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An algorithm for writing any latin interchange as a sum of intercalates

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Abstract

A latin interchange is a pair of disjoint partial latin squares of the same shape and order which are row–wise and column–wise mutually balanced. In this paper we document a simple algorithm which enables one to write a latin interchange as the sum of 2×2 latin interchanges; that is as the sum of intercalates.

1 Introduction and preliminaries

A latin square L of order n is an $n \times n$ array with entries chosen from a set $N = \{1, \ldots, n\}$ in such a way that each element of N occurs precisely once in each row and column of the array. For ease of exposition, a latin square L will be represented by a set of ordered triples $\{(i, j; L_{ij}) \mid$ where element L_{ij} occurs in cell (i, j) of the array$\}$.

A partial latin square P of order n is an $n \times n$ array with entries chosen from a set $N = \{1, \ldots, n\}$ in such a way that each element of N occurs at most once in each row and at most once in each column of the array. Hence there are cells in the array that may be empty, but the positions

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that are filled have been so as to conform with the latin property of array. Once again a partial latin square may be represented as a set of ordered triples. However in this case we will include triples of the form $(i, j; \emptyset)$ and read this to mean that cell (i, j) of the partial latin square is empty. The set of cells $\mathcal{S}_P = \{(i, j) \mid (i, j; P_{ij}) \in P, \text{ for some } P_{ij} \in N\}$ is said to determine the shape of P and $|\mathcal{S}_P|$ is said to be the volume of the partial latin square. That is, the volume is the number of nonempty cells. For each row r, $1 \leq r \leq n$, we let $\mathcal{R}_P^r$ denote the set of entries occurring in row r of P. Formally, $\mathcal{R}_P^r = \{P_{rj} \mid P_{rj} \in N \wedge (r, j; P_{rj}) \in P\}$. Similarly, for each column c, $1 \leq c \leq n$, we define $\mathcal{C}_P^c = \{P_{ic} \mid P_{ic} \in N \wedge (i, c; P_{ic}) \in P\}$.

A latin interchange, $I = (P, Q)$, of volume s is an ordered set of two partial latin squares, of order n, such that

1. $\mathcal{S}_P = \mathcal{S}_Q$,
2. for each $(i, j) \in \mathcal{S}_P$, $P_{ij} \neq Q_{ij}$,
3. for each r, $1 \leq r \leq n$, $\mathcal{R}_P^r = \mathcal{R}_Q^r$, and
4. for each c, $1 \leq c \leq n$, $\mathcal{C}_P^c = \mathcal{C}_Q^c$.

Thus a latin interchange is a pair of disjoint partial latin squares of the same shape and order, which are row–wise and column–wise mutually balanced. We refer to the shape of a latin interchange I as the shape of the individual components P and Q.

EXAMPLE 1.1 Below is an example of two partial latin square which together form a latin interchange of order 5 and of volume 19. To conserve space it will be our practice to display a latin interchange by superimposing one partial latin square on top of the other, and using subscripts to differentiate the entries of the second from those of the first, as shown below.

·	·	2	3	1
·	2	·	1	4
1	·	5	4	3
5	4	1	·	2
4	1	3	2	5

·	·	1	2	3
·	1	·	4	2
4	·	3	1	5
1	2	5	·	4
5	4	2	3	1

·	·	2_1	3_2	1_3
·	2_1	·	1_4	4_2
1_4	·	5_3	4_1	3_5
5_1	4_2	1_5	·	2_4
4_5	1_4	3_2	2_3	5_1

The concept of a latin interchange in a latin square is similar to the concept of a mutually balanced set or a trade (see [9]) in a block design. The same as trades, the discussion of latin interchanges is related to intersection problems. For example, they are relevant to the problem of finding the possible number of intersections for latin squares (see [7], [6], [2], and [1]).

EXAMPLE 1.2 The following array shows each of the partial latin squares given in Example 1.1 can be embedded in a latin square of order 6, which indicates that there exist two latin squares of order 6 having 36-19=17 elements in common.

6	5	2_1	3_2	1_3	4
3	2_1	6	1_4	4_2	5
1_4	6	5_3	4_1	3_5	2
5_1	4_2	1_5	6	2_4	3
4_5	1_4	3_2	2_3	5_1	6
2	3	4	5	6	1

Also latin interchanges arise naturally in the discussion of critical sets in latin squares (see for example [10]). The determination of critical sets has been shown to be an NP–complete problem [4].

Latin interchanges have been studied by other authors. Fu and Fu [6] used the term "disjoint and mutually balanced" (DMB) partial latin squares, Keedwell [10] used "critical partial latin square" (CPLS), while Donovan et al. [5] used the term "latin interchange". Adams et al. [1] suggest the terminology 2-way latin trade for consistency with similar concepts in other combinatorial structures such as block designs, graph colouring, cycle systems, etc. See for instance [9], [13], [8], and [3] for further use of trades.

Let r_i denote the number of non-empty cells in row i and c_j denote the number of non-empty cells in column j in a latin interchange I. Then it is obvious that $\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j = |\mathcal{S}_I|$, and the type of the latin interchange I is defined to be

$$\binom{c_1 + c_2 + c_3 + \ldots + c_n}{r_1 + r_2 + r_3 + \ldots + r_n}.$$

The type of the latin interchange in Example 1.1 is

$$\binom{3+3+4+4+5}{3+3+4+4+5}.$$

Note that the type describes the number of non–empty cells in each of the columns and rows of I. Since the empty rows and columns of I give very little useful information, wherever possible they are deleted, and the latin interchange I is taken to be a partial latin square of order n, where

$n = max\{r, c\}$. Then the type is written as

$$\binom{c_1 + c_2 + c_3 + \ldots + c_c}{r_1 + r_2 + r_3 + \ldots + r_r}.$$

A latin interchange of volume 4 and type $\binom{2+2}{2+2}$, which is unique (up to isomorphism), is said to be an intercalate.

Next we define the sum of two latin interchanges. To this end we introduce the definition of a generalized interchange. A generalized interchange $I = (P, Q)$ is defined as a latin interchange but with the proviso that for any r, $1 \le r \le n$, (c, $1 \le c \le n$) the sets $\mathcal{R}_P^r$ and $\mathcal{R}_Q^r$, ($\mathcal{C}_P^c$ and $\mathcal{C}_Q^c$) may be multisets, but we still require that for all r, $1 \le r \le n$, (c, $1 \le c \le n$) $\mathcal{R}_P^r = \mathcal{R}_Q^r$, ($\mathcal{C}_P^c = \mathcal{C}_Q^c$).

Let $I = (P, Q)$ and $I' = (P', Q')$ be two generalized interchanges. Assume that for each cell $(i, j) \in \mathcal{S}_I \cap \mathcal{S}_{I'}$ (that is, a cell where both I and I' are nonempty) we have $P_{ij} \neq P'_{ij}$, $Q_{ij} \neq Q'_{ij}$, and $\{P_{ij}, P'_{ij}\} \cap \{Q_{ij}, Q'_{ij}\} \neq \emptyset$. Then the sum of I and I' is defined to be a generalized interchange $I + I' = (S, T)$, where the partial squares S and T are such that:

- for each cell $(i, j) \in \mathcal{S}_I \cap \mathcal{S}_{I'}$:

$$S = \{(i, j; \{P_{ij}, P'_{ij}\} \setminus \{Q_{ij}, Q'_{ij}\}) |\ (i, j) \in \mathcal{S}_I \cap \mathcal{S}_{I'}\},$$

$$T = \{(i, j; \{Q_{ij}, Q'_{ij}\} \setminus \{P_{ij}, P'_{ij}\}) |\ (i, j) \in \mathcal{S}_I \cap \mathcal{S}_{I'}\};$$

- for each cell (i, j) for which at least one of the interchanges, say I, is empty while P'_{ij} and $Q'_{ij} \in N$: $(i, j; P'_{ij}) \in S$ and $(i, j; Q'_{ij}) \in T$; and finally
- for each cell (i, j) that both of the interchanges I and I' are empty, then (i, j) in $I + I'$ is also empty.

EXAMPLE 1.3 For example the latin interchange of Example 1.1 is equal to the sum of the following generalized interchanges.

·	·	2_1	3_2	1_3
·	2_1	·	1_4	4_2
1_4	·	5_3	4_1	3_5
5_1	4_2	1_5	·	2_4
4_5	1_4	3_2	2_3	5_1

=

·	·	·	3_2	2_3
·	·	·	2_4	4_2
2_4	·	5_3	4_2	3_5
5_2	4_2	2_5	·	2_4
4_5	2_4	3_2	2_3	5_2

+

·	·	2_1	·	1_2
·	2_1	·	1_2	·
1_2	·	·	2_1	·
2_1	·	1_2	·	·
·	1_2	·	·	2_1

Khanban and Mahmoodian [11] define a linear sum and introduce a linear space which contains all generalized interchanges. Their operation, when induced on the set of generalized interchanges, is the same as the definition above. Mahdian and Mahmoodian [12] show that there is a basis for that linear space which consists of intercalates. In this note we introduce a very simple algorithm to write each latin interchange as a sum of intercalates.

2 The algorithm

In this section we state our results and, in the proofs, we introduce an algorithm for writing a latin interchange as sum of intercalates. A latin interchange is called a cycle if there are exactly two filled cells in each nonempty row or nonempty column; that is, a cycle has type $\begin{pmatrix} 2+2+2+\ldots+2 \\ 2+2+2+\ldots+2 \end{pmatrix}$. An intercalate is a cycle of volume 4.

LEMMA 2.1 *Every latin interchange which is a cycle can be written as sum of intercalates.*

Proof. We proceed by induction on s, the volume of the latin interchange. For the smallest value of $s = 4$ the cycle itself is an intercalate. Now let $C = (P, Q)$ be a cycle of volume $s > 4$. For conveniences we refer to the elements of C as 4-tuples, for example $(i, j; (P_{ij}, Q_{ij}))$. Reorder the rows so that row 1 is nonempty. So for some column j, $(1, j; (P_{1j}, Q_{1j})) \in C$. Let $P_{1j} = a$ and $Q_{1j} = b$ and it follows that for some column k, $(1, k; (b, a)) \in C$ and for some row i, $(i, j; (b, a)) \in C$. Then we see that C contains the following nonempty cells.

		j		k	
	·	a_b	·	b_a	·
	·	·	·	·	·
i		b_a			
	·	·	·	·	·

If cell (i, k) is nonempty then C can be written as the sum

$$\{(1, j; (a, b)), (1, k; (b, a)), (i, j; (b, a)), (i, k; (a, b))\} \ + \\ C \setminus \{(1, j; (a, b)), (1, k; (b, a)), (i, j; (b, a)), (i, k; (a, b))\},$$

which is the sum of two cycles. If cell (i, k) is empty then C can be written as the sum

$$\{(1, j; (a, b)), (1, k; (b, a)), (i, j; (b, a)), (i, k; (a, b)\} \ + \ C',$$

where

$$C' = (C \setminus \{(1,j;(a,b)),(1,k;(b,a)),(i,j;(b,a))\}) \cup \{(i,k;(b,a))\},$$

and it is easy to see that C' is a cycle as it is a latin interchange of type

$$\begin{pmatrix} 2+2+2+\ldots+2 \\ 2+2+2+\ldots+2 \end{pmatrix}.$$

In either case we have C equal to the sum of an intercalate with a cycle C', where $|C'| \leq |C| - 2$, and the statement follows by the inductive hypothesis. ■

EXAMPLE 2.1 The cycle in Example 1.3 can be decomposed as

$$C_1 = \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 2_1 & \cdot & 1_2 \\ \hline \cdot & 2_1 & \cdot & 1_2 & \cdot \\ \hline 1_2 & \cdot & \cdot & 2_1 & \cdot \\ \hline 2_1 & \cdot & 1_2 & \cdot & \cdot \\ \hline \cdot & 1_2 & \cdot & \cdot & 2_1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 2_1 & \cdot & 1_2 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & 1_2 & \cdot & 2_1 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & 2_1 & \cdot & 1_2 & \cdot \\ \hline 1_2 & \cdot & \cdot & 2_1 & \cdot \\ \hline 2_1 & \cdot & \cdot & \cdot & 1_2 \\ \hline \cdot & 1_2 & \cdot & \cdot & 2_1 \\ \hline \end{array} = \cdots$$

THEOREM 2.1 *Every generalized interchange can be written as sum of intercalates.*

Proof. We show that any generalized interchange can be written as sum of cycles. Then the statement follows by Lemma 2.1. We proceed by induction on s, the volume of generalized interchange. For the smallest value of s which is 4, the generalized interchange itself is an intercalate. Now let $I = (P,Q)$ be a generalized interchange of volume $s > 4$. Let a_1 be the smallest element which appears in the first nonempty row, r_1, of I and assume that (a_1,a_2) appears in row r_1 of I in a column say c_1, that is $(r_1,c_1;(a_1,a_2)) \in I$. There exists a_3, where (a_3,a_1) appears in row r_1 and in a column, say c_2 of I. There exists a_4, where (a_1,a_4) appears in column c_2 and in a row, say r_2 of I. Following this process we find a_5 where (a_5,a_1) appears in row r_2 and column c_3 of I, there exists a_6, where (a_1,a_6) appear in column c_3 and row r_3 of I, etc. This process will end when we find a_k where (a_k,a_1) appears in column c_1 of I (see the following pattern).

$$\begin{array}{cccccc} \cdot & \cdot & (a_1,a_2) & \cdot & \cdot & (a_3,a_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (a_1,a_{k-1}) & \cdot & (a_k,a_1) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & (a_5,a_1) & \cdot & (a_1,a_4) \\ * & * & \cdot & \cdot & \cdot & \cdot \\ \cdot & (a_6,a_1) & \cdot & (a_1,a_6) & \cdot & \cdot \end{array}$$

Note that since it is a generalized interchange, we may meet a row or a column more than once. In that case we omit the redundant paths and continue the process making sure that each row and each column (except row r_1 and column c_1) is visited only once (see the next example). Now I is equal to the sum of the following cycle plus a generalized interchange with a volume r, where $r < s$. And the statement follows by the induction hypothesis.

$$\begin{array}{cccccc} \cdot & \cdot & (a_1,a_2) & \cdot & \cdot & (a_2,a_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (a_1,a_2) & \cdot & (a_2,a_1) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & (a_2,a_1) & \cdot & (a_1,a_2) \\ * & * & \cdot & \cdot & \cdot & \cdot \\ \cdot & (a_2,a_1) & \cdot & (a_1,a_2) & \cdot & \cdot \end{array}$$

■

EXAMPLE 2.2 In the following we illustrate our algorithm on the latin interchange of Example 1.1. Example 1.3 shows the first step of the algorithm. It is interesting to see how it continues. Here, to facilitate understanding, we write in an initial row a pair of numbers which indicate the order and the volume of the latin interchange, respectively. If for latin interchanges I, J, and C we have $I = J + C$, then we also may write $J = I - C$.

$(5,17)$ - a cycle (C_2)				
·	·		3_2 - 2_3	2_3 - 3_2
·		·	2_4	4_2
2_4	·	5_3	4_2	3_5
5_2	4_2	2_5	·	2_4
4_5	2_4	3_2	2_3 - 3_2	5_2 - 2_3

$+$

Cycle C_2				
·	·	·	3_2	2_3
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	2_3	3_2

$=$

$(5,14)$				
·	·			
·		·	2_4	4_2
2_4	·	5_3	4_2	3_5
5_2	4_2	2_5	·	2_4
4_5	2_4	3_2		5_3

$+ C_2.$

At this point if we continue our process on the (5,14) latin interchange, we visit row four twice:

$(5,14)$ - a cycle ?				
·	·			
·		·	2_4 - 4_2	4_2 - 2_4
2_4	·	5_3	4_2	3_5
5_2	4_2 - 2_4	2_5 - ?	·	2_4 - 4_2
4_5	2_4 - 4_2	3_2 - 4_2		5_3

$=$?

As this is undesirable we delete the redundant path:

$(5,14)$ - a cycle (C_3)				
$\cdot$	$\cdot$			
$\cdot$		$\cdot$	2_4 - 4_2	4_2 - 2_4
2_4 - 4_2	$\cdot$	5_3	4_2 - 2_4	3_5
5_2 - 2_4	4_2	2_5	$\cdot$	2_4 - 4_2
4_5	2_4	3_2		5_3

$+$

Cycle C_3				
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	2_4	4_2
2_4	$\cdot$	$\cdot$	4_2	$\cdot$
4_2	$\cdot$	$\cdot$	$\cdot$	2_4
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

$=$

$(5,9)$				
$\cdot$	$\cdot$			
$\cdot$		$\cdot$		
	$\cdot$	5_3		3_5
5_4	4_2	2_5		
4_5	2_4	3_2		5_3

$+ C_3$.

$(5,9)$ - a cycle (C_4)				
$\cdot$	$\cdot$			
$\cdot$		$\cdot$		
	$\cdot$	5_3 - 3_5		3_5 - 5_3
5_4	4_2	2_5		
4_5	2_4	3_2 - 5_3		5_3 - 3_5

$+$

Cycle C_4				
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	5_3	$\cdot$	3_5
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	3_5	$\cdot$	5_3

$=$

$(5,6)$				
$\cdot$	$\cdot$			
$\cdot$		$\cdot$		
	$\cdot$			
5_4	4_2	2_5		
4_5	2_4	5_2		

$+ C_4$.

$(5,6)$ - a cycle (C_5)				
$\cdot$	$\cdot$			
$\cdot$		$\cdot$		
	$\cdot$			
5_4	4_2 - 2_4	2_5 - 4_2		
4_5	2_4 - 4_2	5_2 - 2_4		

$+$

Cycle C_5				
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	4_2	2_4	$\cdot$	$\cdot$
$\cdot$	2_4	4_2	$\cdot$	$\cdot$

$=$

$(5,4)$: (C_6)				
$\cdot$	$\cdot$			
$\cdot$		$\cdot$		
	$\cdot$			
5_4		4_5		
4_5		5_4		

$+ C_5$.

So the latin interchange of Example 1.1, $I(5,19)$, is decomposed as sum of 6 cycles namely, $C_1 + C_2 + C_3 + C_4 + C_5 + C_6$.

COROLLARY 2.1 *Every latin interchange can be written as sum of intercalates.*

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