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Defining Sets in Total Coloring of Graphs

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Abstract

In a given graph G, a set of elements (vertices and edges) D with an assignment of colors is said to be a *defining set of the total coloring of* G, if there exists a unique extension of the colors of D to a $\chi''(G)$-coloring of the elements of G. The concept of a defining set has been studied, in various subjects, for block designs and also for vertex coloring of graphs. In this note we determine the size of minimum total defining sets for some families of graphs.

AMS subject classification: 05
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1 Introduction

In this note we consider (simple) graphs which are finite, undirected, with no loops or multiple edges. For the concepts not defined here, we refer [1]. The minimum number of different colors required to color the elements (the vertices and the edges) of a graph G, such that no two associated elements have the same color is called the *total chromatic number* of G, and is denoted by $\chi''(G)$. In a given graph G, a set of elements D, $D \subset V \cup E$, with an assignment of colors is said to be a *defining set of total coloring*, or simply, total defining set, if there exists a unique extension of the colors of D to a $\chi''(G)$-coloring of the elements of G. A total defining set with minimum cardinality is called a *minimum total defining set*, and it is denoted by $S''(G)$, and its cardinality is denoted by $d(G, \chi'')$. This parameter for path P_n and a cycle C_{3n}, is $d(P_n, \chi'') = d(C_{3n}, \chi'') = 2$. The concept of defining set has been studied to some extend, for block designs, see [6], for vertex coloring of graphs, see [4], and under other name, critical set, for latin squares, see [2] and [5].

The following results are deduced from the definition of the total defining set and the proof of them is not difficult.

(i) For any graph G, the set $V \cup E$ is obviously a defining set of G. So $d(G, \chi'')$ exists and $d(G, \chi'') \geq 0$. In fact, $d(G, \chi'') = 0$ if and only if G is null graph.

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(ii) $S''(G)$ is not necessarily unique.

(iii) $d(G,\chi'') \geq \chi''(G) - 1$.

(iv) Let w be an element of G whose total degree is d. It means that if w is a vertex of G, then $d = 2deg_G w$, and if $w = uv$ is an edge of G, then $d = deg_G u + deg_G v$. If $\chi''(G) \geq d + 1$, then w is necessarily a member of any total defining set of G.

(v) For any graph G, $S''(G)$ can not contain only vertices or only edges. For example K_{2n} is a such graph. But for C_{3n}, $S''(C_{3n})$ can contain only vertices, also can contain only edges.

The following definition and a theorem due to Mahdian and Mahmoodian [3] are very useful in some of our results.

A graph G, with v vertices, has the property $M(2)$, if for any list of colors $S_1, S_2, \cdots, S_v$ (S_i is a list of colors available at vertex i), with $|S_i| \geq 2$; having a proper vertex coloring for G implies that there exists also a different proper vertex coloring for G.

Theorem A. [3] *A graph has the property $M(2)$ if and only if every block of which is either a cycle, a complete graph, or a complete bipartite graph.*

For every graph G a graph $T(G)$, called the total graph of G, is defined in such a way that $\chi''(G) = \chi(T(G))$; where $\chi(G)$ is the vertex chromatic number of G. Thus every minimum total defining set of G is a minimum vertex defining set of $T(G)$ and conversely.

The following theorem from [4] is useful in our proofs.

Theorem B. [4] *For any graph G we have*

$$d(G,\chi) \geq |V(G)| - \frac{|E(G)|}{\chi(G) - 1}.$$

($d_v(G)$ is the cardinality of the minimum vertex defining set of G).

2 Defining sets in total coloring

In the following theorem the size of minimum total defining set in cycles C_n is given.

Theorem 1. *For cycles we have,*

(i) $d(C_n,\chi'') = 2$ *if* $n \equiv 0 \pmod 3$

(ii) $d(C_n,\chi'') = \lceil \frac{2n}{3} \rceil$ *if* $n \equiv 2 \pmod 3$

(iii) $d(C_n,\chi'') = \lceil \frac{2n}{3} \rceil + 1$ *if* $n \equiv 1 \pmod 3$.

Proof. We know $\chi''(C_{3n}) = 3$, $\chi''(C_{3n+1}) = \chi''(C_{3n+2}) = 4$.

(i) In this case, it is enough to consider the following total defining set given in Figure 1.

(ii) In this case, first we claim that from every three associated elements of C_n at least one of them must be in every total defining set of C_n. For, assume that three associated elements have

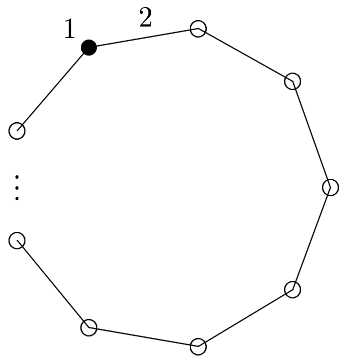

Figure 1: A minimum total defining set for C_{3n}

not been colored yet, and the colors of other elements of C_n are given. These three associated elements of C_n constitute a cycle of size 3 in its total graph. And the set of colors available at each of its vertices are of size 2 each. By Theorem A, we obtain two different total colorings for C_n, a contradiction. Therefore $d(C_n, \chi'') \geq \lceil \frac{2n}{3} \rceil$. (Of course this inequality can be obtained from Theorem B also). To show equality we give a total defining set of size $\lceil \frac{2n}{3} \rceil$ as in Figure 2.

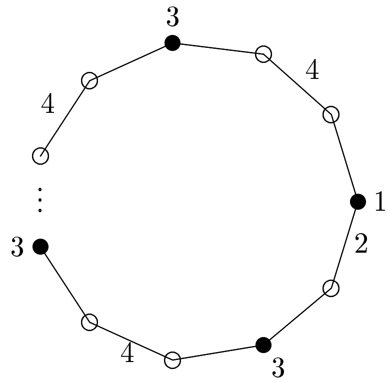

Figure 2: A minimum total defining set for C_{3n+2}

(*iii*) As in the case (*ii*), $d(C_n, \chi'') \geq \lceil \frac{2n}{3} \rceil$. If there exists a total defining set S'' of size $d(C_n, \chi'') \geq \lceil \frac{2n}{3} \rceil$, the remaining elements of C_n constitute a cycle in its total graph. And the set of colors available at each of its vertices are of size 2 each. By Theorem A, we obtain two different total colorings for C_n, but this contradicts the assumption that S'' is a total defining set. Thus $d(C_n, \chi'') \geq \lceil \frac{2n}{3} \rceil + 1$. To show equality we give a total defining set of size $\lceil \frac{2n}{3} \rceil + 1$ as in Figure 3.

□

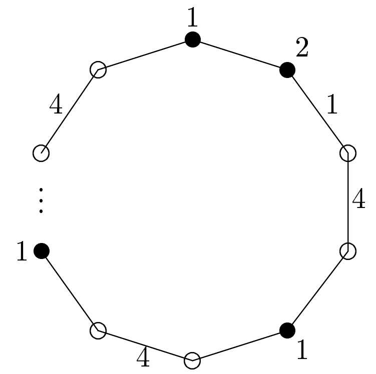

Figure 3: A minimum total defining set for C_{3n+1}

Theorem 2. *Every total defining set for the graph K_{2n}, $n \in N$, is a total defining set for the graph K_{2n+1} as well.*

Proof. Note that $\chi''(K_{2n}) = \chi''(K_{2n+1}) = 2n+1$. Let $1, 2, \cdots, 2n+1$ be the given $2n+1$ different colors. First each proper total coloring of K_{2n} is as follows. Class 1 consists of n independent edges of K_{2n} each of which has color 1. Class k, $2 \leq k \leq 2n+1$, consists of the vertex v_{k-1} along with $n-1$ independent edges of K_{2n} non of which is incident with v_{k-1}; the color of each of these elements is assumed to be k. Since K_{2n} is a subgraph of K_{2n+1}, along with the proper total coloring above, is in turn a total defining set for K_{2n+1}. To see this first note that in K_{2n} for each i, $1 \leq i \leq 2n$, only one of the given $2n+1$ colors is not allocated to the vertex v_i and the edges incident with v_i. Assign this missing color to the edge $v_i v_{2n+1}$ of K_{2n+1}. Now assign color 1 to the vertex v_{2n+1} to produce a proper total coloring for K_{2n+1}. □

The following example shows that, in general, the converse of the above theorem is not true.

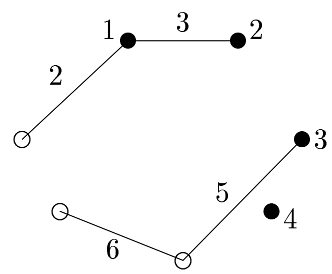

Figure 4: A minimum total defining set for K_7

In the graph K_7 the set of elements (vertices and edges) with the color numbers given as in Figure 4, is a total defining set but no restriction of it can be a total defining set for K_6.

We have $d(K_n, \chi'') = \chi''(K_n) - 1$, for $n < 6$. And for K_6 by a computer program (written by Kamran Bavar) it is shown that $d(K_6, \chi'') = 8$. Also by that program it has been checked that there does not exist a total defining set of size less than 8 for K_7, thus by Theorem 2, $d(K_7, \chi'') = 8$.

A wheel W_n is a graph obtained from C_n by adding a new vertex and edges joining it to all the vertices of the cycle. For wheels we have,

Theorem 3. *For $n > 6$, $d(W_n, \chi'') = 3n - 4$.*

Proof. It can be easily seen that $\chi''(W_n) = n+1$, for $n > 3$, ($\chi''(W_3) = 5$). As it was pointed out in Section 1(iv), for $n > 6$ each element of W_n which is on C_n, is necessarily a member of any total defining set of W_n. And we claim that among the n remaining edges of W_n, at least $n - 4$ of them must belong to any total defining set of W_n. For, assume that five of those edges have not been colored yet, and the other elements of W_n have been colored. These five edges constitute a complete graph K_5 in the total graph. And the set of colors available at each of its vertices are of size 2 each. By Theorem A, we obtain two different total colorings for W_n. Therefore $d(W_n, \chi'') \geq 3n - 4$. To show equality, we give a total defining set of size $3n - 4$ as in Figure 5. □

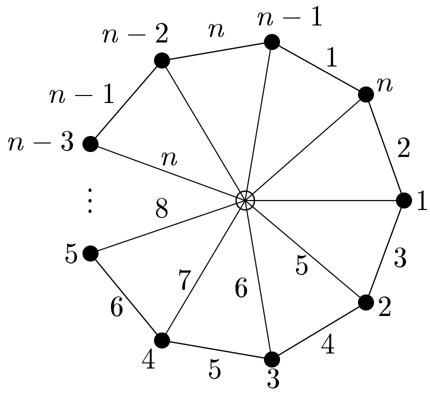

Figure 5: A minimum total defining set for W_n

Results mentioned above are preliminary results. There are many natural questions remained unanswered yet. We hope readers get interested and answer these questions.

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