MR1376713 (96m:05026) Mahmoodian, E. S.; Soltankhah, Nasrin Intersections of 2-(v,4,1) directed designs. J. Combin. Math. Combin. Comput. 20 (1996), 225–236

Intersections of 2-$(v, 4, 1)$ Directed Designs

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Abstract

The intersection problem for a pair of transitive triple systems (or 2-$(v, 3, 1)$ directed designs) is solved by Lindner and Wallis and independently by H.L. Fu in 1982–1983. In this paper we determine the intersection problem for a pair of 2-$(v, 4, 1)$ directed designs.

1 Introduction

Let $0 < t \leq k \leq v$ and $\lambda > 0$ be integers, and V be a set of v elements. Each ordered k-tuple of distinct elements of V is called a *block*. In this note by an n-tuple of V, we mean an ordred n-subset of V. With these meanings of 'block' and 'n-tuple', which we shall use throughout this paper, we make the following definition. A t-(v, k, λ) *directed design* (or simply a t-(v, k, λ)DD) is a pair $(V, \mathcal{B})$, where V is a v-set, and $\mathcal{B}$ is a collection of blocks, such that each t-tuple of V appears in precisely λ blocks. Note that a t-tuple is said to appear in a k-tuple, if its components are contained in that block as a set, and they appear with the same order. For example the 4-tuple $abcd$ contains the ordered pairs ab, ac, ad, bc, bd, and cd.

The problem of determining the possible number of common blocks between two designs with the same parameters is studied extensively. For a recent survey on this problem see Billington [1]. Lindner and Wallis [8] and independently H.L. Fu [5] settled the spectrum of possible intersection sizes for 2-$(v, 3, 1)$DDs (transitive triple systems) for all admissible v. Also the spectrum of possible intersection sizes for ordinary designs $S(2, 3, v)$ and $S(2, 4, v)$ is settled by Lindner and Rosa [7] and by Colbourn, Hoffman and Lindner [4] respectively. In this paper, we solve the intersection problem for 2-$(v, 4, 1)$DDs. The existence problem of 2-$(v, 4, \lambda)$DDs has been solved in [9]. The necessary and sufficient condition for the existence of a 2-$(v, 4, 1)$DD is $v \equiv 1 \pmod 3$.

The number of blocks in a 2-$(v, 4, 1)$DD is equal to $b_v = \frac{v(v-1)}{6}$. Let $J_D(v) = \{0, 1, \ldots, b_v - 2, b_v\}$, and let $I_D(v)$ denote the set of all possible integers m, such that there exist two 2-$(v,4,1)$DDs with exactly m common blocks. It is clear that $I_D(v) \subseteq J_D(v)$. We prove the following:

Main Theorem. For each $v \equiv 1 \pmod 3$, $v \neq 7$, $I_D(v) = J_D(v)$ and $I_D(7) = \{0, 1, 7\}$.

For the rest of this section we state some definitions which are needed in the sequel.

Let $K = \{k_1, \ldots, k_l\}$ be a set of numbers. A 2-(v, K, λ) *design* is a v-set V and a collection of k_i-subsets also called blocks, such that every 2-subset of V appears exactly λ times in the blocks. These designs are also called pairwise balanced designs (PBD).

We define a group divisible design as in Hanani [6]. Let V be a v-set such that $V = \bigcup_{i=1}^{t} G_i$, $G_i \cap G_j = \emptyset$, $|G_i| \in M$ for all i. G_i's are called *groups*. A *group divisible design*, $GD(k, \lambda, M; v)$, is a collection of k-subsets of a v-set V also called blocks such that each block intersects each group in at most one element and a pair of elements of V from different groups occurs in exactly λ blocks.

A *directed group divisible design* $DGD(k, \lambda, M; v)$ (or simply a DGD) is a group divisible design GD in which every block is ordered and each ordered pair formed from elements of different groups occurs in the same number of blocks. If $M = \{m\}$ then we simply write $DGD(k, \lambda, m; v)$.

A (v, k, t) *directed trade* (or simply a (v, k, t)DT) of volume s consists of two disjoint collections T' and T'', each of s blocks, such that each t-tuple occurs in the same number of blocks T' as of T''. Such a DT is usually denoted by $T = T' - T''$.

Let D be a t-(v, k, λ)DD and $T = T' - T''$ be a (v, k, t)DT. If D contains the collection of blocks of T'', then by substituting the blocks of T' for the blocks of T'' in the design, we obtain a new t-(v, k, λ)DD which is denoted by $D + T$. This method of "trade off" is used frequently in this paper.

2 Some small cases

In this section we discuss some small cases needed for general constructions.

$\mathbf{I_D(4) = J_D(4)}$

Let D_1 and D_2 be two 2-$(4, 4, 1)$DD on the set $\{0, 1, 2, 3\}$, given below.
D_1 : 0123, 3210 ; D_2 : 1023, 3201. We have $|D_1 \cap D_1| = 2$, $|D_1 \cap D_2| = 0$. □

$\mathbf{\{0, 1, 7\} \subseteq I_D(7)}$

Let D_1 be a 2-$(7, 4, 1)$DD with the base block (6 0 3 5) mod 7, and D_2 with the base block (5 3 0 6) mod 7, and let α be a permutation given by $\alpha = (01425)$ on the set $\{0, 1, \ldots, 6\}$. We have $|D_1 \cap D_1| = 7$, $|D_1 \cap D_2| = 0$, and $|D_1 \cap D_2\alpha| = 1$. □

$\mathbf{I_D(10) = J_D(10)}$

Let D_1 be the following 2-(10, 4, 1)DD, on the set $\{0, 1, \ldots, 9\}$.

1324	2156	3517	4189	8610	7901	2938	
4207	0582	6972	0436	5309	7683	9654	8745.

Now we list some small (10, 4, 2)DTs:

Directed Trade	Blocks removed	Blocks added
T	8610 7901	8601 7910
T_*	7901 6972	9701 6792
T_1	2156 3517	2516 3157
T_2	2938 7683	2983 7638
T_3	1324 4207	1342 2407
T_4	8745 9654	8754 9645

Let $D_2 = D_1 + T$ and $D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 13$, $|D_2 \cap D_3| = 12$, and for $i = 1, 2, 3, 4$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 15 - (2i + 2);$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 15 - (2i + 3).$$

For the following permutations on the elements of each block of D_3 we have

Permutation α	Intersection number of D_1 and $D_3\alpha$
(0123456789)	0
(03985267)	1
(047)	2
(0423)(15)(67)(89)	3

This results in $I_D(10) = J_D(10)$. □

$\mathbf{I_D(13) = J_D(13)}$

Let D_1 be the following 2-(13, 4, 1)DD on the set $\{0, 1, \ldots, 9, a, b, c\}$.

$123a$	$456a$	$789a$	$147b$	$258b$	$369b$
$b321$	$b654$	$b987$	$c741$	$c852$	$c963$
$159c$	$267c$	$348c$	3570	2490	1680
0951	0762	0843	$a753$	$a942$	$a861$

$abc0$

$0cba$

Now we list some small $(13,4,2)$DTs:

Directed Trade	Blocks removed	Blocks added
T	$789a$ $b987$	$879a$ $b978$
T_*	$789a$ $a942$	$78a9$ $9a42$
T_1	$123a$ $b321$	$213a$ $b312$
T_2	$456a$ $b654$	$465a$ $b564$
T_3	$147b$ $c741$	$417b$ $c714$
T_4	$258b$ $c852$	$285b$ $c582$
T_5	$369b$ $c963$	$396b$ $c693$
T_6	$159c$ 0951	$519c$ 0915
T_7	$267c$ 0762	$627c$ 0726
T_8	$348c$ 0843	$438c$ 0834
T_9	3570 $a753$	3750 $a573$
T_{10}	1680 $a861$	6180 $a816$
T_{11}	$abc0$ $0cba$	$bac0$ $0cab$
T_{12}	2490 $a942$	4290 $a924$

Let $D_2 = D_1 + T,\ D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 24,\ |D_2 \cap D_3| = 23$, and

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 26 - (2i+2) \quad i = 1, \ldots, 12;$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 26 - (2i+3) \quad i = 1, \ldots, 11.$$

This results in $I_D(13) = J_D(13)$. □

3 Recursive Constructions

We introduce two constructions, which will be applied in constructing designswith required intersection sizes.

Construction 1. If there exists a group divisible design $(G, \mathcal{B})$ of order v with block size 4 and groups each of size congruent to 0 (mod 3), then there exists a 2-$(2v+1,4,1)$DD.

Proof. Let $(G, \mathcal{B})$ be a group divisible design on the element set V. We form a 2-$(2v+1,4,1)$DD on the element set $V \times Z_2 \cup \{\infty\}$ as follows. For each block $b \in \mathcal{B}$, say $b = \{x,y,z,w\}$, we form a $DGD(4,1,2;8)$ on $b \times Z_2$, such that its groups are $\{x\} \times Z_2, \{y\} \times Z_2, \{z\} \times Z_2, \{w\} \times Z_2$. This DGD exists, as we will see later on. Now for each group g of G, we substitute a 2-$(2|g|+1,4,1)$DD on $(g \times Z_2) \cup \{\infty\}$. ■

Construction 2. If there exists a group divisible design $(G, \mathcal{B})$ of order v with block size 4 and groups of size 2 and 5, then there exists a 2-$(2v, 4, 1)$DD.

Proof. Let $(G, \mathcal{B})$ be such a GD on the element set V. We form a 2-$(2v, 4, 1)$DD on the element set $V \times Z_2$. For each block $b \in \mathcal{B}$, say $b = \{x, y, z, w\}$ we place a $DGD(4, 1, 2; 8)$ on $b \times Z_2$, such that its groups are $\{x\} \times Z_2$, $\{y\} \times Z_2$, $\{z\} \times Z_2$, $\{w\} \times Z_2$. For each group $g \in G$ we place a 2-$(10, 4, 1)$DD if $|g| = 5$, and we place a 2-$(4, 4, 1)$DD if $|g| = 2$ on $g \times Z_2$. ∎

In applying constructions 1 and 2 we need some GDs and DGDs. We may use a $GD(4, 1, \{3, 6\}; v)$, a $GD(4, 1, \{2, 5\}; v)$, and a $DGD(4, 1, 2; 8)$.

A $GD(4, 1, \{3, 6\}; v)$ exists for all $v \equiv 0 \pmod 3$, $v \neq 9, 18$. For $v \equiv 0, 3 \pmod{12}$, they may be obtained by omitting an element from a 2-$(v + 1, 4, 1)$ design. For $v \equiv 6, 9 \pmod{12}$ they may be obtained by taking an element of the block of size 7 in a 2-$(v + 1, \{4, 7^*\}, 1)$ design with only one block of size 7 [2], and omitting this element from all the blocks which contain it.

A $GD(4, 1, \{2, 5^*\}; v)$ with one group of size 5 exists for all $v \equiv 5 \pmod 6$, $v \neq 11, 17$ by [2], and a $GD(4, 1, 2; v)$ exists for all $v \equiv 2 \pmod 6$, $v \neq 8$ by [3]. Then it can be deduced that a $GD(4, 1, \{2, 5\}; v)$ exists for all $v \equiv 2, 5 \pmod 6$, $v \neq 8, 11, 17$.

We can construct a $DGD(4, 1, 2; 8)$ on the set $\{0, 1, \ldots, 7\}$ as follows:
groups: 12, 34, 56, 07.
blocks: 5103, 4016, 3175, 6714, 2360, 5247, 0425, 7632.

Consider all $DGD(4, 1, 2; 8)$s with the same groups. Let $I_G(8)$ be the set of all possible integers m, such that there exist two such DGDs with exactly m common blocks.

$\mathbf{I_G(8) = \{0, 1, 2, 3, 4, 5, 6, 8\}}$

Consider the $DGD(4, 1, 2; 8)$ constructed above, and let D_1 be its blocks. Now we list some small directed trades:

Directed Trade	Blocks removed	Blocks added
T	5103 4016	5013 4106
T_*	4016 0425	0416 4025
T_1	3175 6714	3715 6174
T_2	2360 7632	2630 7362
T_3	5247 0425	2547 0452

Let $D_2 = D_1 + T$ and $D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 6$, $|D_2 \cap D_3| = 5$, and

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 8 - (2i + 3) \quad i = 1, 2 \ ;$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 8 - (2i+2) \quad i = 1, 2, 3.$$

This results in $I_G(8) = \{0, 1, 2, 3, 4, 5, 6, 8\}$. □

Lemma 1. Let $(G, \mathcal{B})$ be a group divisible design of order v with b blocks, each of size 4, and $r+s$ groups, r of size 3 and s of size 6. For $1 \leq i \leq b$, let $a_i \in I_G(8)$. For $1 \leq i \leq r$, let $c_i \in I_D(7)$; for $1 \leq i \leq s$, let $d_i \in I_D(13)$. Then there exist two 2-$(2v+1, 4, 1)$DDs intersecting in precisely

$$\sum_{i=1}^{b} a_i + \sum_{i=1}^{r} c_i + \sum_{i=1}^{s} d_i$$

blocks.

Proof. Using construction 1, take two copies of the same group divisible design $(G, \mathcal{B})$ and construct on them two 2-$(2v+1, 4, 1)$DDs. Corresponding to each of the blocks $B_1, \ldots, B_b$, place on $B_i \times Z_2$ in the two systems $DGD(4, 1, 2; 8)$s having the same groups, and a_i blocks in common. Corresponding to groups G_i of size 3, place 2-$(7, 4, 1)$DDs with c_i blocks in common, and for groups H_i of size 6, place 2-$(13, 4, 1)$DDs with d_i blocks in common. □

For the $2v$ construction we also have a similar lemma.

Lemma 2. Let $(G, \mathcal{B})$ be a group divisible design of order v with b blocks, each of size 4, and $r+s$ groups, r of size 2 and s of size 5. For $1 \leq i \leq b$, let $a_i \in I_G(8)$. For $1 \leq i \leq r$, let $c_i \in I_D(4)$; for $1 \leq i \leq s$, let $d_i \in I_D(10)$. Then there exist two 2-$(2v, 4, 1)$DDs intersecting in precisely

$$\sum_{i=1}^{b} a_i + \sum_{i=1}^{r} c_i + \sum_{i=1}^{s} d_i$$

blocks.

4 Applying recursions

In this section, we prove the following main theorems.

Theorem 1. For $v \equiv 1$ or $7 \pmod{12}$, $v \neq 7, 19, 37$, $I_D(v) = J_D(v)$.

Proof. There are four possibilities for v: $v = 2(12k)+1$, $v = 2(12k+3)+1$,$v = 2(12k+6)+1$ or $v = 2(12k+9)+1$. Now we may apply construction 1. All the required DGDs and GDs exist. By Lemma 1 and the fact that $\{0, 1, 7\} \subseteq I_D(7)$, $I_D(13) = J_D(13)$ and $I_G(8) = \{0, 1, 2, 3, 4, 5, 6, 8\}$, we deduce $I_D(v) = J_D(v)$ for $v \equiv 1$ or $7 \pmod{12}$, $v \neq 7, 19, 37$. ■

Theorem 2. For $v \equiv 4$ or $10 \pmod{12}$, $v \neq 16, 22, 34$ $I_D(v) = J_D(v)$.

Proof. In this case, either v is $v = 2(6k+2)$ or $v = 2(6k+5)$. We may apply construction 2. By Lemma 2 and the fact that $I_D(4) = J_D(4)$, $I_D(10) = J_D(10)$ and $I_G(8) = \{0,1,2,3,4,5,6,8\}$, we deduce $I_D(v) = J_D(v)$ for $v \equiv 4$ or $10 \pmod{12}$, $v \neq 16, 22, 34$. ∎

5 Remaining small cases

Six small orders $\{7, 16, 19, 22, 34, 37\}$ remain. In this section we handle these small cases.

$\mathbf{I_D(7) = \{0, 1, 7\}}$

For $v = 7$ we show that there exist only two non-isomorphic directed designs of this order. Using this result we obtain the intersection numbers.
The existence of exactly two non-isomorphic 2-$(7,4,1)$DD is shown in the following three steps.

(i) For a given 2-$(7,4,1)$DD on the set of elements $\{0, 1, \ldots, 6\}$, we may consider a matrix of size 7×4, whose rows are the blocks of this design. Let x be an element and x_i be the number of appearances of x in the *i-th* column $(1 \leq i \leq 4)$ of the matrix. We count the number of all ordered pairs such as xy and yx respectively, for a constant x. We have

$$3x_1 + 2x_2 + x_3 = 6 \text{ and } x_2 + 2x_3 + 3x_4 = 6$$

respectively. From these equations it follows that $0 \leq x_1 \leq 2$ and $0 \leq x_4 \leq 2$. Since the 2-$(7,4,2)$ design obtained from a 2-$(7,4,1)$DD is symmetric every two blocks have two elements in common. Thus $x_1 = 2$ or $x_4 = 2$ is impossible. Thus $0 \leq x_1 \leq 1$ and $0 \leq x_4 \leq 1$. We solve these two equations for $x_1 = 0$ and for $x_1 = 1$.

$(1): \quad x_1 = 0, \quad x_2 = 2, \quad x_3 = 2, \quad x_4 = 0$

$(2): \quad x_1 = 0, \quad x_2 = 3, \quad x_3 = 0, \quad x_4 = 1$

$(3): \quad x_1 = 1, \quad x_2 = 0, \quad x_3 = 3, \quad x_4 = 0$

$(4): \quad x_1 = 1, \quad x_2 = 1, \quad x_3 = 1, \quad x_4 = 1$

Clearly for each fixed column (i) we have

$$\sum_{0 \leq x \leq 6} x_i = 7$$

Let a_j be the number of elements with frequencies as in solution (j) above, $(j =$

$1, 2, 3, 4)$. Then for the first and fourth columns we have.

$$0 \times a_1 + 0 \times a_2 + a_3 + a_4 = 7$$
$$0 \times a_1 + 1 \times a_2 + 0 \times a_3 + 1 \times a_4 = 7$$

By solving these equations along with $a_1 + a_2 + a_3 + a_4 = 7$, we obtain $a_1 = a_2 = a_3 = 0$ and $a_4 = 7$. Thus for each element x, $0 \leq x \leq 6$ we have

$$x_1 = x_2 = x_3 = x_4 = 1.$$

(ii) By the above result and by an easy argument one may show that, if there exists a block of the form $xyzw$, then no two adjacent elements in this block, say xy, can be adjacent in any other block.

(iii) Now if 1234 is a block in a 2-$(7, 4, 1)$DD, and if the second element of the second block is 1, then in column 1 of this block we must have 3 or 4.

If it is 3, then a unique 2-$(7, 4, 1)$DD, may be constructed as follows:

D_1 : 1234 3156 2610 0541 5302 6425 4063

If that element is 4, then a unique 2-$(7, 4, 1)$DD, may be constructed as follows:

D_2 : 1234 4156 5310 2061 6403 0542 3625

For any permutation $\alpha \in S_7$ we have $|D_1\alpha \cap D_2| = 0$ or 1. Thus D_1 and D_2 are non-isomorphic. And for any permutation α we have $|D_1\alpha \cap D_1| = 0, 1$ or 7 and $|D_2\alpha \cap D_2| = 0, 1$ or 7. Therefore we deduce $I_D(7) = \{0, 1, 7\}$. □

Note. One may produce two non-isomorphic cyclic 2-$(7, 4, 1)$DDs with base blocks (5 3 0 6) and (6 0 3 5) mod 7 respectively. These designs are isomorphic to D_1 and D_2 respectively.

$\mathbf{I_D(16) = J_D(16)}$

Let D_1 be the following 2-$(16, 4, 1)$DD, on the set $\{0, 1, \ldots, 9, a, b, c, d, e, f\}$.

1248	2359	346a	457b	568c	679d	78ae	ea21	fb32	c431
d542	e653	f764	89bf	19ac	2abd	3bce	4cdf	cb95	dca6
edb7	fec8	fd91	15de	26ef	137f	16b0	27c0	38d0	ba84
8751	9862	a973	0b61	0c72	0d83	49e0	5af0	0e94	0fa5

Now we list some small $(16, 4, 2)$DTs:

Directed Trade	Blocks removed	Blocks added
T	1248 $ea21$	2148 $ea12$
T_*	1248 $ba84$	1284 $ba48$
T_1	2359 $fb32$	3259 $fb23$
T_2	$346a$ $c431$	$436a$ $c341$
T_3	$457b$ $d542$	$547b$ $d452$
T_4	$568c$ $e653$	$658c$ $e563$
T_5	$679d$ $f764$	$769d$ $f674$
T_6	$89bf$ $cb95$	$8b9f$ $c9b5$
T_7	$19ac$ $dca6$	$19ca$ $dac6$
T_8	$2abd$ $edb7$	$2adb$ $ebd7$
T_9	$3bce$ $fec8$	$3bec$ $fce8$
T_{10}	$4cdf$ $fd91$	$4cfd$ $df91$
T_{11}	$15de$ 8751	$51de$ 8715
T_{12}	$26ef$ 9862	$62ef$ 9826
T_{13}	$137f$ $a973$	$173f$ $a937$
T_{14}	$16b0$ $0b61$	$61b0$ $0b16$
T_{15}	$27c0$ $0c72$	$72c0$ $0c27$
T_{16}	$38d0$ $0d83$	$83d0$ $0d38$
T_{17}	$49e0$ $0e94$	$490e$ $e094$
T_{18}	$5af0$ $0fa5$	$a5f0$ $0f5a$
T_{19}	$78ae$ $ba84$	$7a8e$ $b8a4$

Let $D_2 = D_1 + T$ and $D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 38$, $|D_2 \cap D_3| = 37$, and

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 40 - (2i+3) \quad i = 1, \ldots, 18;$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 40 - (2i+2) \quad i = 1, \ldots, 19.$$

This results in $I_D(16) = J_D(16)$. □

$\mathbf{I_D(19) = J_D(19)}$

Let D_1 be a 2-$(19, 4, 1)$DD on the set $\{0, 1, \ldots, 18\}$, with base blocks (0 3 12 1), (12 0 4 18), (17 3 0 13) mod 19 ([9]). Now we list some small $(19, 4, 2)$DTs:

$T:$ $\quad T' = \{0\ 3\ 12\ 1,\ 17\ 3\ 0\ 13\};\ T'' = \{3\ 0\ 12\ 1,\ 17\ 0\ 3\ 13\}.$

T_* $\quad T'_* = \{17\ 3\ 0\ 13,\ 11\ 18\ 3\ 17\};\ T''_* = \{3\ 17\ 0\ 13,\ 11\ 18\ 17\ 3\}.$

$T_i:$ $\quad T'_i = \{17+i\ 3+i\ 0+i\ 13+i,\ 0+i\ 3+i\ 12+i\ 1+i\};$
$T''_i = \{17+i\ 0+i\ 3+i\ 13+i,\ 3+i\ 0+i\ 12+i\ 1+i\} \quad i = 1, \ldots, 18.$

$T_{l+19}:$ $\quad T'_{l+19} = \{3+l\ 0+l\ 12+l\ 1+l,\ 12+l\ 0+l\ 4+l\ 18+l\};$
$T''_{l+19} = \{3+l\ 12+l\ 0+l\ 1+l,\ 0+l\ 12+l\ 4+l\ 18+l\} \quad l = 0, \ldots, 17.$

Let $D_2 = D_1 + T$ and $D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 55$, $|D_2 \cap D_3| = 54$, and

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 57 - (2i+3) \quad i = 1, \ldots, 18;$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 57 - (2i+2) \quad i = 1, \ldots, 18;$$

$$|(D_2 + \sum_{j=1}^{l} T_{j+19}) \cap D_1| = 57 - (l+40) \quad l = 0, \ldots, 17.$$

This results in $I_D(19) = J_D(19)$. □

$\mathbf{I_D(22) = J_D(22)}$

In [2] it is shown that there exists a 2-$(22, \{4, 7^*\}, 1)$ design. If we replace the block of size 7 by a 2-$(7,4,1)$DD and put a 2-$(4,4,1)$DD on each block of size 4, then we obtain a 2-$(22,4,1)$DD. From the fact that $\{0,1,7\} \subseteq I_D(7)$ and $I_D(4) = J_D(4)$, we deduce $J_D(22) - \{b-5, b-3\} \subseteq I_D(22)$, where b is the number of blocks of the design. For the remaining intersection numbers, we may use a recursive construction as follows.

We have $22 = 3 \times 7 + 1$. We construct a 2-$(7,4,1)$DD on the set $A = \{1, \ldots, 7\}$ and we take a Kirkman triple system of order 15 on the set $B = \{8, \ldots, 22\}$. Let $P_1, \ldots, P_7$ be parallel classes of this system. The 4-tuples of D_1, the desired 2-$(22,4,1)$DD on the set $A \cup B$ are:

(i) the 4-tuples of 2-$(7,4,1)$DD;

(ii) the 4-tuples $xyzi$, $(i+1)zyx$ mod 7 such that $\{x,y,z\}$ is a triple in P_i, $i = 1, \ldots, 7$.

Note that with any prior order on the triples of P_i's the resulting design is a 2-$(22,4,1)$DD.

Now we introduce some directed trades on D_1. If $\{x,y,z\}$, $\{a,b,c\} \in P_1$ and $\{x',y',z\} \in P_2$, then D_1 may be constructed so that the following blocks belong to

D_1:

$$xyz1,\ abc1,\ 2zyx,\ 2cba,\ x'y'z2,\ 3zy'x'$$

Now take these small $(22,4,2)$DTs:

Directed Trade	Blocks removed	Blocks added
T	$xyz1\ \ 2zyx\ \ x'y'z2$	$yxz1\ \ z2xy\ \ x'y'2z$
T_1	$abc1\ \ 2cba$	$bac1\ \ 2cab$

Let $D_2 = D_1 + T$. We have $|D_1 \cap D_2| = b-3$, $|D_1 \cap (D_2+T_1)| = b-5$. Therefore $I_D(22) = J_D(22)$. □

$\mathbf{I_D(34) = J_D(34)}$

By an argument similar to above, using the 2-$(34, \{4, 7^*\}, 1)$ design constructed in [2], it can be shown that $J_D(34) - \{b-5, b-3\} \subseteq I_D(34)$. For the remaining two values, we construct a 2-$(34,4,1)$DD as in [9]. Take a 2-$(11,5,1)$DD with the base block (3 5 1 4 9) mod 11. On each block $b = xyzuw$ of this directed design we form a 2-$(16,4,1)$ design on the set $b \times Z_3 \cup \{\infty\}$, such that it contains the quadruples $\{x\} \times Z_3 \cup \{\infty\}, \{y\} \times Z_3 \cup \{\infty\}, \{z\} \times Z_3 \cup \{\infty\}, \{u\} \times Z_3 \cup \{\infty\}, \{w\} \times Z_3 \cup \{\infty\}$ and we put an order on the quadruples of this design, such that its quadruples have the order induced by block b. By this method we may construct a 2-$(34,4,1)$DD, D_1: such that the following blocks belong to D_1,

$$(3,1)(5,1)(1,2)(4,1),\ (5,1)(3,1)(6,1)(11,2),$$
$$(3,2)(5,2)(1,3)(4,2),\ (5,2)(3,2)(6,2)(11,3),\ (6,1)(8,2)(4,1)(1,2).$$

Now take these small $(34,4,2)$DTs.

$T:\quad T' = \{(3,1)(5,1)(1,2)(4,1),\ (5,1)(3,1)(6,1)(11,2),\ (6,1)(8,2)(4,1)(1,2)\};$
$\quad\quad T'' = \{(5,1)(3,1)(4,1)(1,2),\ (3,1)(5,1)(6,1)(11,2),\ (6,1)(8,2)(1,2)(4,1)\}.$

$T_1:\quad T_1' = \{(3,2)(5,2)(1,3)(4,2),\ (5,2)(3,2)(6,2)(11,3)\};$
$\quad\quad T_1'' = \{(5,2)(3,2)(1,3)(4,2),\ (3,2)(5,2)(6,2)(11,3)\}.$

Let $D_2 = D_1 + T$. We have $|D_1 \cap D_2| = b-3$, $|D_1 \cap (D_2+T_1)| = b-5$. Therefore $I_D(34) = J_D(34)$. □

$\mathbf{I_D(37) = J_D(37)}$

To construct 2-$(37,4,1)$DD's we use a general recursive construction described below.

Construction 3. If there exists a 2-$(v,4,1)$ design, then there exists a 2-$(3v-2,4,1)$DD.

Proof. Let D be a 2-$(v, 4, 1)$ design on the set $\{1, \ldots, v-1\} \cup \{\infty\}$. If a block $b \in D$ contains ∞, say $b = \{x, y, z, \infty\}$, then we replace b by a 2-$(10, 4, 1)$DD on the set $(\{x, y, z\} \times Z_3) \cup \{\infty\}$. If b does not contain ∞, say $b = \{x, y, z, w\}$, then we replace b by a $GD(4, 1, 3; 12)$ on the set $b \times Z_3$, such that its groups are $\{x\} \times Z_3$, $\{y\} \times Z_3$, $\{z\} \times Z_3$, $\{w\} \times Z_3$ and on each block of this $GD(4, 1, 3; 12)$ we form a 2-$(4, 4, 1)$DD. ∎

Since $37 = 3 \times 13 - 2$, we may use construction 3 for 2-$(37, 4, 1)$DD. Since $I_D(4) = J_D(4)$ and $I_D(10) = J_D(10)$, therefore we can deduce $I_D(37) = J_D(37)$. □

References

[1] E.J. Billington, The intersection problem for combinatorial designs, *Congressus Numerantium* (to appear).

[2] A.E. Brouwer, Optimal packings of K_4's into a K_n, *J. Combin. Theory,* Series A**26** (1979), 278–297.

[3] A.E. Brouwer, H. Hanani and A. Schrijver, Group divisible designs with block size four, *Discrete Math.* **20** (1977), 1–10.

[4] C.J. Colbourn, D.G. Hoffman and C.C. Lindner, Intersections of $S(2, 4, v)$ designs, *Ars Combin.* **33** (1992), 97–111.

[5] H.L. Fu, Directed triple systems having a prescribed number of triples in common, *Tamkang J. Math.* **14** (1983), 85–90.

[6] H. Hanani, Balanced incomplete block designs and related designs, *Discrete Math.* **11** (1975), 255–369.

[7] C.C. Lindner and A. Rosa, Steiner triple systems having a prescribed number of triples in common, *Canad. J. Math.* **27** (1975), 1166–1175. Corrigendum: *Canad. J. Math.* **30** (1978), 890.

[8] C.C. Lindner and W.D. Wallis, Embeddings and prescribed intersection of transitive triple systems, *Annals of Discrete Math.* **15** (1982), 265–272.

[9] D.J. Street and J.R. Seberry, All DBIBDs with block size four exist, *Utilitas Math.* **18** (1980), 27–34.