau$$



Firing Rate Model

$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest})\int_{-\infty}^{t} \eta(t-\tau)r(\tau)d\tau$$

$$u_i(t) = \sum_{j=1}^{N} w_{ij} \int_{-\infty}^{t} \eta(t-\tau)r_j(\tau)d\tau$$



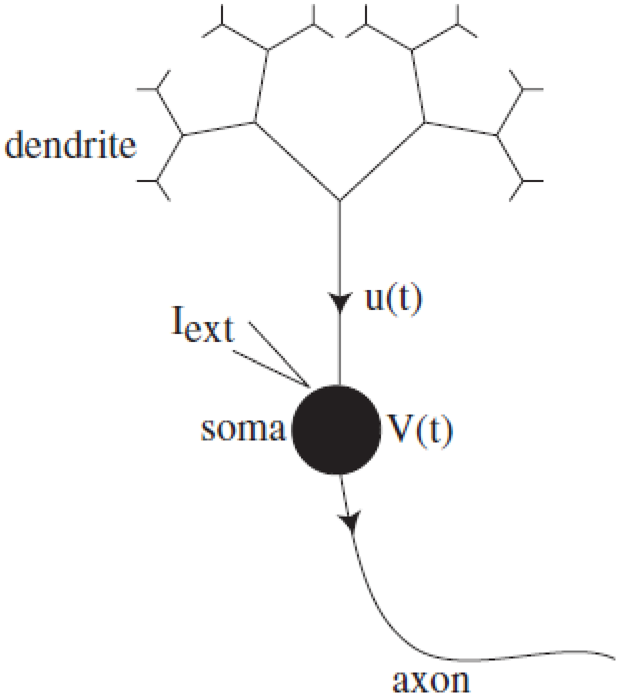
Firing Rate Model

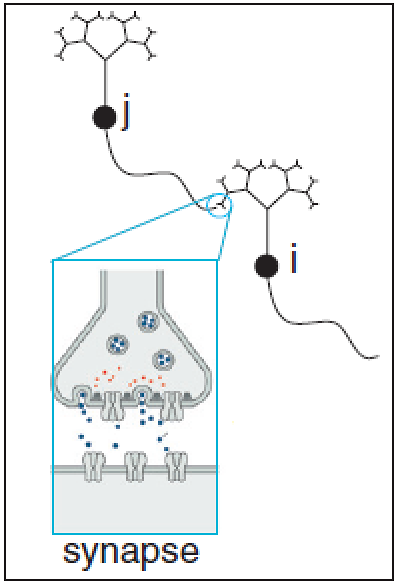
$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest})\int_{-\infty}^{t} \eta(t-\tau) r(\tau) d\tau$$

$$u_i(t) = \sum_{j=1}^{N} w_{ij} \int_{-\infty}^{t} \eta(t-\tau) r_j(\tau) d\tau$$

$$\mathcal{L}u_i = \sum_{j=1}^{N} w_{ij} F(u_j(t))$$

$$\mathcal{L} = 1 + \frac{1}{\alpha}\frac{d}{dt}$$





Cable Theory for the Dendrite

$$\frac{\partial V(x,t)}{\partial t} = -\frac{V(x,t)}{\tau} + D\frac{\partial^2 V(x,t)}{\partial x^2} + r_m I(x,t)$$

$$-\frac{1}{r}\frac{\partial v}{\partial x}(0,t) = \sigma\Big(v(0,t) - V(t)\Big)$$

$$I_{syn} = \sigma\Big(v(0,t) - V(t)\Big)$$

Dendrite processing

$$V(0,t) = r_m \int_{-\infty}^{t} \int_{0}^{+\infty} G(0,y,t-s) I(y,s)\, dy ds$$
$$- \sigma r \int_{-\infty}^{t} G(0,0,t-s)\Big(v(0,s) - V(s)\Big) ds$$

$$G(x,y,t) = G_0(x-y,t) + G_0(x+y,t)$$

$$G_0(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{t}{\tau} - \frac{x^2}{4Dt}\right)$$

Dendrite Filtering of Synaptic Input

$$I_{syn}(t) = \sigma r_m \int_{-\infty}^{t} \int_{0}^{+\infty} G(0, y, t-s) I(y, s)\, dy ds$$

Dendrite Filtering of Synaptic Input

$$I_{syn}(t) = \sigma r_m \int_{-\infty}^{t} \int_{0}^{+\infty} G(0, y, t-s) I(y, s)\, dy ds$$

$$I(x,t) = \delta(x - z) w_0 \sum_m \eta(t - T_m)$$

Dendrite Filtering of Synaptic Input

$$I_{syn}(t) = \sigma r_m \int_{-\infty}^{t} \int_{0}^{+\infty} G(0, y, t - s) I(y, s)\, dy ds$$

$$I(x, t) = \delta(x - z) w_0 \sum_m \eta(t - T_m)$$

$$I_{syn}(t) = \sigma r_m w_0 \sum_m \int_0^{t - T^m} G(0, z, t - T^m - s) \eta(s) ds = \sum_m \Phi(t - T^m)$$

Dendrite Filtering of Synaptic Input

$$I_{syn}(t) = \sigma r_m \int_{-\infty}^{t} \int_{0}^{+\infty} G(0, y, t-s) I(y, s)\, dy ds$$

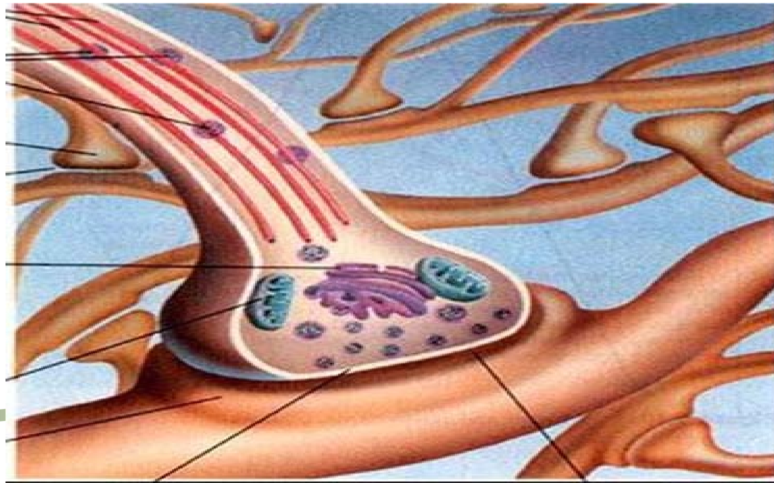
$$I(x,t) = \delta(x-z) w_0 \sum_m \eta(t - T_m)$$

$$I_{syn}(t) = \sigma r_m w_0 \sum_m \int_0^{t-T^m} G(0, z, t - T^m - s) \eta(s) ds = \sum_m \Phi(t - T^m)$$

$$I_{syn}(t) = \overline{g}(V_{syn} - V_{rest}) \sum_m \eta(t - T^m)$$

History Dependent Effect

$$I_{syn}(t) = w_0 \sum_m A(T^m)\eta(t - T^m)$$



History Dependent Effect

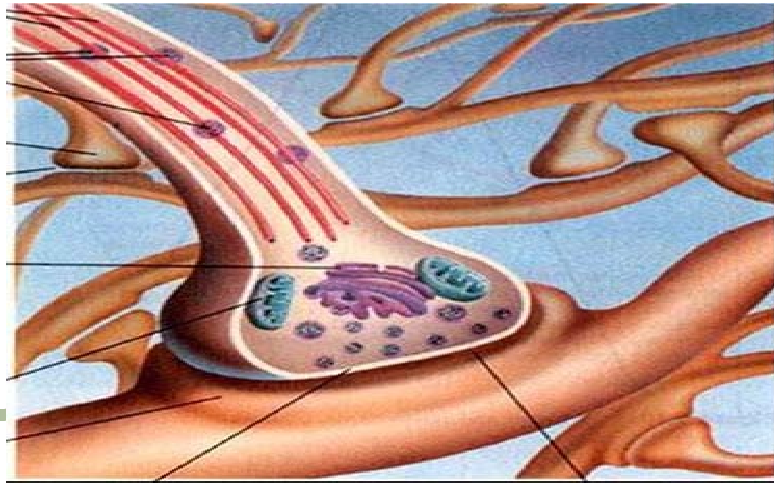
$$I_{syn}(t) = w_0 \sum_m A(T^m)\eta(t - T^m)$$

Depression: $(\gamma < 1)$

$$A \rightarrow \gamma A$$

Facilitation: $(\gamma > 1)$

$$A \rightarrow A + \gamma - 1$$



History Dependent Effect

$$I_{syn}(t) = w_0 \sum_m A(T^m)\eta(t - T^m)$$

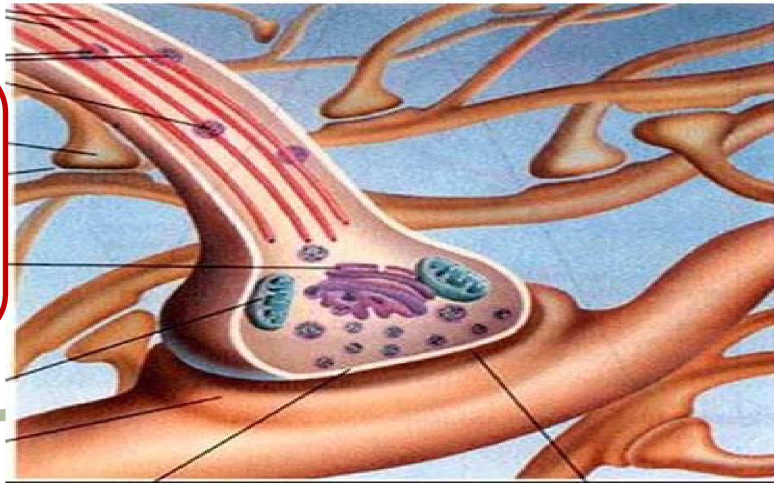
Depression: $(\gamma < 1)$

$$A \rightarrow \gamma A$$

Facilitation: $(\gamma > 1)$

$$A \rightarrow A + \gamma - 1$$

$$\frac{dA}{dt} = \frac{1 - A}{\tau_A} - (1 - \gamma)\sum_m \left[A(T^m)\right]^\beta \delta(t - T^m)$$



History Dependent Effect

$$I_{syn}(t) = w_0 \sum_m A(T^m)\eta(t - T^m)$$

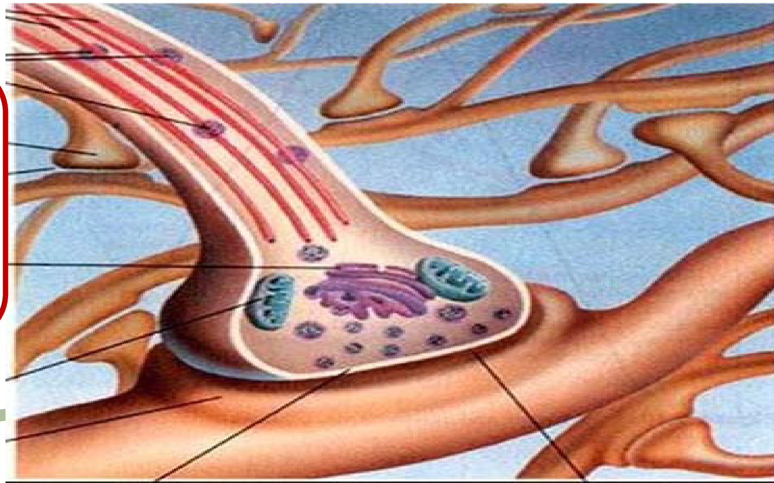
Depression: $(\gamma < 1)$

$$A \to \gamma A \qquad \beta = 1$$

Facilitation: $(\gamma > 1)$

$$A \to A + \gamma - 1 \quad \beta = 0$$

$$\frac{dA}{dt} = \frac{1 - A}{\tau_A} - (1 - \gamma)\sum_m \left[A(T^m)\right]^{\beta} \delta(t - T^m)$$



Synaptic Depression and Facilitation

$$\frac{dA_{ij}}{dt} = \frac{1 - A_{ij}}{\tau_A} - (1-\gamma)\left[A_{ij}(t)\right]^{\beta} r_j(t)$$

$$u_i(t) = \sum_{j=1}^{N} w_{ij} \int_{-\infty}^{t} \eta(t-\tau) A_j(\tau) r_j(\tau) d\tau$$

Synaptic Depression and Facilitation

$$\frac{dA_{ij}}{dt} = \frac{1 - A_{ij}}{\tau_A} - (1 - \gamma)\left[A_{ij}(t)\right]^{\beta} r_j(t)$$

$$u_i(t) = \sum_{j=1}^{N} w_{ij} \int_{-\infty}^{t} \eta(t - \tau) A_j(\tau) r_j(\tau) d\tau$$

$$\mathcal{L} u_i = \sum_{j=1}^{N} w_{ij} A_j(t) F(u_j(t))$$

Adaptation Model

$$C\frac{dV}{dt} = -I_{con} + u_i - c_i$$

Adaptation Model

$$C\frac{dV}{dt} = -I_{con} + u_i - c_i$$

$$r_i = F(u_i - c_i)$$

Adaptation Model

$$C\frac{dV}{dt} = -I_{con} + u_i - c_i$$

$$r_i = F(u_i - c_i)$$

$$\mathcal{L}u_i = \sum_{j=1}^{N} w_{ij} F(u_j - c_j)$$

Adaptation Model

$$C\frac{dV}{dt} = -I_{con} + u_i - c_i$$

$$r_i = F(u_i - c_i)$$

$$\mathcal{L}u_i = \sum_{j=1}^{N} w_{ij}F(u_j - c_j)$$

$$\frac{dc_i}{dt} = -\frac{c_i}{\tau_c} + \gamma_c F(u_i - c_i)$$

Axonal Propagation Delay Model

$$I_{syn}(t) = w_0 \sum_m \eta(t - T^m - \delta) = w_0 \int_{-\infty}^{t} \eta(t-\tau) r(\tau - \delta) d\tau$$

Axonal Propagation Delay Model

$$I_{syn}(t) = w_0 \sum_m \eta(t - T^m - \delta) = w_0 \int_{-\infty}^{t} \eta(t-\tau) r(\tau - \delta) d\tau$$

$$u_i(t) = \sum_{j=1}^{N} w_{ij} \int_{-\infty}^{t} \eta(t-\tau) r_j(\tau - \delta_{ij}) d\tau$$

Axonal Propagation Delay Model

$$I_{syn}(t) = w_0 \sum_m \eta(t - T^m - \delta) = w_0 \int_{-\infty}^{t} \eta(t-\tau) r(\tau - \delta) d\tau$$

$$u_i(t) = \sum_{j=1}^{N} w_{ij} \int_{-\infty}^{t} \eta(t-\tau) r_j(\tau - \delta_{ij}) d\tau$$

$$\mathcal{L} u_i = \sum_{j=1}^{N} w_{ij} F(u_j(t - \tau_{ij}))$$

Neural Field Model

$$(1 + \frac{1}{\alpha}\frac{d}{dt})u_i = \sum_{j=1}^{N} w_{ij} F(u_j(t))$$

$$(1 + \frac{1}{\alpha}\frac{d}{dt})u(x,t) = \int_{-\infty}^{+\infty} w(x,y) F(u(y,t)) dy$$

Neural Field Model

$$(1+\frac{1}{\alpha}\frac{d}{dt})u_i = \sum_{j=1}^{N} w_{ij} F(u_j(t))$$

$$(1+\frac{1}{\alpha}\frac{d}{dt})u(x,t) = \int_{-\infty}^{+\infty} w(x,y)F(u(y,t))dy$$

Homogeneous Field $w(x,y) = w(\left|x-y\right|)$

$$\mathcal{L}u = w * F(u)$$

Neural Field Model

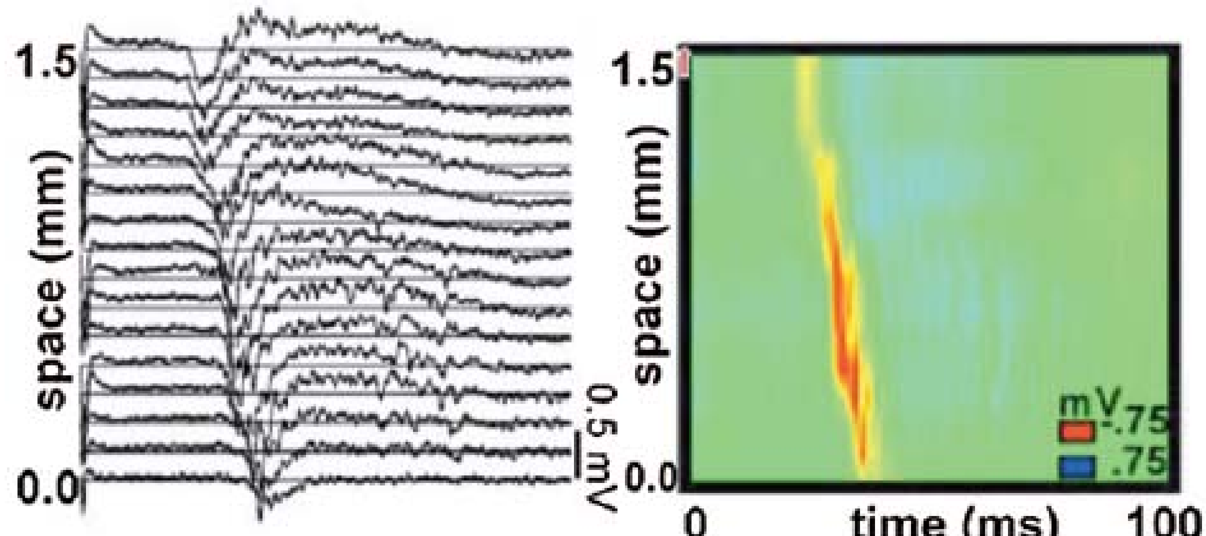
$$(1+\frac{1}{\alpha}\frac{d}{dt})u_i = \sum_{j=1}^{N} w_{ij} F(u_j(t))$$

$$(1+\frac{1}{\alpha}\frac{d}{dt})u(x,t) = \int_{-\infty}^{+\infty} w(x,y)F(u(y,t))dy$$

Homogeneous Field $\quad w(x,y) = w(\left|x-y\right|)$

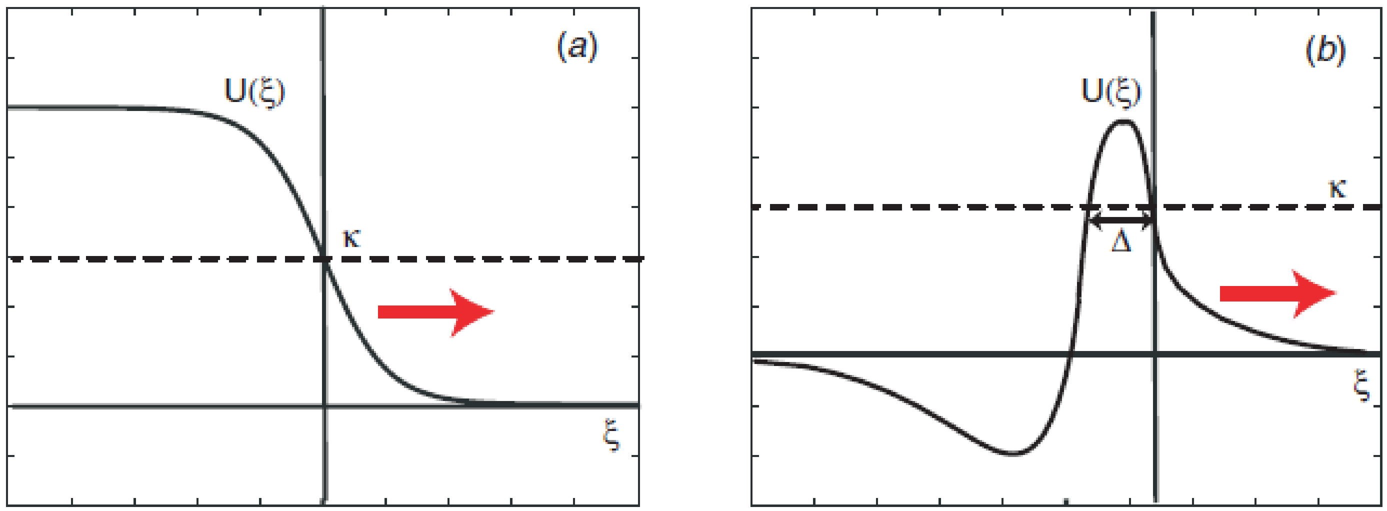
$$\boxed{\mathcal{L}u = w * F(u)} \Rightarrow \boxed{u = \eta \otimes w * F(u)}$$

Traveling Waves



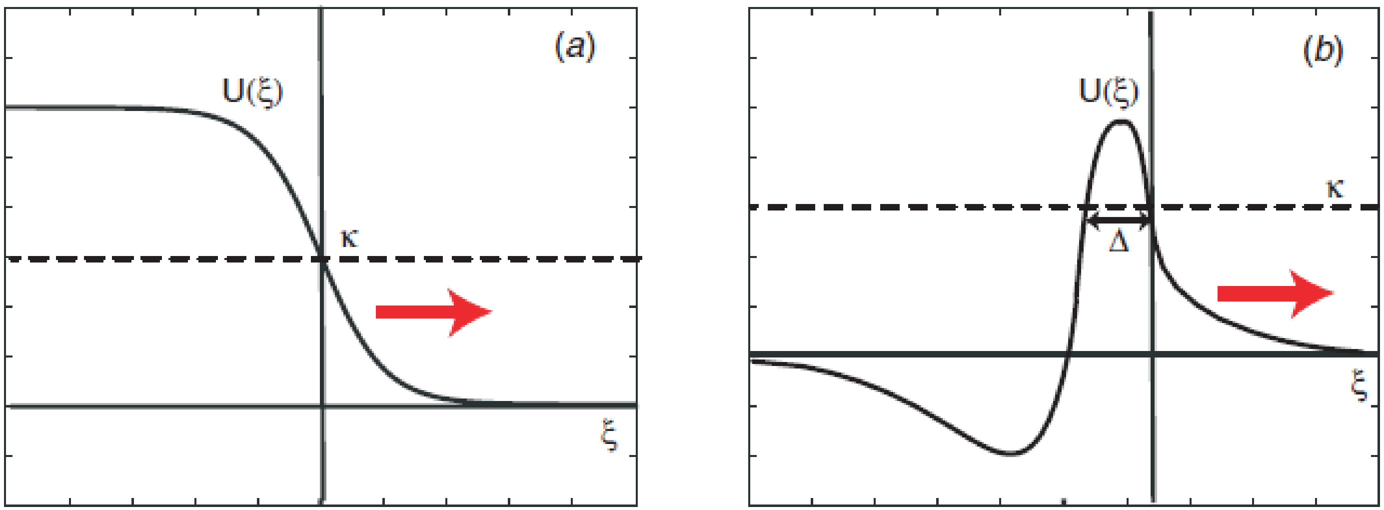
$$\frac{\partial}{\partial t}u(x,t) = -u(x,t) + \int_{-\infty}^{+\infty} w(x-y)F(u(y,t))dy$$

Traveling Waves



$$u(x,t) = U(x - ct)$$

Traveling Waves



$$u(x,t) = U(x - ct)$$

$$\xi = x - ct \quad \Rightarrow \quad -cU'(\xi) + U(\xi) = \int_{-\infty}^{0} w(\xi - \eta)d\eta = W(\xi)$$

Stability

$$U_t = \mathcal{F}(U), \qquad \mathcal{F} : \mathcal{X} \to \mathcal{X}$$

The solution $\mathcal{F}(U_0(x)) = 0$ is stable when for every initial condition $u(x,0)$ enough close to $U_0(x)$, the solution satisfies

$$\Vert u(x,t) - U_0(x) \Vert_{\mathcal{X}} < \delta$$

Linear Stability

$$V_t = \mathcal{L} V, \quad \mathcal{L} = \mathcal{F}'(U_0) : \mathcal{X} \to \mathcal{X}$$

$$Spec(\mathcal{L}) = \{\lambda \,|\, (\mathcal{L} - \lambda I)^{-1} \; is \; not \; a \; bounded \; operator\}$$

Linear Stability

$$V_t = \mathcal{L}V, \qquad \mathcal{L} = \mathcal{F}'(U_0) : \mathcal{X} \to \mathcal{X}$$

$$Spec(\mathcal{L}) = \{\lambda \,|\, (\mathcal{L} - \lambda I)^{-1} \textit{ is not a bounded operator}\}$$

Stability Condition:

$$\max\{\mathrm{Re}\,\lambda : \lambda \in Spec(\mathcal{L})\} \leq -\sigma < 0$$

Stability of Traveling Wave

$$u(x,t) = U(x - ct, t)$$

$$U_t = cU_\xi - U + \int_{-\infty}^{+\infty} w(\xi - \eta) F(U(\eta,\mathrm{t}))\, d\eta$$

Stability of Traveling Wave

$$u(x,t) = U(x - ct, t)$$

$$U_t = cU_\xi - U + \int_{-\infty}^{+\infty} w(\xi - \eta) F(U(\eta,\mathrm{t}))\, d\eta$$

$$\mathcal{L}V = cV_\xi - V + \int_{-\infty}^{+\infty} w(\xi - \eta) F'(U(\eta))\mathrm{V}(\eta)\, d\eta$$

Stability of Traveling Wave

$$u(x,t) = U(x - ct, t)$$

$$U_t = cU_\xi - U + \int_{-\infty}^{+\infty} w(\xi - \eta) F(U(\eta,\mathrm{t}))\, d\eta$$

$$\mathcal{L}V = cV_\xi - V + \int_{-\infty}^{+\infty} w(\xi - \eta) F'(U(\eta))\mathrm{V}(\eta)\, d\eta$$

$$\mathcal{L}(U_\xi) = 0$$

Stability of Traveling Wave

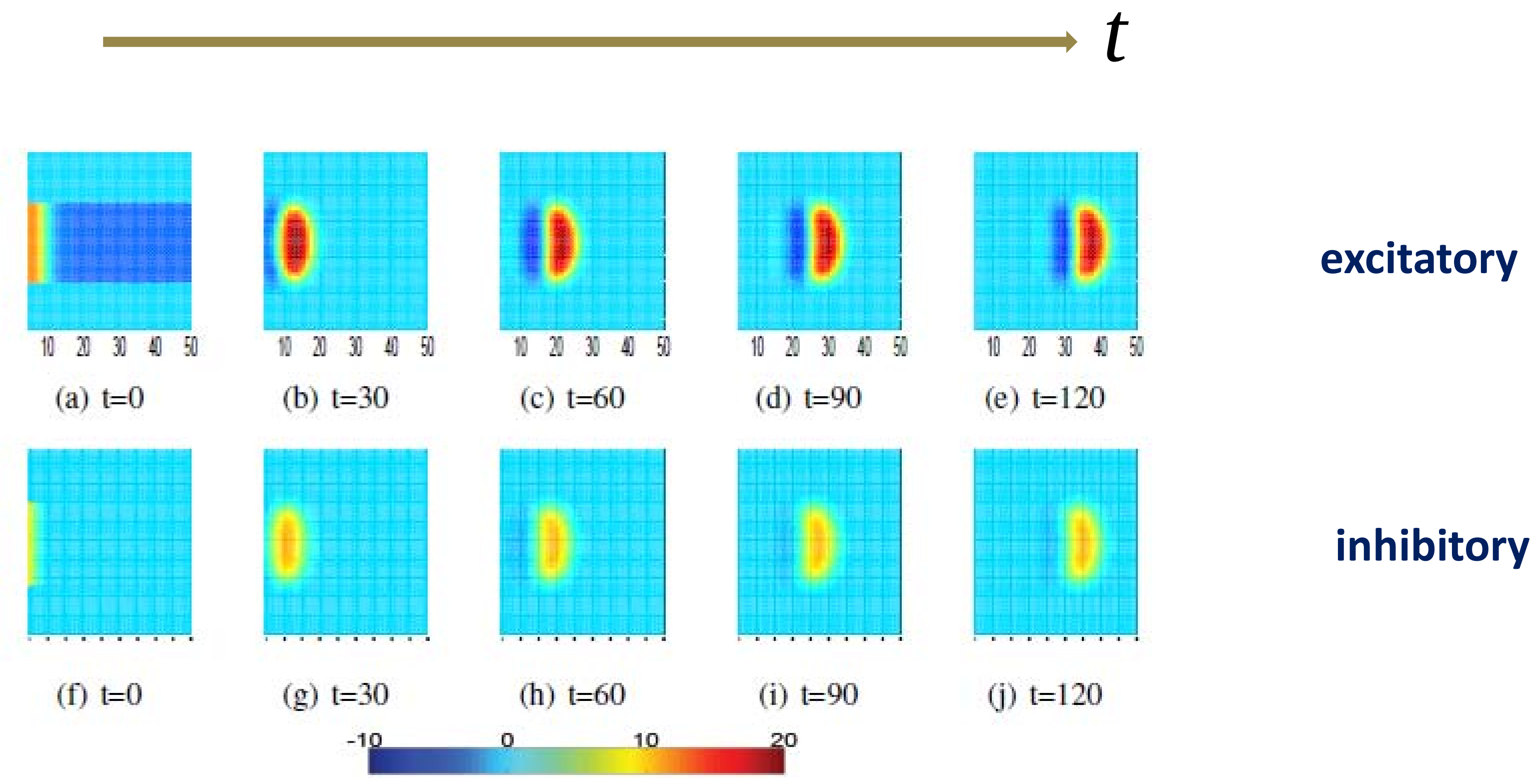
$$u(x,t) = U(x - ct, t)$$

$$U_t = cU_\xi - U + \int_{-\infty}^{+\infty} w(\xi - \eta) F(U(\eta,\mathrm{t}))\, d\eta$$

$$\mathcal{L}V = cV_\xi - V + \int_{-\infty}^{+\infty} w(\xi - \eta) F'(U(\eta))\mathrm{V}(\eta)\, d\eta$$

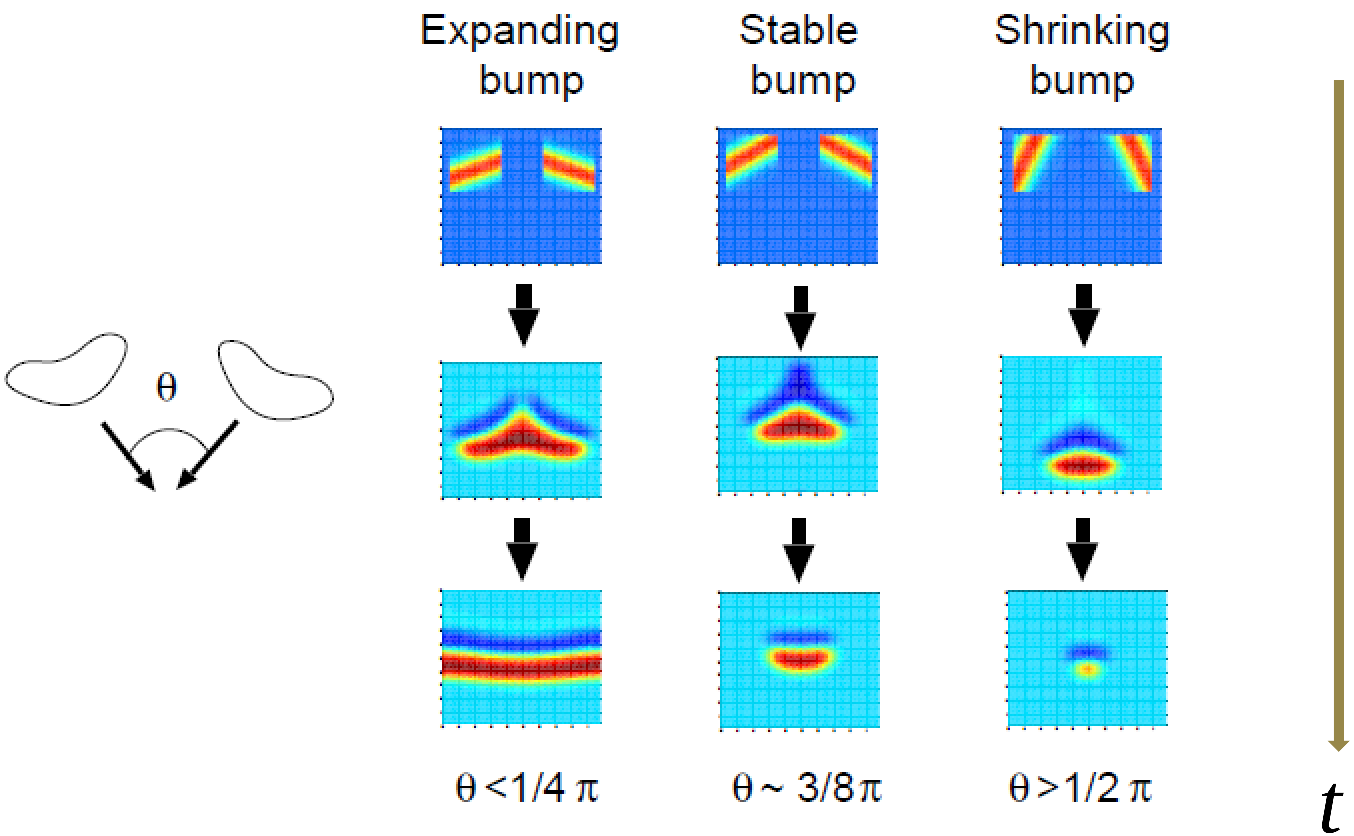
$$\mathcal{L}U_\xi = 0 \Rightarrow 0 \in Spec(\mathcal{L})$$

Traveling bump

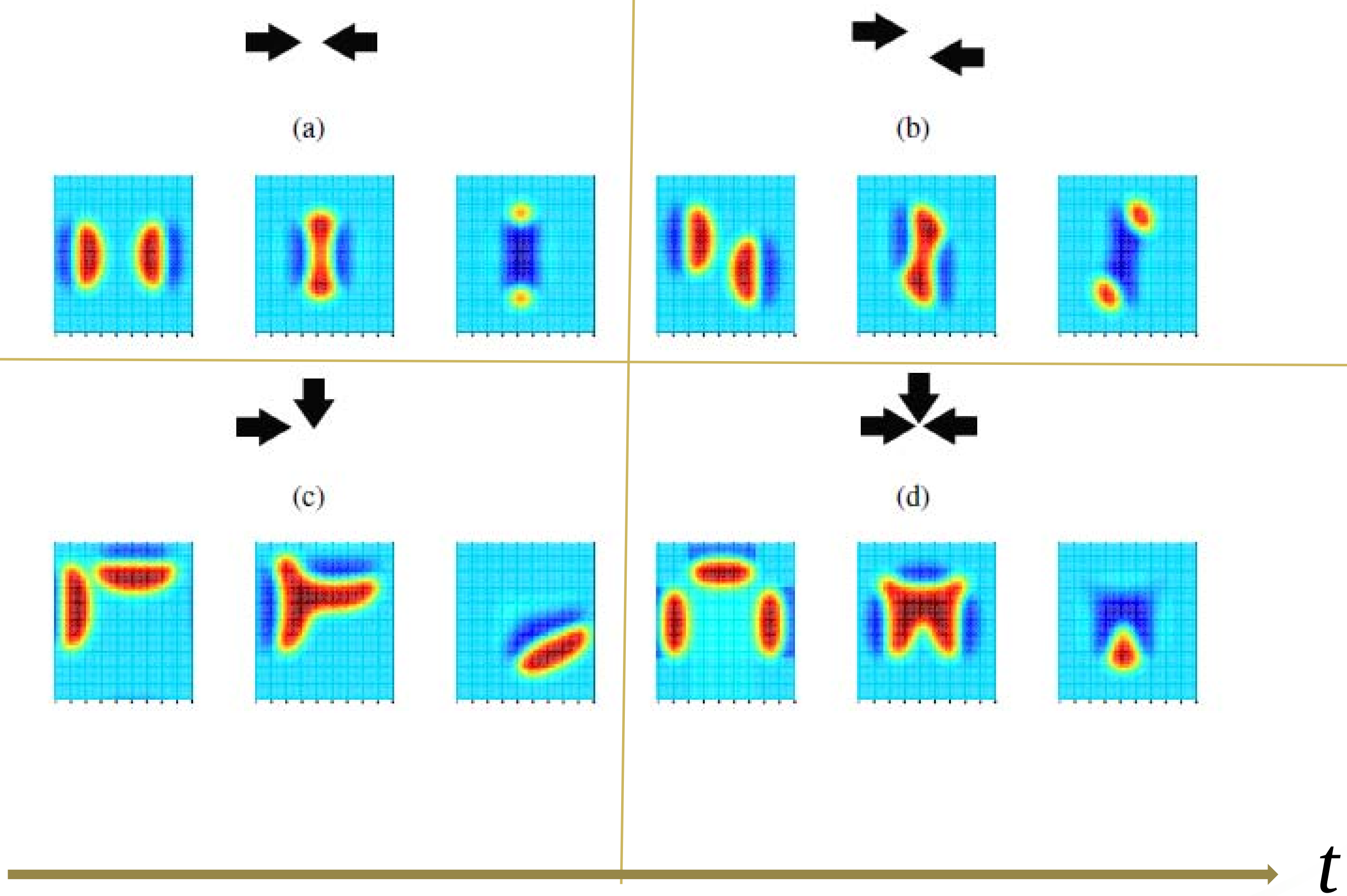


Traveling bumps and their collisions (Lu, Sato & Amari, 2011)

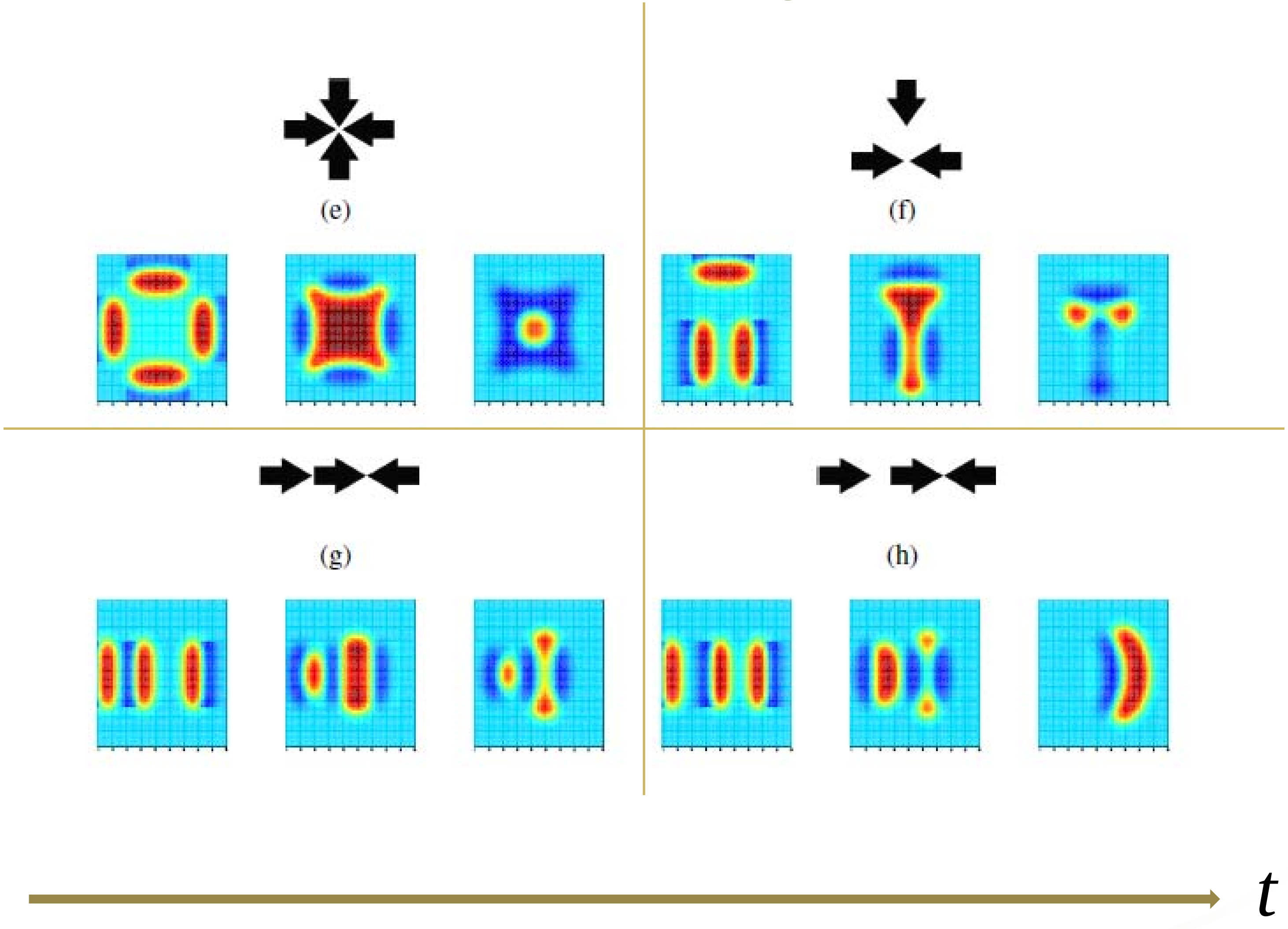
Collision of bumps (1)



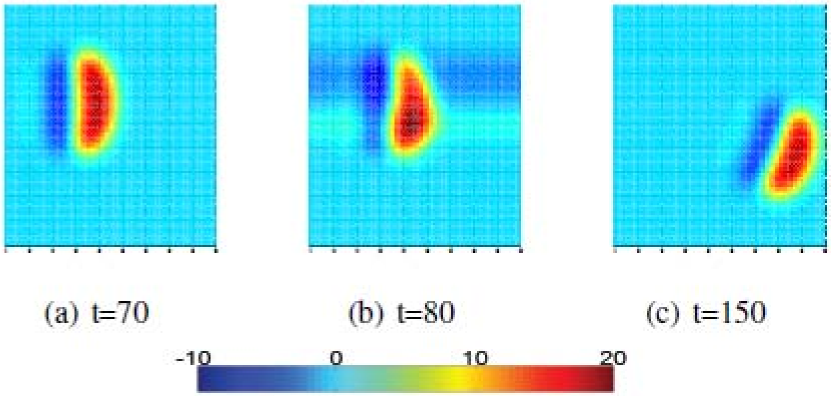
Collision of bumps (2)



Collision of bumps (3)

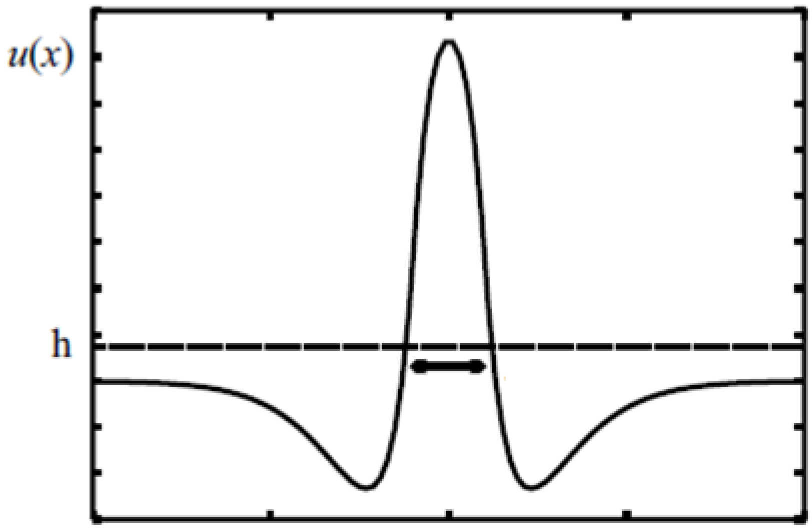


Control of moving bumps



(a) t=70 (b) t=80 (c) t=150

Bump: Stationary Pulse Solution



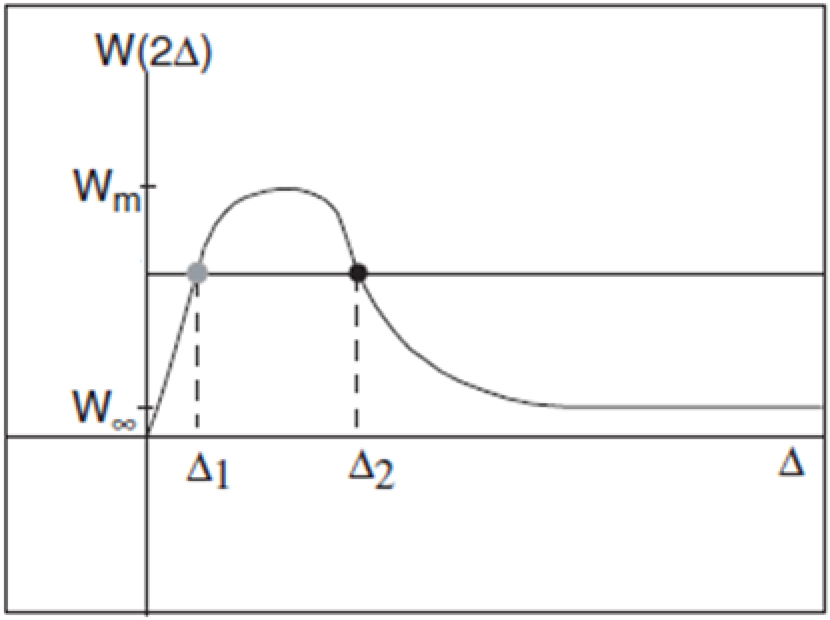
$$\frac{\partial}{\partial t}u(x,t) = -u(x,t) + \int_{-\infty}^{+\infty} w(x-y)F(u(y,t))dy$$

$$u(x,t) = p(x) > h \quad \Leftrightarrow \quad \left|x\right| < \Delta$$

Existence of Bumps

$$p(x) = \int_{|y|<\Delta} w(x-y)dy = W(x)$$

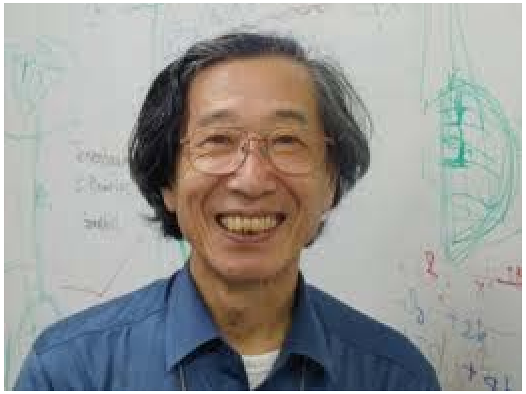
$$p(\pm\Delta) = h$$



Bump with the smaller width is unstable
and the other one is stable.

Reference

Paul C Bressloff,
Department of Mathematics
University of Utah



Shun-ichi Amari
RIKEN Brain Science Institute

Stephen Coombes
School of Mathematical Sciences
University of Nottingham