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On defining numbers of k -chromatic k -regular graphs

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Abstract

In a given graph G , a set S of vertices with an assignment of colors is a defining set of the vertex coloring of G , if there exists a unique extension of the colors of S to a $\chi(G)$ -coloring of the vertices of G . A defining set with minimum cardinality is called a smallest defining set (of vertex coloring) and its cardinality, the defining number, is denoted by $d(G, \chi)$. We study the defining number of regular graphs. Let $d(n, r, \chi = k)$ be the smallest defining number of all r -regular k -chromatic graphs with n vertices, and $f(n, k) = \frac{k-2}{2(k-1)}n + \frac{2+(k-2)(k-3)}{2(k-1)}$. Mahmoodian and Mendelsohn (1999) determined the value of $d(n, k, \chi = k)$ for all $k \leq 5$, except for the case of $(n, k) = (10, 5)$. They showed that $d(n, k, \chi = k) = \lceil f(n, k) \rceil$, for $k \leq 5$. They raised the following question: Is it true that for every k , there exists $n_0(k)$ such that for all $n \geq n_0(k)$, we have $d(n, k, \chi = k) = \lceil f(n, k) \rceil$?

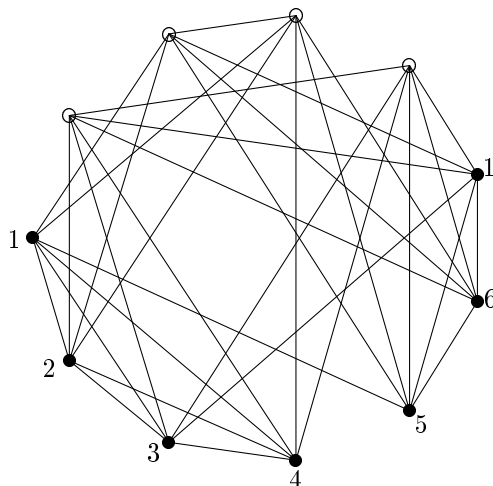
Here we determine the value of $d(n, k, \chi = k)$ for each k in some congruence classes of n . We show that the answer for the question above, in general, is negative. Also here, for $k = 6$ and $k = 7$ the value of $d(n, k, \chi = k)$ is determined except for one single case, and it is shown that $d(10, 5, \chi = 5) = 6$.

KEYWORDS: regular graphs, colorings, defining sets, uniquely completable pre-coloring

1 Introduction

A k -coloring of a graph G is an assignment of k different colors to the vertices of G such that no two adjacent vertices receive the same color. The (vertex) chromatic number of a graph G , denoted by $\chi(G)$, is the minimum number k , for which there exists a k -coloring for G . A graph G with $\chi(G) = k$ is called a k -chromatic graph. In a given graph G , a set of vertices S with an assignment of colors is called a defining set of vertex coloring, if there exists a unique extension of the colors of S to a $\chi(G)$ -coloring of the vertices of G .

EXAMPLE 1.1 In the following figure the set of bold vertices with the assigned colors is a defining set for the given graph.



A defining set with minimum cardinality is called a **smallest defining set** (of vertex coloring) and its cardinality is the **defining number**, denoted by $d(G, \chi)$. For example in the case of a bipartite graph, this number is obviously equal to the number of connected components. For Petersen graph P , $d(P, \chi) = 4$. In Section 5 we will show that the defining set given in Example 1.1 is a smallest defining set. There are some results on the defining numbers in [6].

The concept of a defining set has been studied, to some extent, for block designs, see [8], and also under another name, a **critical set**, for latin squares, see [1] and [3]. In [4] this concept is extended to graphs (see also [3]). Morrill and Pritikin [7] generalized this concept for any k -coloring of graphs for $k \geq \chi(G)$.

We study the defining number of regular graphs. Let $d(n, r, \chi = k)$ be the smallest defining number of all r -regular k -chromatic graphs with

n vertices. Mahmoodian and Mendelsohn [5], determined the value of $d(n, r, \chi = k)$ for each r and for $k = 2$ and 3 . Let $f(n, k) = \frac{k-2}{2(k-1)}n + \frac{2+(k-2)(k-3)}{2(k-1)}$. In [5] it is proved that for $k \leq 5$, $d(n, k, \chi = k) = \lceil f(n, k) \rceil$, except for the case of $(n, k) = (10, 5)$. The following question is raised in [5]:

QUESTION *Is it true that for every k , there exists $n_0(k)$ such that for all $n \geq n_0(k)$, we have $d(n, k, \chi = k) = \lceil f(n, k) \rceil$?*

We determine the value of $d(n, k, \chi = k)$ for the following cases:

- For each even k and for n , such that $n \equiv k + 3 \pmod{2(k-1)}$,
or $n \equiv 4 \pmod{2(k-1)}$.
- For each odd k and for n , such that $n \equiv k + 3 \pmod{2(k-1)}$.

These results show that the answer to the question above, in general, is negative. Also here, for $k = 6$ and 7 the value of $d(n, k, \chi = k)$ are determined, except for the case of $(n, k) = (26, 7)$. Also we show that $d(10, 5, \chi = 5) = 6$.

In the rest of this section we give some necessary definitions and state some results from [5] which will be used later on. The following theorem which is a slight generalization of Theorem 5 in [5], can be proved similarly.

THEOREM A *Let G be a k -regular k -chromatic graph with $|V(G)| = n$, and let S be a defining set for G . Then $|S| = \frac{k-2}{2(k-1)}n + \frac{e+c}{k-1}$, where e is the number of edges of the subgraph induced by S and c is the number of components in the induced subgraph $\langle V(G) - S \rangle$.*

To prove this theorem, as in [5], one may note that the induced subgraph $\langle V(G) - S \rangle$ is a forest and therefore $|E(G - S)| = |V(G - S)| - c$. This implies the assertion. Also since $\langle V(G) - S \rangle$ is a forest, it is 2-colorable, which implies that the chromatic number of $\langle S \rangle$ is at least $k - 2$. Since $\langle S \rangle$ has at least one edge between every two color classes, it has at least $\binom{k-2}{2}$ edges in total, i.e. $e \geq \binom{k-2}{2}$. Then the following corollary follows.

COROLLARY A [5] *Theorem A implies that,*

$$d(n, k, \chi = k) \geq \left\lceil \frac{k-2}{2(k-1)}n + \frac{2+(k-2)(k-3)}{2(k-1)} \right\rceil = \lceil f(n, k) \rceil.$$

If some of the vertices of a k -colorable graph G have pre-assigned colors, then for each of other vertices of G , there is a list of available colors induced by this pre-coloring.

DEFINITION [5] A defining set S with an assignment of colors in graph G , is called a **strong defining set**, if there exists an ordering $\{v_1, v_2, \dots, v_{n-|S|}\}$ of the vertices of $\langle V(G) - S \rangle$ such that, in the induced list of colors in each of the subgraphs $\langle V(G) - S \rangle$, $\langle V(G) - (S \cup \{v_1\}) \rangle$, $\langle V(G) - (S \cup \{v_1, v_2\}) \rangle$, \dots , and $\langle V(G) - (S \cup \{v_1, v_2, \dots, v_{n-|S|}) \rangle$, there exists at least one vertex whose list of colors is of cardinality 1.

LEMMA A [5] *Any defining set of a k -regular k -chromatic graph is strong.*

LEMMA B [5] *Let G be a k -regular k -chromatic graph with n vertices. Then there exists a k -regular graph H with $n + 2(k - 1)$ vertices, and a set $S' \subset V(G)$ of size $d(G, \chi) + (k - 2)$ with assignment of colors for which there exists a unique extension of colors of S' to a k -coloring of H .*

2 Some necessary lemmas

The following results are useful in our discussion.

LEMMA 2.1 *Let G be a k -regular k -chromatic graph with n vertices and let S be a defining set for G . Then the minimum number of edges necessary to determine the color of all vertices is $\binom{k-2}{2} + (n - |S|)(k - 1)$.*

PROOF Since the chromatic number of induced subgraph $\langle S \rangle$ is at least $k - 2$, it has at least $\binom{k-2}{2}$ edges. On the other hand since S is a strong defining set, therefore there must appear $k - 1$ different colors in the neighborhood of each vertex of $V(G) - S$. Thus if we consider these vertices ordered as in the definition of strong defining set, we note that the minimum number of edges necessary is $\binom{k-2}{2} + (n - |S|)(k - 1)$. ■

DEFINITION Let G be a k -chromatic graph and let S be a defining set for G . Then a set $F(S)$ of edges is called **nonessential edges**, if the chromatic number of $G - F(S)$, the graph obtained from G by removing the edges in $F(S)$, is still k , and S is also a defining set for $G - F(S)$.

The following corollary is immediate from Lemma 2.1.

COROLLARY 2.1 *With the conditions of Lemma 2.1 the number of nonessential edges in G is at most $\frac{nk}{2} - \binom{k-2}{2} - (n - |S|)(k - 1)$.*

LEMMA 2.2 *With the conditions of Lemma 2.1 and assuming that S is a proper subset of $V(G)$, the number of edges in the induced subgraph $\langle S \rangle$ satisfies the inequality,*

$$\binom{k-2}{2} \leq |E(\langle S \rangle)| \leq \frac{nk}{2} - (n - |S|)(k - 1) - 1.$$

PROOF We already discussed the left hand side inequality earlier. For the right hand side, we note that since the graph $\langle V(G) - S \rangle$ is a forest, $|E(\langle V(G) - S \rangle)| \leq n - |S| - 1$. The number of edges not in $\langle S \rangle$ is at least equal to $(n - |S|)k - |E(\langle V(G) - S \rangle)| \geq (n - |S|)k - (n - |S| - 1)$. ■

LEMMA 2.3 *A connected k -chromatic graph with the minimum number of edges is necessarily a K_k .*

PROOF Let G be a k -chromatic graph. Then G has at least k vertices. Since between every two color classes there exists at least one edge, G has at least $\binom{k}{2}$ edges. Suppose that G has more than k vertices. Since G is connected, the degree of each vertex is a positive number. We show that in each color class there exists at least one vertex of degree $k - 1$. If the degree of all vertices in a color class, say with color k , is less than $k - 1$, then we can change the color of each vertex to one of the other $k - 1$ colors. This contradicts that G is k -chromatic. So each color class has only one vertex and the graph is a K_k . ■

The following lemma is straight forward.

LEMMA 2.4 *Let G be a k -chromatic graph G which does not contain a K_k . Then G has at least $k + 2$ vertices and $\binom{k+2}{2} - 5$ edges.*

The following lemma shows the importance of the function $f(n, k)$, defined in Corollary A, in our discussion.

LEMMA 2.5 *Let G be a k -regular k -chromatic graph with n vertices and let S be a defining set for G such that $|S| = f(n, k)$. Then $\langle S \rangle$ is a union of a K_{k-2} and $|S| - k + 2$ isolated vertices. Moreover $\langle V(G) - S \rangle$ is a tree.*

PROOF By Theorem A we have $|S| = \frac{k-2}{2(k-1)}n + \frac{e+c}{(k-1)}$. Since $e \geq \binom{k-2}{2}$ and $c \geq 1$ we obtain $|S| \geq \frac{k-2}{2(k-1)}n + \frac{2+(k-2)(k-3)}{2(k-1)} = f(n, k)$. Since the equality holds, c and e must be as small as possible. So $e = \binom{k-2}{2}$ and $c = 1$, i.e. $\langle V(G) - S \rangle$ is a tree. Thus by Lemma 2.3 the statement follows. ■

Let H be a graph with a k -coloring and $\deg_H v \leq k$, for all $v \in V(H)$. For each vertex v of H we call $k - \deg_H v$, the capacity of v . For each i ($1 \leq i \leq k$) let $x_i(H) = \sum (k - \deg_H v)$ be the capacity of color i in H , where the sum is taken over all vertices with color i in H . Sometimes we write x_i for $x_i(H)$, for short, when it is clear from the context what the subgraph H is.

In the following lemma some useful inequalities are given.

LEMMA 2.6 Let G be a k -regular k -chromatic graph with n vertices, S a defining set for G , e' the number of nonessential edges in G , and let y_i denote the number of vertices with color i in $V(G) - S$. Then we have:

$$(i) \ y_i \geq \lceil \frac{n - |S| - x_i(\langle S \rangle)}{2} \rceil, \quad i = 1, 2, \dots, k; \text{ and}$$

$$(ii) \ x_i(\langle S \rangle) \leq n - |S| + e', \quad i = 1, 2, \dots, k.$$

In particular, if H is a spanning subgraph of $\langle S \rangle$ then:

$$(i') \ y_i \geq \lceil \frac{n - |S| - x_i(H)}{2} \rceil, \quad i = 1, 2, \dots, k.$$

PROOF (i) There is a unique extension of colors of S to $V(G)$. Since by Lemma A any defining set of G is a strong defining set, there exists $k - 1$ distinct colors appearing in the neighborhood of each vertex in $V(G) - S$. Let W_i be the set of vertices of color i in $V(G) - S$. All of the vertices in $V(G) - (S \cup W_i)$ have different colors from i . To determine the color of each of these vertices, we need to have at least one vertex of color i in its neighborhood, which is either in S or in W_i . There are at most $x_i(\langle S \rangle)$ times when these vertices are in S and at most y_i times when they are in W_i . Because, for each vertex $v \in V(G) - S$, $k - 1$ edges incident to v are used to determine the color of v itself and at most one remaining edge may be used to determine the color of an adjacent vertex to v . Thus $n - |S| - y_i = |V(G) - (S \cup W_i)| \leq x_i(\langle S \rangle) + y_i$. Therefore $y_i \geq \lceil \frac{n - |S| - x_i(\langle S \rangle)}{2} \rceil$.

(ii) The capacity of color i in $\langle S \rangle$ may be used to determine the color of at most $n - |S|$ vertices in $V(G) - S$. And the vertices with color i in $\langle S \rangle$ are incident to at most e' nonessential edges. Therefore $x_i(\langle S \rangle) \leq n - |S| + e'$.

(i') The statement follows from the fact that $x_i(H) \geq x_i(\langle S \rangle)$, for each color i . ■

We also need the following lemma for our results.

LEMMA 2.7 Let G be a k -regular k -chromatic graph with n vertices, then $n \geq 2k - 1$.

PROOF Let G be a k -regular k -chromatic graph with n vertices. If G contains no K_k as a subgraph, then by the main Theorem of [2] we have $n \geq 2k - 1$.

Otherwise if G contains a K_k as a subgraph. By deleting the vertices of K_k from G , there remains $n - k$ vertices. The induced subgraph H on these $n - k$ vertices has $\frac{(n-k)k-k}{2}$ edges. Because each vertex of K_k adjacent to H by exactly one edge. This results that:

$$\binom{n-k}{2} \geq \frac{(n-k)k-k}{2} \implies n \geq 2k.$$

3 A construction algorithm

To construct a k -regular k -chromatic graph on n vertices with a defining set S , $|S| \geq k - 1$, we introduce the following algorithm. This algorithm will be used frequently in the rest of this paper.

At the beginning, let H_0 be a graph which consists of a K_{k-2} on the vertex set $U = \{u_1, \dots, u_{k-2}\}$ and $|S| - k + 2$ isolated vertices $V = \{v_1, \dots, v_{|S|-k+2}\}$. Assign color i to u_i , for $i = 1, 2, \dots, k - 2$; and for the vertices in V assign colors $1, 2, \dots, k$, such that at least one of the colors $k - 1$ and k be used. For each i ($1 \leq i \leq k$) determine x_i , the capacity of color i in H_0 . We add $n - |S|$ new vertices $W = \{w_1, \dots, w_{n-|S|}\}$ to H_0 as follows.

In step j ($1 \leq j \leq n - |S|$) assume that i is a color in H_{j-1} which has a minimum capacity so far. Among all such colors we choose the smallest i . Add a vertex w_j to the vertices of H_{j-1} and join this vertex to $k - 1$ vertices in H_{j-1} whose colors are distinct and is different from i . By doing this the color of w_j is forced to be i . Call this graph H_j . In graph H_j , compared with H_{j-1} , the capacity of each color, except i , is decreased by 1, while the capacity of color i is increased by 1. This last 1 is due to the capacity of w_j .

In each step the aim is that newly increased capacities be used and also try to create a K_k by extending the original K_{k-2} . This is to make sure that the resulting graph is k -chromatic. In the last step the resulting graph $H_{n-|S|}$ has n vertices and the sum of capacities for all colors is equal to $\epsilon = 2[\frac{nk}{2} - \binom{k-2}{2} - (n - |S|)(k - 1)]$. Now, this algorithm will produce a graph with the desired properties, if we can add as many as $\frac{\epsilon}{2}$ edges to the set of vertices with positive capacities. Actually here we have a kind of "graphical degree sequence" problem. The constraint is that two vertices can be joined together if they have positive capacities and different colors.

If a graph is produced by this algorithm, it will be k -regular k -chromatic with a defining set of size $|S|$.

4 General results

In this section we discuss $d(n, k, \chi = k)$ for the values of n for which $f(n, k)$ is an integer. The results show that the answer to Question 1 in [5], in general, is negative. First we note the following trivial lemma.

LEMMA 4.1 *he value of $f(n, k)$ is an integer if either of the following cases holds:*

- (i) k is even and $n \equiv k + 3 \pmod{2(k-1)}$, or $n \equiv 4 \pmod{2(k-1)}$;
- (ii) k is odd and $n \equiv k + 3 \pmod{2(k-1)}$.

By Lemma 4.1, with the conditions given in the following theorems, $f(n, k)$ is an integer.

THEOREM 4.1 *For each even k ($k \geq 6$) and for $n \equiv k + 3 \pmod{2(k-1)}$, we have $d(n, k, \chi = k) = f(n, k) + 1$.*

PROOF By Corollary A we have $d(n, k, \chi = k) \geq f(n, k)$. First we show that equality is impossible. Let $n = 2(k-1)l + (k+3)$ and G be a k -regular, k -chromatic graph with n vertices. Note that by Lemma 2.7, $l \geq 1$. Assume that there exists a defining set S of size $f(n, k)$ for G . We show a contradiction. By Lemma 4.1 and Lemma 2.5 the graph $\langle S \rangle$ consists of the union of $|S| - k + 2$ isolated vertices and a K_{k-2} . Suppose without loss of generality that the vertices of K_{k-2} are colored by $1, 2, \dots, k-2$. So for $\langle S \rangle$ we have:

$$\begin{aligned} x_i &= 3 + km_i, & \text{for } i = 1, 2, \dots, k-2; \text{ and} \\ x_i &= km_i, & \text{for } i = k-1 \text{ and } k, \end{aligned}$$

where m_i is the number of isolated vertices of color i in $\langle S \rangle$. On the other hand we have $\sum_{i=1}^k x_i = 3(k-2) + k(|S| - k + 2)$, $|S| = (k-2)l + (k-1)$, and $n - |S| = kl + 4$.

For each $i = 1, 2, \dots, k-2$, the number x_i is odd, while x_{k-1} and x_k are even. Thus by Lemma 2.6 we have:

$$\begin{aligned} y_i &\geq \frac{n - |S| - x_i + 1}{2}, & \text{for } i = 1, 2, \dots, k-2; \text{ and} \\ y_i &\geq \frac{n - |S| - x_i}{2}, & \text{for } i = k-1, k. \end{aligned}$$

Adding these inequalities together we obtain

$$\sum_{i=1}^k y_i \geq k \frac{n - |S|}{2} - \frac{1}{2} \sum_{i=1}^k x_i + \frac{k-2}{2}.$$

But $\sum_{i=1}^k y_i = n - |S|$. So $n - |S| \geq k \frac{n - |S|}{2} - \frac{1}{2} \sum_{i=1}^k x_i + \frac{k-2}{2}$. Substituting for $\sum_{i=1}^k x_i$ and $n - |S|$ from above, we obtain: $kl + 4 \geq kl + \frac{k+4}{2}$, which is a contradiction to $k \geq 6$.

Next, by Lemma B, it is sufficient to construct a graph with the conditions given in the statement, and with the minimum possible n (i.e. for $l = 1$) which has a defining set of size $f(n, k) + 1$. So the parameters are $n = 3k + 1$ and $|S| = 2k - 2$. Now we employ the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that $c(u_i) = c(v_i) = i$; for $i = 1, 2, \dots, k-2$, and $c(v_i) = i$; for $i = k-1$ and k . So

$$\begin{aligned} x_i(H_0) &= k + 3, & \text{for } i = 1, 2, \dots, k-2; \text{ and} \\ x_i(H_0) &= k, & \text{for } i = k-1, k. \end{aligned}$$

The new vertices to be added are $W = \{w_1, \dots, w_{k+3}\}$. After $k+3$ steps

the capacity of color i , for $i = 1, 2, \dots, k-2$, is equal to 2; for $i = k-1$ is equal to 3; and for $i = k$ is equal to 1. Here a 1 in the capacity of color $k-1$ is due to a vertex in W say w_j , and the other capacities are due to the isolated vertices of H_0 . Now by adding the edges of a path $w_j v_{k-2} v_{k-1} v_1 \dots v_{k-3} v_k$, we obtain a k -regular graph. By the algorithm it is clear that $c(w_1) = k-1$, and w_1 is joined to all of the vertices of U and to v_k . Also $c(w_2) = k$, and w_2 is joined to w_1 and to all of the vertices of U . So we have a K_k on $U \cup \{w_1, w_2\}$ and the constructed graph is one of the desired form. \blacksquare

THEOREM 4.2 *For each $k \equiv 0 \pmod{4}$, ($k \geq 8$) and $n \equiv 4 \pmod{2(k-1)}$, we have $d(n, k, \chi = k) > f(n, k)$. Moreover for each $n = 2(k-1)l + 4$, where $l \geq k/4$, we have $d(n, k, \chi = k) = f(n, k) + 1$.*

PROOF The proof of the first part is exactly the same as the proof of Theorem 4.1. To show the second part of the statement, again by Lemma B it suffices to construct a graph with the conditions given in the statement, and with the minimum possible n (i.e. for $l_0 = k/4$) which has a defining set of size $f(n, k) + 1$. So the parameters are $n = 2l_0(k-1) + 4$ and $|S| = (k-2)l_0 + k/2 + 1$. Now we apply the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that

$$\begin{aligned} c(u_i) &= i, & \text{for } i = 1, 2, \dots, k-2; \text{ and} \\ c(v_{3(i-1)+j}) &= j, & \text{for } i = 1, 2, \dots, l_0, \text{ and } j = 1, 2, 3. \end{aligned}$$

Also there are $(k-3)(l_0-1)$ vertices left in $V(H_0) = S$. For each l_0-1 of these vertices we assign a color j ($j = 4, \dots, k$). So the capacity of the colors in the beginning of the algorithm are as follows;

- $x_i(H_0) = kl_0 + 3$, for $i = 1, 2, 3$;
- $x_i(H_0) = k(l_0 - 1) + 3$, for $i = 4, 5, \dots, k-2$; and
- $x_i(H_0) = k(l_0 - 1)$, for $i = k-1, k$.

After applying the algorithm, capacities will be as in the following table.

color i	1	2	3	4	...	$\frac{k}{4} + 2$	$\frac{k}{4} + 3$...	$k-2$	$k-1$	k
$x_i(H_{n- S })$	$2l_0$	$2l_0$	$2l_0$	2	...	2	0	...	0	1	1

Here a 1 in the capacity of color $k/4 + 2$ is due to a vertex of W , and the other capacities are due to isolated vertices of H_0 . Now, if $k > 8$, by adding some edges to $H_{n-|S|}$ we obtain a k -regular graph. These edges are a path of length 2 on the vertices with colors $k/4 + 2, k/4 + 1, k/4 + 2$, a path of length $k/4 - 1$ on the vertices with colors $k-1, 4, 5, \dots, k/4, k$, respectively, and l_0 triangles (cycles of length 3) on the vertices with colors 1, 2, and 3.

If $k = 8$, by adding the edges of 2 triangles (cycles of length 3) on the vertices with colors 1, 2, and 3, joining two vertices with color 4 and 7, and finally by joining two vertices with color 4 and 8 we obtain a k -regular graph. As we have seen at the end of the proof for Theorem 4.1 this is the desired graph. ■

THEOREM 4.3 For each $k \equiv 2 \pmod{2(k-1)}$, ($k > 6$), and $n \equiv 4 \pmod{2(k-1)}$, we have:

- (i) $d(n, k, \chi = k) > f(n, k)$ for $n = 2(k-1)l+4$, where $l < (k+6)/4$;
- (ii) $d(n, k, \chi = k) = f(n, k)$ for $n = 2(k-1)l+4$, where $l \geq (k+6)/4$.

If $k = 6$, (i) holds only for $l = 1$, and (ii) holds for $l \geq 2$.

PROOF By Corollary A we have $d(n, k, \chi = k) \geq f(n, k)$.

(i) We show that equality is impossible. For $n = 2(k-1)l+4$, let G be a k -regular k -chromatic graph with n vertices. Note that $l \geq 1$. Assume that there is a defining set S of size $f(n, k)$ for G . We show a contradiction. By Lemma 4.1 and Lemma 2.5, the graph $\langle S \rangle$ consists of $|S| - k + 2$ isolated vertices and a K_{k-2} . Suppose that the vertices of K_{k-2} are colored by $1, 2, \dots, k-2$. So for $\langle S \rangle$ we have $x_i = 3 + km_i$; for $1, 2, \dots, k-2$, and $x_i = km_i$; for $i = k-1$ and k , where m_i is the number of isolated vertices of color i in $\langle S \rangle$. On the other hand by Corollary 2.1 the number of nonessential edges is at most 1. Also by Lemma 2.6, $x_i \leq n - |S| + 1$; for $1, 2, \dots, k$. Thus $x_i = 3 + km_i \leq kl - k/2 + 5$, for $i = 1, 2, \dots, k-2$, results that $m_i \leq l-1$; and also $x_i = km_i \leq kl - k/2 + 5$, implies that for $i = k-1$ and k , $m_i \leq l-1$. Therefore

$$|S| = f(n, k) = (k-2)l + \frac{k}{2} \leq (k-2)l + 2(l-1)$$

which implies that $l \geq \frac{k+6}{4}$. This is a contradiction to $l < (k+6)/4$.

(ii) Again by Lemma B it is sufficient to construct a graph with the conditions given in the statement, and with the minimum possible n , i.e. for $l_0 = (k+6)/4$, which has a defining set of size $f(n, k)$. So the parameters are $n = 2l_0(k-1) + 4$ and $|S| = (k-2)l_0 + k/2$. Now we apply the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that $c(u_i) = i$ for $i = 1, 2, \dots, k-2$, and to $l_0 - 2$ of the isolated vertices in S we assign color k . This will leave $(l_0 - 1)(k-1)$ isolated vertices which we partition into $(k-1)$ classes of $l_0 - 1$ vertices each. We assign colors $i = 1, 2, \dots, k-1$ to these classes. So capacities of the colors in the beginning of the algorithm are as follows:

$$x_i(H_0) = k(l_0 - 1) + 3, \text{ for } i = 1, 2, \dots, k-2;$$

$$\begin{aligned}x_{k-1}(H_0) &= k(l_0 - 1); \text{ and} \\x_k(H_0) &= k(l_0 - 2).\end{aligned}$$

After applying the algorithm, the capacity of color i ($i = 1, 2, \dots, k-2$) is equal to 0, and for $i = k-1$ and k is equal to 1. Here the capacity of color $k-1$ is due to a vertex in W , and the other capacity is due to an isolated vertex of H_0 . These two vertices are not adjacent. Now by joining them together, we obtain a k -regular graph. This is the desired graph.

If in the proof of (i) we let $k = 6$, we see that only for $l = 1$ there is a contradiction. For $l \geq 2$ the statement is shown in Theorem 5.1 (see $n = 24$). \blacksquare

THEOREM 4.4 *For each odd number k ($k \geq 7$), and $n \equiv k + 3 \pmod{2(k-1)}$, we have:*

$$(i) \quad d(n, k, \chi = k) = f(n, k) + 1 \quad \text{for } n = 2(k-1)l + k + 3, \text{ where } l < (k-3)/2;$$

$$(ii) \quad d(n, k, \chi = k) = f(n, k) \quad \text{for } n = 2(k-1)l + k + 3, \text{ where } l \geq (k-3)/2.$$

PROOF (i) Proof of impossibility of $d(n, k, \chi = k) = f(n, k)$ is similar to the proof of Theorem 4.3. Here we have $m_i \leq l$; for $i = 1, 2, \dots, k$. So each of the colors $1, 2, \dots, k-2$ appears at most $l+1$ times, while each of the colors $k-1$ and k appears at most l times on S . If each of the colors $k-1$ and k appears exactly l times on S , then at least $k - (2l+1)$ colors from $1, 2, \dots, k-2$ appear $l+1$ times each, and the remaining $2l-1$ of them appear l times each. So

$$\begin{aligned}\sum_{i=1}^k y_i &\geq \sum_{i=1}^k \lceil \frac{n - |S| - x_i}{2} \rceil \\ &= \sum_{i=1}^{k-(2l+1)} \lceil \frac{n - |S| - x_i}{2} \rceil + \sum_{i=k-(2l+1)+1}^{k-2} \lceil \frac{n - |S| - x_i}{2} \rceil \\ &\quad + \sum_{i=k-1}^k \lceil \frac{n - |S| - x_i}{2} \rceil.\end{aligned}$$

We note that in the last expression, in the first summation each summand is a non-integer, while in the second and the third summations all summands are integers. Now by substituting we have

$$\sum_{i=1}^k y_i \geq (k - (2l+1)) \lceil \frac{1}{2} \rceil + (2l-1) \frac{k+1}{2} + 2 \times 2 = kl + \frac{k-2l+5}{2}.$$

On the other hand $\sum_{i=1}^k y_i = n - |S| = kl + 4$. Therefore $kl + 4 \geq kl + \frac{k-2l+5}{2}$.

And this is in contradiction to $l < (k-3)/2$.

If any of the colors appears less than l times, then at least one of the colors must appear $l+1$ times. Thus the summands on the right hand side in the above will be increased, which clearly results in the failure of the inequality.

Now we show that $d(n, k, \chi = k) = f(n, k) + 1$, for $l < (k-3)/2$. For each $n = 2l(k-1) + k + 3$, where $l < (k-3)/2$ we construct a k -regular k -chromatic graph on n vertices with a defining set of size $f(n, k) + 1$. We apply the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that

$$\begin{aligned} c(u_i) &= i, \text{ for } i = 1, 2, \dots, k-2; \\ c(v_1) &= k-1, \text{ and} \\ c(v_2) &= k. \end{aligned}$$

There will remain $l(k-2)$ isolated vertices that we partition them into $(k-2)$ classes of l elements each, and assign the colors $i = 1, 2, \dots, k-2$ to these classes. So the capacity of the colors in the beginning of the algorithm are as follows:

$$\begin{aligned} x_i(H_0) &= kl + 3, \text{ for } i = 1, 2, \dots, k-2; \text{ and} \\ x_i(H_0) &= k, \text{ for } i = k-1, k. \end{aligned}$$

If l is even, after applying the algorithm, capacity of each vertex with color i ($i = 1, \dots, k$) is equal to 2. Here one of the capacities of color k is due to a vertex of W , and the other capacities are due to isolated vertices of H_0 . Now, by adding the edges of a path of length k on the vertices with colors $k, 1, 2, \dots, k-1, k$, respectively, we obtain a k -regular graph.

If l is odd, the capacity of color i for $i = 1, \dots, k-2$ is equal to 2; for $i = k-1$ is equal to 3, and for $i = k$ is equal to 1. Here one of the capacities of color $k-1$ is due to a vertex of W and the other capacities are due to the isolated vertices of H_0 . Now by adding the edges of a path of length k on the vertices with colors $k, k-1, 1, 2, \dots, k-2, k-1$, respectively, we obtain a k -regular graph.

(ii) Again by Lemma B it is sufficient to construct a graph with the conditions given in the statement, and with the minimum possible n (i.e. for $l_0 = (k-3)/2$) which has a defining set of size $f(n, k)$. So the parameters are $n = 2l_0(k-1) + k + 3$ and $|S| = (k-2)l_0 + k - 1$. We apply the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that $c(u_i) = i$; for $i = 1, 2, \dots, k-2$, and for each l_0 of isolated vertices in S we assign a color i ($i = 1, 2, k-1, k$). There will be $(l_0 - 1)(k-4)$ isolated vertices left over, which we partition into $(k-4)$

classes of $l_0 - 1$ vertices each, and assign the colors $i = 3, 4, \dots, k - 2$ to these classes. So the capacity of colors in the beginning of the algorithm are as follows:

After applying the algorithm, capacity of color i for $i = 3, \dots, k$, is equal to 0, and for $i = 1$ and 2 is equal to 1. Here the capacity of one of the colors is due to a vertex of W and the other capacity is due to an isolated vertex of H_0 , and they are not adjacent. Now by joining these two vertices, we obtain a k -regular graph. This is the desired graph. ■

5 Cases $k = 6$ and $k = 7$

In [5] one of the small cases, i.e. $d(10, 5, \chi = 5)$, is left undetermined. By a computer program we searched all 5-regular 5-chromatic graphs with 10 vertices and found out that there is no such a graph with defining set of size 5. But we can construct a graph G with $d(G, \chi) = 6$ as follows. Take two disjoint copies of K_5 with vertex set $\{u_1, u_2, \dots, u_5\}$ and $\{v_1, v_2, \dots, v_5\}$ and add 5 more edges, $u_1v_1, u_2v_2, \dots, u_5v_5$.

In the rest of this section we find $d(n, k, \chi = k)$ for $k = 6$ and 7, leaving only one single undetermined case.

THEOREM 5.1 *We have*

$$d(n, 6, \chi = 6) = \begin{cases} \lceil f(n, 6) \rceil & \text{for } n \not\equiv 9 \pmod{10} \text{ and } n \geq 15; \\ \lceil f(n, 6) \rceil + 1 & \text{otherwise.} \end{cases}$$

PROOF To show the first part, by Lemma B it suffices to construct 6-regular 6-chromatic graphs with n vertices, for some small values of n , say $n = 15, 16, 17, 18, 20, 21, 22, 23$, and 24, each with a defining set of size $f(n, 6)$. In each case we apply the algorithm of Section 3. Here H_0 consists of a K_4 and $f(n, 6) - 4$ isolated vertices. The color of vertices of K_4 is taken to be 1, 2, 3, and 4. The color of isolated vertices are taken such a way that the capacity of each color, in each case, is as in the following table.

$n = 15, 16$	i	1	2	3	4	5	6
	$x_i(H_0)$	9	9	3	3	6	6
$n = 17, 18$	i	1	2	3	4	5	6
	$x_i(H_0)$	9	9	9	3	6	6
$n = 20, 21$	i	1	2	3	4	5	6
	$x_i(H_0)$	9	9	9	9	6	6
$n = 22, 23$	i	1	2	3	4	5	6
	$x_i(H_0)$	15	9	9	9	6	6

$n = 24$	i	1	2	3	4	5	6
	$x_i(H_0)$	9	9	9	9	6	12

For the second part of the statement, note that by Theorem 4.1 we have $d(n, 6, \chi = 6) = f(n, 6) + 1$, for $n \equiv 9 \pmod{10}$, $n \geq 19$. The cases $n = 11, 12, 13$, and 14 are discussed in the appendix. ■

THEOREM 5.2 *We have*

$$d(n, 7, \chi = 7) = \begin{cases} \lceil f(n, 7) \rceil & \text{for } n \text{ even and } n \neq 14, 16, 22, 26; \\ \lceil f(n, 7) \rceil + 1 & \text{for } n = 16, 22; \\ \lceil f(n, 7) \rceil + 2 & \text{for } n = 14. \end{cases}$$

PROOF To show the first part, by Lemma B and Theorem 4.4, it suffices to construct 7-regular 7-chromatic graphs with n vertices for $n = 18, 20, 24, 28$, and 38 , each with a defining set of size $f(n, 7)$. In each case we apply the algorithm of Section 3. Here H_0 consists of a K_5 and $f(n, 7) - 5$ isolated vertices. The vertices of K_5 are taken to be colored 1, 2, 3, 4, and 5. The isolated vertices are colored such a way that the capacity of each color, in each case, is as shown in the following table.

$n = 18$	i	1	2	3	4	5	6	7
	$x_i(H_0)$	10	10	10	3	3	7	7
$n = 20$	i	1	2	3	4	5	6	7
	$x_i(H_0)$	10	10	10	10	3	7	7
$n = 24$	i	1	2	3	4	5	6	7
	$x_i(H_0)$	10	10	10	10	10	7	7
$n = 28$	i	1	2	3	4	5	6	7
	$x_i(H_0)$	17	17	10	10	10	7	7
$n = 38$	i	1	2	3	4	5	6	7
	$x_i(H_0)$	17	17	17	17	10	14	14

For the second part of the statement, note that by Theorem 4.4 (i) we have $d(22, 7, \chi = 7) = f(22, 7) + 1$. The cases $n = 14$ and 16 are discussed in the appendix. ■

Appendix

Case $k = 6$.

$n = 11$ First we show that it is impossible to have $d(11, 6, \chi = 6) = 6$. On the contrary assume that G is a 6-regular 6-chromatic graph with 11 vertices and S is a defining set of size 6 for G . Then $\langle S \rangle$ is a graph on 6 vertices with maximum degree 6 and chromatic number greater

than or equal to 4. By Lemma 2.2 the graph $\langle S \rangle$ has at most 7 edges. Thus by Lemma 2.4 it contains a K_4 . By Corollary 2.1 the number of nonessential edges is at most 2. Thus by Lemma 2.6 we have $x_i \leq 7$, for $1 \leq i \leq 6$. Assume that vertices of K_4 are colored 1, 2, 3, and 4. Therefore the colors of other two vertices must be 5 and 6. Let H be a spanning subgraph of $\langle S \rangle$ which consists of a K_4 and two isolated vertices. Thus $x_1 = x_2 = x_3 = x_4 = 3$, and $x_5 = x_6 = 6$. Therefore by Lemma 2.6, $y_i \geq 1$ for $i = 1, 2, 3$, and 4. This implies that if we extend the coloring of S to the vertices of G , then the colors 1, 2, 3, and 4 must appear at least once on the vertices of $V(G) - S$. There are two cases to be considered.

Case 1 One of the colors 5 or 6, say 5, has appeared in $V(G) - S$. Then we have $y_1 = y_2 = y_3 = y_4 = y_5 = 1$, and $y_6 = 0$. This implies that the capacities of the colors in graph H' , which is the union of H and the set of all edges which are necessary to determine the color of vertices in $V(G) - S$, are as follows: $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = 3$, and $x_6 = 1$. So the induced subgraph on the nonessential edges has 3 capacities on the color 5 and one capacity of color 6, which is impossible.

Case 2 None of the colors 5 and 6 appear on the vertices of $V(G) - S$ ($y_5 = y_6 = 0$). Then without loss of generality we have $y_1 = 2$ and $y_2 = y_3 = y_4 = 1$. This implies that the capacities of the colors in graph H' , which is the union of H and the set of all edges which are necessary to determine the color of vertices in $V(G) - S$, are as follows $x_1 = 2$, $x_2 = x_3 = x_4 = 0$, and $x_5 = x_6 = 1$. So the induced subgraph on the nonessential edges has color 1 with capacity 2 and color 5 and 6, with capacity 1 each. Then the unique vertices with color 5 and 6 in G can not be adjacent. But then G can be recolored by using only 5 colors. This is a contradiction.

The graph of Example 1.1 shows that $d(11, 6, \chi = 6) = 7$.

$n = 12$ First we show that $d(12, 6, \chi = 6) > 7$. In contrary assume that G is a 6-regular 6-chromatic graph with 12 vertices and S is a defining set of size 7 for G . Then $\langle S \rangle$ is a graph on 7 vertices with maximum degree 6 and chromatic number greater than or equal to 4. By Lemma 2.2 the graph $\langle S \rangle$ has at most 10 edges. We consider two cases.

Case 1 The graph $\langle S \rangle$ contains a K_4 . By Corollary 2.1 the number of nonessential edges is at most 5, and then by Lemma 2.6, $x_i \leq 10$, for $1 \leq i \leq 6$. Assume that the vertices of K_4 are colored 1, 2, 3, and 4. Therefore the colors 5 and 6 appear at most twice in three other vertices. Let H be a spanning subgraph of $\langle S \rangle$ which consists of a K_4 and three isolated vertices. According to the colors of the isolated vertices we have two cases to consider.

Case 1.1 One of the colors 5 or 6, say 5, has appeared once on the isolated vertices. Then without loss of generality the capacities of colors in H are

$x_1 = x_2 = 9$, $x_3 = x_4 = 3$, $x_5 = 6$, and $x_6 = 0$. Therefore by Lemma 2.6, $y_3 \geq 1$, $y_4 \geq 1$, and $y_6 \geq 3$. This implies that if we extend the coloring of S to the vertices of G , then the colors 3 and 4 must appear once and the color 6, must appear three times on the vertices of $V(G) - S$. This implies that the capacities of the colors in graph H' , which is the union of H and the set of all edges which are necessary to determine the color of vertices in $V(G) - S$, are as follows: $x_1 = x_2 = 4$ and $x_5 = x_6 = 1$.

So the induced subgraph on the nonessential edges has 4 capacities on the colors 1 and 2, and one capacity of color 5 and 6, each. Then the unique vertex of color 5 in G can not be adjacent to one of the colors 1 or 2. But then G can be recolored by using only 5 colors. This is a contradiction.

Case 1.2 The colors 5 and 6 have appeared once in isolated vertices, each. Then without loss of generality we have $x_1 = 9$, $x_2 = x_3 = x_4 = 3$, and $x_5 = x_6 = 6$. Therefore by Lemma 2.6, $y_i \geq 1$ for $i = 2, 3, 4$. This implies that if we extend the coloring of S to the vertices of G , then the colors 2, 3, and 4 must appear at least once on the vertices of $V(G) - S$. And two other vertices of $V(G) - S$ may be colored by some of the colors 1, 2, 3, 4, 5, or 6. But in each case by a computer program we see that the 6-regular graphs obtained in this way are not 6-chromatic.

Case 1.3 One of the colors 5 or 6, say 5, has appeared twice in isolated vertices. Then we have $x_5 = 12$. If we extend the coloring of S to the vertices of G , then the induced subgraph on the nonessential edges has capacity at least 7 on the color 5 and at most three capacities on the other colors, which is impossible.

Case 2 The graph $\langle S \rangle$ does not contain a K_4 . Then it must be a union of a wheel, W_5 , and an isolated vertex. According to the colors of vertices of $\langle S \rangle$ we have two cases to consider.

Case 2.1 One of the colors, say 6, does not appear in S . So $x_6 = 0$. Now if the isolated vertex u , has the same color as w , the vertex of degree 5, then without loss of generality there are three colors, say 2, 3, and 4, such that $x_2 = x_3 = x_4 = 3$. Thus by Lemma 2.6, $y_i \geq 1$ for $i = 2, 3, 4$, and $y_6 \geq 3$. In other words $V(G) - S$ has at least 6 vertices, which is impossible.

If u and w have different colors, then clearly $x_1 = 1$, where 1 is assumed to be the color of vertex w . Also for one of the other colors, say 2, we have $x_2 = 3$. Therefore by Lemma 2.6, $y_1 \geq 2$, $y_2 \geq 1$ and $y_6 \geq 3$. This implies that there are at least 6 vertices in $V(G) - S$, which is impossible.

Case 2.2 All of the colors appear in S . Now if u and w have the same color, then $x_1 = 7$, and $x_i = 3$ for $2 \leq i \leq 6$. Thus by Lemma 2.6, $y_i \geq 1$ for $2 \leq i \leq 6$. This implies that if we extend the coloring of S to the vertices of G , then the colors 2, 3, 4, 5, and 6 must appear once on the vertices of $V(G) - S$. This implies that the capacities of the colors in graph H' , which is the union of H and the set of all edges which are necessary to determine

the color of vertices in $V(G) - S$, are as follows.

$$x_1 = 2, \text{ and } x_2 = \dots = x_6 = 0.$$

So the induced subgraph on the nonessential edges has 2 capacities on the color 1, which is impossible.

If u and w have different colors, then without loss of generality we have $x_1 = 1$, $x_2 = x_3 = x_4 = x_5 = 3$, and $x_6 = 9$. Therefore by Lemma 2.6, $y_1 \geq 2$, and $y_i \geq 1$ for $i = 2, 3, 4, 5$. This implies that there are at least 6 vertices in $V(G) - S$, which is impossible.

The proof of $d(12, 6, \chi = 6) = 8$ is similar to the case of $d(10, 5, \chi = 5) = 6$, which was discussed in the beginning of this section.

$n = 13$ First we show that it is impossible to have $d(13, 6, \chi = 6) = 7$. On the contrary assume that G is a 6-regular 6-chromatic graph with 13 vertices and S is a defining set of size 7 for G . Then $\langle S \rangle$ is a graph on 7 vertices with maximum degree 6 and chromatic number greater than or equal to 4. By Lemma 2.2 the graph $\langle S \rangle$ has at most 8 edges. Thus by Lemma 2.4 it contains a K_4 . By Corollary 2.1 of Lemma 2.1 the number of nonessential edges is at most 3. Thus by Lemma 2.6 we have $x_i \leq 9$, for $1 \leq i \leq 6$. Assume that the vertices of K_4 are colored 1, 2, 3, and 4. Therefore the colors 5 and 6 appear at most once in three other vertices of S . Let H be a spanning subgraph of $\langle S \rangle$ which consists of a K_4 and three isolated vertices. According to the colors of isolated vertices we have two cases to consider.

Case 1 One of the colors 5 or 6, say 5, has appeared once on an isolated vertex. Then without loss of generality we have $x_1 = x_2 = 9$, $x_3 = x_4 = 3$, $x_5 = 6$, and $x_6 = 0$. Therefore by Lemma 2.6, $y_3 \geq 2, y_4 \geq 2$, and $y_6 \geq 3$. This means that there are at least 7 vertices in $V(G) - S$, which is impossible.

Case 2 The colors 5 and 6 have each appeared once on an isolated vertex. Then without loss of generality we have $x_1 = 9$, $x_2 = x_3 = x_4 = 3$, and $x_5 = x_6 = 6$. Therefore by Lemma 2.6, $y_i \geq 2$ for $i = 2, 3, 4$. This implies that if we extend the coloring of S to the vertices of G , then each of the colors 2, 3, and 4 must appear exactly two times on the vertices of $V(G) - S$. This implies that the capacities of the colors in graph H' , which is the union of H and the set of all edges which are necessary to determine the color of vertices in $V(G) - S$, are as follows.

$$x_1 = 3, x_2 = x_3 = x_4 = 1, \text{ and } x_5 = x_6 = 0.$$

So the induced subgraph on the nonessential edges has 3 capacities on the color 1 and one capacity on colors 2, 3, and 4, each. Then the unique

vertices of color 5 and 6 in G can not be adjacent. But then G may be recolored by using only 5 colors. This is a contradiction.

Now we show that $d(13, 6, \chi = 6) = 8$. It suffices to construct a 6-regular 6-chromatic graph with 13 vertices with a defining set of size 8. We apply the algorithm of Section 3. Here H_0 consists of a K_4 and 4 isolated vertices. The colors of vertices of K_4 are taken to be 1, 2, 3, and 4. The colors of isolated vertices are taken in such a way that the capacity of each color is as shown in the following table.

i	1	2	3	4	5	6
$x_i(H_0)$	9	9	3	3	6	6

$n = 14$ By Theorem 4.3 we have $d(14, 6, \chi = 6) > 7$. It suffices to construct a 6-regular 6-chromatic graph with 14 vertices with a defining set of size 8. We apply the algorithm of Section 3. Here H_0 consists of a K_4 and 4 isolated vertices. The vertices of K_4 are taken to be colored 1, 2, 3, and 4. The isolated vertices are colored in such a way that the capacity of each color is as shown in the following table.

i	1	2	3	4	5	6
$x_i(H_0)$	9	9	3	3	6	6

Case $k = 7$.

$n = 14$ First we show that there does not exist a 7-regular 7-chromatic graph on 14 vertices which does not contain any K_7 as a subgraph. For this, we prove that there exists only one unique 7-critical graph with $n = 14$, maximum degree 7, which contains no K_7 as a subgraph, but this graph is not extendable to a 7-regular graph with $n = 14$.

To prove our claim, let G be a 7-critical graph with maximum degree 7 and with 14 vertices. Let u be a vertex of G . Then $G - u$ is a 6-chromatic graph with 13 vertices. We color $G - u$ with 6 colors and add vertex u with the assigned color 7 to it. Suppose S is a maximal independent set in G . Let $G' = G - S$. Since in each 6-coloring of $G - u$, there exists a color which appears at least three times, we have $|S| \geq 3$.

Now there are two cases to be considered:

Case 1. G' contains no K_6 as a subgraph.

In this case by the main theorem of [2] we have $|V(G')| \geq 11$. On the other hand $|V(G')| = |V(G)| - |S| \leq 11$. Therefore $|V(G')| = 11$. Again by Theorem 5.1 in [2], such a graph has at least 5 vertices of degree 6 and the rest of vertices are of degree 5. To obtain a 7-regular graph from this graph we can add at most $6 + 2.5 = 17$ edges on these vertices. But for extending G' to G , we need to add at least 18 edges from the vertices of S to the vertices of G' , for G is 7-critical, $\delta \geq 6$, so each vertex is of degree 6 or 7.

Case 2. G' contains a K_6 as a subgraph.

By deleting the vertices of K_6 from G , there remains 8 vertices, which at least three of them are independent vertices. Now consider the induced subgraph on these 8 vertices. It can be checked that this graph must be the graph $K_5 \vee \overline{K_3}$, where “ \vee ” means the *join* of two graphs. For, otherwise we can find a 6-coloring for G . Since G is 7-critical, each vertex is of degree 6 or 7, so for extending $K_6 \cup (K_5 \vee \overline{K_3})$ to G , the only way is that, we join each of three independent vertices to each distinct pair of vertices of K_6 . But then, this 7-critical graph which has maximum degree 7 and 14 vertices, can not be extended to a 7-regular 7-chromatic graph with 14 vertices which has no K_7 as a subgraph.

These two cases result that there does not exist a 7-regular 7-chromatic graph with 14 vertices which has no K_7 as a subgraph. Therefore, up to isomorphism each 7-regular 7-chromatic graph with 14 vertices consists of two disjoint copies of K_7 and a 1-factor between them. It is easy to check that the size of a smallest defining set of such graph is equal to 10.

$n = 16$ The proof the impossibility of $d(16, 7, \chi = 7) = 9$, is the same as in the proof of case $n = 11$ in Theorem 5.1. Now we show that $d(16, 7, \chi = 7) = 10$. It suffices to construct a 7-regular 7-chromatic graph with 16 vertices with a defining set of size 10. We apply the algorithm of Section 3. Here H_0 consists of a K_5 and 5 isolated vertices. The vertices of K_5 are taken to be colored 1, 2, 3, 4, and 5. The isolated vertices are colored in such a way that the capacity of each color is as shown in the following table.

i	1	2	3	4	5	6	7
$x_i(H_0)$	10	10	10	10	3	7	0

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References

- [1] J.A. BATE AND G.H.J. VAN REES, *The size of the smallest strong critical set in a latin square*, *Ars Combinatoria*, **53** (1999), 73–83.
- [2] A. BEUTELSPACHER AND P.R. HERING, *Minimal graphs for which the chromatic number equals to the maximal degree*, *Ars Combinatoria*, **18** (1983), 201–216.
- [3] A.D. KEEDWELL, *Critical sets for Latin squares, graphs and block designs: a survey*, (Festschrift for C. St. J. A. Nash-Williams) *Congr. Numer.* **113**, (1996), 231–245.
- [4] E.S. MAHMOODIAN, *Some problems in graph colorings*, in *Proc. 26th Annual Iranian Math. Conference*, S. Javadpour and M. Radjabalipour, eds., (Kerman, Iran, Mar. 1995), Iranian Math. Soc., University of Kerman, 215–218.
- [5] E.S. MAHMOODIAN AND E. MENDELSON, *On defining numbers of vertex coloring of regular graphs*, *Discrete Mathematics*, **197/198** (1999), 543–554.
- [6] E.S. MAHMOODIAN, R. NASERASR, AND M. ZAKER, *Defining sets of vertex colorings of graphs and latin rectangles*, *Discrete Mathematics*, **167/168** (1997), 451–460.
- [7] T. MORRILL AND D. PRITIKIN, *Defining sets and list-defining sets in graphs*, 1997 (preprint).
- [8] A.P. STREET, *Defining sets for block designs: An update*, in *Combinatorics Advances*, C. J. Colbourn and E. S. Mahmoodian, eds., *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London, (1995), 307–320.

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