

# On Dynamic Monopolies of Cubic Graphs

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## Abstract

Majority based recoloring processes over graphs are used to model the spread of fault in distributed computing and communication networks. We consider two of the most common variations: the reversible process and the irreversible process. The reversible majority based recoloring process starts on a graph whose vertices are initially colored black and white and at each round, each vertex recolors itself with the color of the majority of its neighbors. The irreversible process is similar to the reversible process except that it forbids white vertices from becoming black. If the process eventually reaches an all-white global state, the set of initially white vertices is called a dynamic monopoly (or a perfect target set).

In this paper, we study the reversible and the irreversible majority based recoloring processes over 3-regular (cubic) graphs and derive upper and lower bounds for the minimum size of a dynamic monopoly for both of these processes.

**Keywords:** Recoloring Processes, Dynamic Monopolies, Cubic Graphs.

## 1. Introduction

Consider a graph  $G(V, E)$ . Let  $N(v)$  denotes the set of neighbors of node  $v$  and  $\deg(v) = |N(v)|$ . A 0/1 initial assignment is a function  $f_0: V(G) \rightarrow \{0, 1\}$  which specifies the color of node  $v$  at round 0 (0 for black and 1 for white). For any 0/1 initial assignment  $f_0$ , let  $f_\tau: V(G) \rightarrow \{0, 1\}$  be the state of nodes at round  $\tau$ . For each node  $v$ , define  $t(v) = \lceil \frac{\deg(v)+1}{2} \rceil$  to be the threshold of node  $v$ .

In the reversible majority based recoloring process:

$$f_\tau(v) = \begin{cases} 0 & \text{if } \sum_{u \in N(v)} f_{\tau-1}(u) < t(v) \\ 1 & \text{if } \sum_{u \in N(v)} f_{\tau-1}(u) \geq t(v) \end{cases}$$

In the irreversible majority based recoloring process:

$$f_\tau(v) = \begin{cases} 0 & \text{if } f_{\tau-1}(v) = 0 \text{ and } \sum_{u \in N(v)} f_{\tau-1}(u) < t(v) \\ 1 & \text{if } f_{\tau-1}(v) = 1 \text{ or } \sum_{u \in N(v)} f_{\tau-1}(u) \geq t(v) \end{cases}$$

Informally speaking, in the reversible majority based recoloring process, at each round, each node recolors itself with the color of the majority of its neighbors. In the irreversible majority based recoloring process, black vertices change their color with the same update rule, but white vertices never change their color. These two processes are widely used to model the spread of fault in distributed computing and communication networks [1]. We study these two processes on cubic graphs where for each node  $v$ ,  $\deg(v) = 3$  and thus  $t(v) = 2$ .

A 0/1 initial assignment  $f_0$  is called a dynamic monopoly (or perfect target set), if for a finite  $\tau$ ,  $f_\tau(v) = 1$  for all  $v \in V(G)$ , i.e., the dynamics will converge to a steady state of all-white. The cost of a dynamic monopoly  $f_0$ , denoted by  $cost(f_0)$ , is the number of nodes  $v$  with  $f_0(v) = 1$ . The cost of this minimum dynamic monopoly is denoted by  $IDM(G)$  and  $RDM(G)$  respectively for the irreversible and reversible

majority based recoloring processes. In this paper, we want to work on the problem proposed by [2], which asks how small a dynamic monopoly can be. This problem is also called target set selection (See e.g. [3]).

There has been a great interest in studying the irreversible majority based recoloring process, thus many bounds for  $IDM(G)$  are known for general and special types of graphs such as torus, hypercube, butterfly, chordal ring and triangular grid (See e.g. [4,5,6,7,8,9,10,3,1]). Best bounds for the irreversible majority based recoloring process in general graphs are due to Chang and Lyu. They first showed that for a directed graph  $G$ ,  $IDM(G) \leq \frac{23}{27}|V(G)|$  [10]. They improved their upper bound to  $\frac{2}{3}|V(G)|$  for directed graphs and  $\frac{|V(G)|}{2}$  for undirected graphs [9]. In [11], it is shown that, the upper bound for directed graphs with no vertex of in-degree zero (such as strongly connected graphs) can be improved to  $\frac{|V(G)|}{2}$ .

In this paper, we show that this upper bound can be improved to  $\frac{3|V(G)|+2}{8}$  for cubic graphs. For this group of graphs, we prove the lower bound of  $\frac{|V(G)|}{4}$ . We also show that both of these bounds are tight.

The best lower bound for  $RDM(G)$  is due to Fazli et al. [12]. They proved that  $RDM(G) \geq \frac{2|V(G)|}{\Delta(G)+1}$  where  $\Delta(G)$  is the maximum degree of  $G$ . They also found tight examples for this lower bound. Applying their result directly for cubic graphs, leads to the lower bound of  $\frac{|V(G)|}{2}$ . In this paper, we construct 3-regular graphs for which this bound is tight. To the best of our knowledge, there exists no tight upper bound for  $RDM(G)$ . In [12], it is shown that  $RDM(G) \leq \frac{n\Delta(G)(\delta(G)+2)}{4\Delta(G)+(\Delta(G)+1)(\delta(G)-2)}$ , where  $\delta(G)$  is the minimum degree of vertices in  $G$ . In this paper, we show that  $RDM(G) \leq \frac{3|V(G)|}{4} + 2$  for cubic graphs. We also construct 3-regular graphs with  $(G) = \frac{2V(G)}{3}$ .

Finally, it must be mentioned that the size of dynamic monopolies in majority based recoloring processes is very sensitive to the tie breaking laws. Tie breaking laws are used when for a node, the number of its black neighbors and its white neighbors are the same. In [7], four different tie breaking laws are proposed: Prefer-White, Prefer-Black, Prefer-Current and Prefer-Flip. Each of these laws chooses a different action for a vertex in the case of a tie (For example, our general model is Prefer-Black, since vertices become black in case of tie).

Very different observations are carried out for other types of tie breaking laws. For example in [13] the authors found examples for Prefer-Current case with dynamic monopoly of constant size for the reversible majority based recoloring process. Consider that, in this paper ties never happen, because the degrees of all vertices in cubic graphs are odd. However, since our general model is Prefer-Black, only bounds for the processes with this tie breaking law are cited to make the comparison of the results easier.

## 2. Bounds for $IDM(G)$

In this section, we derive an upper bound and a lower bound for  $IDM(G)$  for every cubic graph  $G$ . We prove that for a

cubic graph  $G$ ,  $\frac{|V(G)|}{4} < IDM(G) \leq \frac{3|V(G)|+2}{8}$ . Toward this end, we first show that for a given graph  $G$ ,  $IDM(G)$  is equal to  $G$ 's decycling number.

**Definition:** For a graph  $G$  and  $S \subset V(G)$ , if  $G[V(G) \setminus S]$  is acyclic, then  $S$  is said to be a decycling set of  $G$ . ( $G[A]$  where  $A \subseteq V(G)$ , denotes the induced subgraph of  $G$  by  $A$ ). The size of a smallest decycling set of  $G$  is called the decycling number of  $G$  and is denoted by  $\phi(G)$ .

Let  $f_0$  be a 0/1 initial assignment. Define  $W(f, \tau)$  and  $B(f, \tau)$  as follows:

$$W(f, \tau) = \{v \in V(G) | f_\tau(v) = 1\}$$

$$B(f, \tau) = \{v \in V(G) | f_\tau(v) = 0\}$$

where  $f_\tau$  is calculated according to the irreversible majority based recoloring process recursively from  $f_0$ .

**Lemma 1:** If  $f_0$  is a dynamic monopoly, then  $G[B(f, 0)]$  is acyclic.

**Proof:** Assume that  $G[B(f, 0)]$  has a cycle  $C$ . Each vertex in  $C$  has two adjacent black vertices, therefore they will never become white. Thus it contradicts the fact that  $f_0$  is a dynamic monopoly.

Assume that  $S \subseteq V(G)$ , define 0/1 initial assignment  $f_0^S: V(G) \rightarrow \{0,1\}$  as follows:

$$f_0^S(v) = \begin{cases} 0 & \text{if } v \notin S \\ 1 & \text{if } v \in S \end{cases}$$

**Lemma 2:** If  $G[S]$  is acyclic, then  $f_0^S$  is a dynamic monopoly where  $\bar{S} = V(G) \setminus S$ .

**Proof:** We use induction to prove that  $G[B(f^{\bar{S}}, \tau)]$  for all  $\tau \geq 0$  is acyclic. Assume that at a given time  $\tau$ ,  $G[B(f^{\bar{S}}, \tau)]$  has no cycle. Therefore, there must be at least two vertices  $u, v \in B(f^{\bar{S}}, \tau)$  which have at least two adjacent vertices in  $W(f^{\bar{S}}, \tau)$ . Thus, in time  $\tau + 1$  these vertices become white. We know that in the irreversible process none of the white vertices becomes black again. Therefore,  $G[B(f^{\bar{S}}, \tau + 1)]$  remains acyclic. From the previous discussion we know that at each step at least two black vertices join  $W(f^{\bar{S}}, \tau)$ . Thus, all vertices will eventually become white in at most  $\frac{|V(G)|}{2}$  rounds.

**Theorem 1:** For a cubic graph  $G$ ,  $IDM(G) = \phi(G)$ .

**Proof:** Let  $f_0^*$  be a dynamic monopoly with  $cost(f_0^*) = IDM(G)$ . From Lemma 1, we know  $G[B(f_0^*, 0)]$  is acyclic. Thus  $D = W(f_0^*, 0)$  is a decycling set. Assume that  $D^*$  is a decycling set with the minimum size. From Lemma 2, we know  $f_0^{D^*}$  is a dynamic monopoly. We have:

$$|D^*| \leq |D| = cost(f_0^*) \leq cost(f_0^{D^*}) = |D^*|.$$

Thus  $D^* = cost(f_0^*)$ .

From [14], we know that for every connected graph  $G$  with  $\Delta(G) \leq 3$ , if  $G \neq K_4$  then its decycling number  $\phi(G)$  is less than or equal to  $\frac{|E(G)|+1}{4}$ .

Therefore by using Theorem 1, we can conclude the following theorem.

**Theorem 2:** For a given cubic graph  $G \neq K_4$ :

$$IDM(G) \leq \frac{3|V(G)| + 2}{8}$$

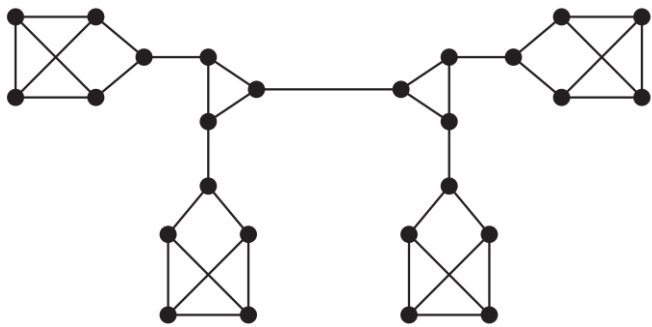


Figure 1. A sample for  $\mathcal{F}(t, k)$ . This family of graphs are tight examples for the proved upper bound of  $IDM(G)$

In the work done by Alon et al. [15], a family of graphs with  $\Delta(G) \leq 3$  and  $a(G) \leq |V(G)| - \frac{|E(G)|}{4} - \frac{1}{4}$  is constructed.  $a(G)$  denotes the maximum size of a subset that induces a forest. If we exclude the vertices of this forest, remaining vertices form a decycling set of  $G$ . Thus in these graphs we have  $\phi(G) \geq \frac{|E(G)|}{4} + \frac{1}{4}$ .

These graphs are shown by  $\mathcal{F}(t, k)$  and consist of  $t$  disjoint triangles and  $k$  disjoint copies of  $K_4$  (a graph with five vertices obtained from  $K_4$  by subdividing an edge) such that the multigraph obtained by contracting each triangle and each copy of  $K_4$  to a single vertex is a tree of order  $t + k$ . In Fig. 1,  $\mathcal{F}(2,4)$  is shown. Infinitely many of these graphs are 3-regular, thus their decycling number is greater than or equal to  $\frac{3|V(G)|+2}{8}$ . Therefore the upper bound of  $\frac{3|V(G)|+2}{8}$  for  $IDM(G)$  is tight.

**Theorem 3:** For infinitely many  $n \in \mathbb{N}$  there exists a cubic graph  $G$  of order  $n$  for which  $IDM(G) = \frac{3n+2}{8}$ .

The next theorem proves the lower bound of  $\frac{|V(G)|}{4}$  for  $IDM(G)$ . This result can also be used as a lower bound for the decycling number of cubic graphs.

**Theorem 4:** For every cubic graph  $G$ , we have  $IDM(G) > \frac{|V(G)|}{4}$

**Proof:** Let  $f_0^*$  be a minimum cost dynamic monopoly of  $G$ . From Lemma 1, we know the graph  $G[B(f^*, 0)]$  is acyclic, so the number of its edges is at most  $|B(f^*, 0)| - 1$ . Therefore,  $|E_G[B(f^*, 0), W(f^*, 0)]| = 3|B(f^*, 0)| - 2|E_G[B(f^*, 0)]| = 3|W(f^*, 0)| - 2|E_G[W(f^*, 0)]|$

where  $E_G[A, B]$  denotes the number of  $G$ 's edges between  $A \subseteq V(G)$  and  $B \subseteq V(G)$  and  $E_G[A]$  shows the number of edges in  $G[A]$ . Then by replacing  $|E_G[B(f^*, 0)]|$  with  $|B(f^*, 0)| - 1$ , we have the following chain of inequalities:

$$\begin{aligned} 3|B(f^*, 0)| - 2(|B(f^*, 0)| - 1) &\leq 3|W(f^*, 0)| - 2|E_G[W(f^*, 0)]| \implies \\ |B(f^*, 0)| + 2 &\leq 3|W(f^*, 0)| \implies \\ |B(f^*, 0)| + |W(f^*, 0)| + 2 &\leq 4|W(f^*, 0)| \implies \\ |V(G)| + 2 &\leq 4|W(f^*, 0)| \implies \\ \frac{|V(G)|}{4} + \frac{1}{2} &\leq |W(f^*, 0)| \implies \\ \frac{|V(G)|}{4} &< |W(f^*, 0)|. \end{aligned}$$

Along with the upper bound, the lower bound is also tight.

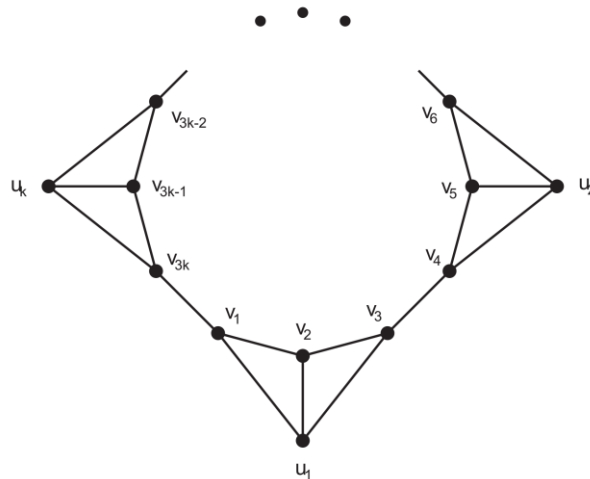


Figure 2. The  $G_k$  graph. This graph is a tight example for the proved lower bound of  $IDM(G)$

**Theorem 5:** For infinite number of  $n \in \mathbb{N}$  there exists a cubic graph  $G$  of order  $n$  in which  $IDM(G) \leq \frac{n}{4} + 1$ .

**Proof:** Consider a graph  $G_k$  of order  $n = 4k$  in which  $V(G_k) = \{v_1, v_2, \dots, v_{3k}\} \cup \{u_1, \dots, u_k\}$ . Let  $V = \{v_1, v_2, \dots, v_{3k}\}$  and  $U = \{u_1, u_2, \dots, u_k\}$ .  $G[V]$  is a cycle. Each vertex  $u_i \in U$  is connected to  $v_{3i-2}$ ,  $v_{3i-1}$  and  $v_{3i}$ . The graph is depicted in Fig. 2. One can see that  $f_0^{U \cup \{v_1\}}$  is a dynamic monopoly of  $G_k$ . Thus  $IDM(G) \leq k + 1 = \frac{n}{4} + 1$ .

### 3. Bounds for $RDM(G)$

In this section, for each cubic graph  $G$ , we derive an upper and a lower bound for  $RDM(G)$ . We prove that for a cubic graph  $G$ ,  $\frac{|V(G)|}{2} \leq RDM(G) \leq \frac{3|V(G)|}{4} + 2$ .

Let  $f_0$  be a 0/1 initial assignment and  $f_\tau$  defined according to the reversible majority based recoloring process. We define  $W(f, \tau)$ ,  $B(f, \tau)$  and  $f_0^S$  like Section 2. The following theorem provides a lower bound for  $RDM(G)$  in general graphs. The proof of this theorem can be found in Theorem 1 of [12].

**Theorem 6:** (Fazli et al. [12]) For every graph,  $RDM(G) \geq \frac{2|V(G)|}{\Delta(G)+1}$ .

This theorem results in a lower bound for cubic graphs which is provided in the following theorem.

**Theorem 7:** For every cubic graph  $G$ ,  $RDM(G) \geq \frac{|V(G)|}{2}$ .

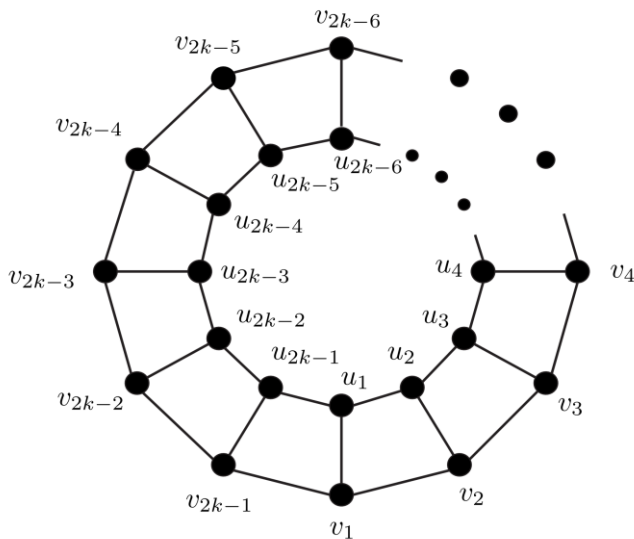


Figure 3. The  $H_k$  graph. This graph is a tight example for the proved lower bound of  $RDM(G)$

The following theorem shows that the lower bound for  $RDM(G)$  is tight.

**Theorem 8:** For infinite number of  $n \in \mathbb{N}$ , there exists a 3-regular graph  $G$  of order  $n$  in which  $RDM(G) = \frac{n}{2}$ .

**Proof:** For an integer  $k \geq 2$ , consider the graph  $H_k$  constructed by two cycles  $u_1, u_2, \dots, u_{2k-1}$  and  $v_1, v_2, \dots, v_{2k-1}$  where we join each vertex  $v_i$  to a corresponding vertex  $u_i$ . This graph is shown in Fig. 3. Define  $S = \{v_1, v_2, \dots, v_{2k-1}\} \cup \{u_1\}$ . One can see that  $f_0^S$  is a dynamic monopoly. Thus  $RDM(G) \leq \frac{n}{2} + 1$ .

Finally, it remains to prove the upper-bound of  $\frac{3|V(G)|}{4} + 2$  for  $RDM(G)$ . In [12], authors show that if  $\alpha |V(H)| \leq RDM(H) \leq \beta |V(H)|$  for every bipartite graph  $H$ , then  $\alpha |V(G)| \leq RDM(G) \leq \beta |V(G)|$  for every graph  $G$ . Therefore it is enough to prove the theorem only for bipartite graphs. So from now on, we assume that the input graph is bipartite.

**Lemma 3:** Assume that each vertex  $v$  with  $f_0(v) = 1$ , has at most one adjacent vertex  $u$  with  $f_0(u) = 0$ . If  $G[B(f, 0)]$  is acyclic, then  $f_0$  is a dynamic monopoly.

**Proof:** We prove the lemma by induction on  $\tau$ . We show that at each step at least one black vertex becomes white, all white vertices will remain white and the induced subgraph by black vertices remains acyclic.

The assertion is trivial at time  $\tau = 0$ . Suppose that the assertion is true at time  $\tau > 0$ . Therefore the induced subgraph by  $B(f, \tau)$  is acyclic, so there exists at least one vertex in  $B(f, \tau)$  for which the number of its adjacent white vertices is at least two. This vertex will become white at time  $\tau + 1$ . By induction hypothesis, at time  $\tau$  there is no white vertex with more than one black neighbor, so all white vertices keep their color. Finally, all white vertices at time  $\tau + 1$  have at most one neighbor in  $G[B(f, \tau)]$ , therefore there is no white vertex at time  $\tau + 1$  with more than one black adjacent vertex. Therefore,  $G[B(f, \tau + 1)]$  remains acyclic.

The following algorithm can provide a dynamic monopoly which satisfies Lemma 3 with low cost.

Let  $M = \emptyset$  and  $i = 0$ . Do the following procedure while  $M \neq V(G)$ :

1. If  $M \neq \emptyset$  then select a vertex  $v$  from  $N(M)$  otherwise select an arbitrary vertex  $v$  from  $V(G)$  ( $N(M)$  is the set of vertices in  $V(G) \setminus M$  which has one neighbor in  $M$ ).
2. Increase  $i$  by one and set  $T_i = \{v\}$ .
3. While there is a vertex  $u$  in  $N(T_i) \setminus M$ , if  $|N(u) \cap N(T_i)| = 0$  and if adding  $u$  to  $T_i$  doesn't create a cycle in  $G[T_i]$ , then add  $u$  to  $T_i$  and set  $f_0(u) = 0$  otherwise add  $u$  to  $M$  and set  $f_0(u) = 1$ .
4. For all vertices  $u \in V(G) \setminus T_i$  from which there exists a path of length at most 2 to a vertex in  $T_i$ , set  $f_0(u)$  to 1 and add  $u$  to  $M$ .

In the  $i$ 'th iteration of the above algorithm, some of the vertices are chosen to be added to  $T_i$  and the value of  $f_0$  for these vertices are set to be 0. The value of  $f_0$  for all vertices with a distance of at most two from the vertices of  $T_i$ , are set to be 1. All of these vertices are marked by adding them to  $M$ . The vertices added to  $M$  will not be considered in further iterations of the algorithm. The algorithm terminates when the value of  $f_0$  is computed for all vertices of  $G$  i.e. when  $M = V(G)$ .

**Lemma 4:** In the first iteration of the algorithm, when  $M = \emptyset$ , we add at most  $4|T_1| + 2$  vertices to  $M$ .

**Proof:** Define  $Q$  to be the set of vertices whose distance to  $T_1$ 's vertices is 2. We show that  $|Q| + |T_1| + |N(T_1)| \leq 4|T_1| + 2$ . Consider that after the first iteration of the algorithm,  $M \subseteq T_1 \cup N(T_1) \cup Q$ . Assume that  $|T_1| = k$ .  $G[T_1]$  is a tree and  $G$  is 3-regular, so  $|N(T_1)| = 3k - 2(k - 1) = k + 2$ . Furthermore, we know from step 3 that each vertex  $v \in N(T_1)$  is in  $M$  and since it is not added to  $T_1$ , then 1) adding  $v$  to  $T_1$  creates a cycle in  $T_1$  or 2)  $|N(v) \cap N(T_1)| \neq \emptyset$ . In case 1,  $v$  has at least 2 neighboring edges to vertices of  $T_1$  and in case 2, it has at least one neighboring edge to  $N(T_1)$  and another to  $T_1$ . Thus each vertex  $v \in N(T_1)$  has at most one neighbor in  $Q$ . Since each vertex in  $Q$  has at least one neighbor in  $N(T_1)$ ,  $|Q| \leq |N(T_1)| = k + 2$ . This leads to  $|Q| + |T_1| + |N(T_1)| \leq k + (k + 2) + (k + 2) = 3k + 4$ . If  $k > 1$ , there is nothing left to prove because in this case,  $4|T_1| + 2 = 4k + 2 \geq 3k + 4$ . We prove  $k > 1$ .  $G$  is bipartite. So the first and the second vertices verified by the algorithm are in different parts of the graph, therefore they have no shared neighboring vertex. This implies that both of them must be added in step 3 of the first iteration of the algorithm and so  $k \geq 2$ .

**Lemma 5:** In the  $i$ 'th iteration of the algorithm where  $i \geq 2$ , we add at most  $4|T_i|$  vertices to  $M$ .

**Proof:** The proof is very similar to Lemma 4. Again define  $k = |T_i|$  and  $Q$  to be the vertices with distance 2 from  $T_i$ 's vertices. The difference is that this iteration starts when  $M \neq \emptyset$ . Therefore at least one vertex in  $T_i$  must have one neighboring edge to the vertices whose  $f_0$  value is determined until the  $(i - 1)$ 'th iteration of the algorithm. Therefore  $|N(T_i)| \leq 3k - 2(k - 1) - 1 = k + 1$ . Thus  $|Q| \leq k + 1$  and  $|T_i| + |N(T_i)| + |Q| \leq 3k + 2$ . If  $k > 1$ , we have  $4|T_i| = 4k \geq 3k + 2$  and the proof is complete. Assume that  $k = 1$  and  $v$  is the only member of  $T_i$ . All of  $v$ 's neighbors must be in  $M$  at

the start of the  $i$ 'th iteration. Otherwise assume that one of  $v$ 's neighbors say  $u$  was not in  $M$  at the start of the  $i$ 'th iteration.  $G$  is bipartite and  $u$  and  $v$  are in different parts of this graph. This implies that they never have a shared neighboring vertex. Therefore  $u$  must be added to  $T_i$  and this contradicts our assumption that  $|T_i| = k = 1$ . The number of vertices in  $Q$  is at most  $|N(T_i)|$ . Thus  $|Q| \leq 3$ . Then when  $|T_i| = 1$  at most  $|Q| + |T_i| \leq 4 = 4|T_i|$  vertices are added to  $M$ . This completes the proof.

**Theorem 9:** For every 3-regular graph  $G$ ,  $RDM(G) \leq \frac{3|V(G)|}{4} + 2$ .

**Proof:** We use Lemma 3 to prove that  $f_0$  is a dynamic monopoly. First, consider that we do not add a vertex to  $T_i$  if it forms a cycle with  $T_i$ 's vertices. Furthermore, because of step 4 it is guaranteed that the minimum distance between the vertices in  $T_i$  and  $T_j$  for every  $i \neq j$  is at least 2. Thus the graph induced by vertices  $v$  with  $f_0(v) = 0$ , which is  $G[\cup_{i>0} T_i]$  is acyclic.

Now, we use contradiction to prove that no vertex  $v \in V(G)$ , with  $f_0(v) = 1$  exists which has two adjacent vertices say  $u, z \in V(G)$  with  $f_0(u) = f_0(z) = 0$ . Suppose that such vertex exists. WLOG assume that  $u$  becomes black first in iteration  $i$ . So  $v \in T_i$ . Since  $v \in N(u) \cap N(z)$  and in step 3 we do not add vertices to  $T_i$  which have a common adjacent vertex with  $T_i$ 's vertices,  $z$  is not added to  $T_i$  in this step. In step 4 of this iteration, we add  $z$  to  $M$ , since its distance to  $u \in T_i$  is at most 2 and set  $f_0(z) = 1$ . Therefore, the algorithm will not select  $z$  in further iterations to set  $f_0(z)$  to 0 and it contradicts our assumption. Thus,  $f_0$  is a dynamic monopoly.

It remains to prove  $cost(f_0) \leq \frac{3|V(G)|}{4} + 2$ . We know  $f_0(v) = 1$  if and only if  $v \notin \cup_{i \geq 1} T_i$ . Thus we have:

$$cost(f_0) = \sum_{v \in V(G)} f_0(v) = |V(G)| - \left| \bigcup_{i \geq 1} T_i \right|$$

Assume that  $M_i$  is the set of vertices added to  $M$  at the end of  $i$ 'th iteration. So  $\cup_{i \geq 1} M_i = V(G)$ . From Lemma 4 we know that  $M_1 \leq 4|T_1| + 2$ . From Lemma 5, we know that for  $i \geq 2$ ,  $M_i \leq 4|T_i|$ . Therefore we have:

$$\begin{aligned} cost(f_0) &= \sum_{i \geq 1} |M_i| - \sum_{i \geq 1} |T_i| \leq 2 + \sum_{i \geq 1} \frac{3|M_i|}{4} \\ &\leq 2 + \frac{3|V(G)|}{4} \end{aligned}$$

We couldn't find tight examples for Theorem 9. The maximum value of  $RDM(G)$  for which we found tight examples is  $\frac{2|V(G)|}{3}$  which is introduced in Theorem 10.

**Theorem 10:** For infinite number of  $n \in \mathbb{N}$ , there exists a 3-regular graph  $G$  with  $n$  vertices such that  $RDM(G) = \frac{2n}{3}$ .

**Proof:** Consider the graph  $H$  shown in Fig. 4 and an integer  $n$  divisible by 6. Join  $\frac{n}{6}$  copies of  $H$  to form a cycle. In every dynamic monopoly of this graph, there exist at least 4 initially white vertices in each copy of  $H$  (for each cycle of even length in  $H$ , there must be at least two initially white vertices). Therefore, at least  $\frac{2n}{3}$  vertices are needed to be initially white.

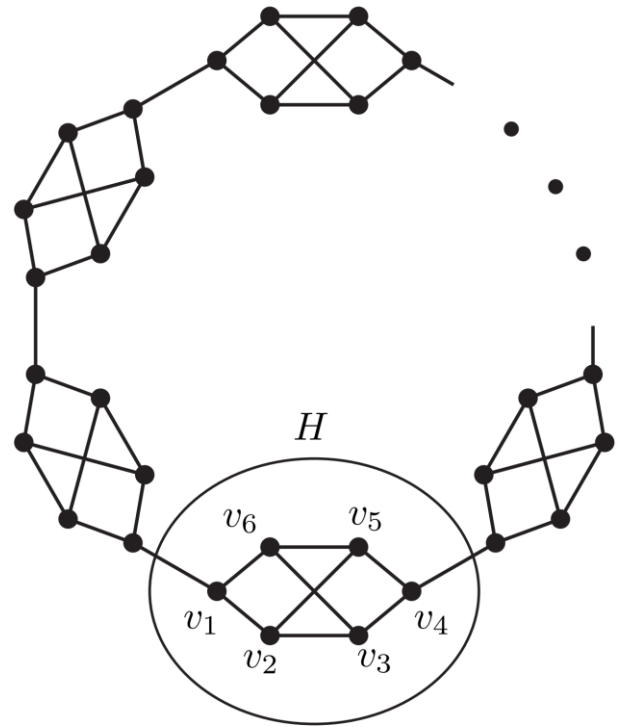


Figure 4. The graph used in Theorem 10.

Table 1. The Maximum Value of  $RDM(G)$  for Connected 3-regular Graphs

$n$	Number of Connected 3-regular Graphs	$Max\{RDM(G)\}$
4	1	3
6	2	4
8	5	6
10	19	6
12	85	8
14	509	8
16	4060	10
18	41301	12

The maximum value of  $RDM(G)$  for all connected 3-regular graphs with at most 20 vertices is calculated and shown in Table 1. Our experiments show that the size of  $RDM(G)$  for none of these graphs is greater than two third of the number of their vertices. Also, Theorem 10 shows that there are infinitely many  $n$  for which there is at least one 3-regular graph of order  $n$  with this proportion of  $RDM(G)$  to number of vertices, admitting that this bound is tight. So, we infer that the following conjecture is true.

**Conjecture 1:** For every 3-regular graph  $G$ ,  $RDM(G) \leq 2|V(G)|/3$ .

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