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Intersections of Directed Quadruple Systems

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Abstract

A t-(v, k, λ) Directed Block Design (or simply a t-(v, k, λ)DD) is a pair $(V, \mathcal{B})$, where V is a v-set and $\mathcal{B}$ is a collection of ordered k-tuples of distinct elements of V, such that every ordered t-tuple of distinct elements of V belongs to exactly λ elements of $\mathcal{B}$. We say that a t-tuple belongs to a k-tuple, if its components appear in that k-tuple as a set, and they appear with the same order. In this paper we solve the intersection sizes for a pair of 3-$(v, 4, 1)$ directed designs.

1 Introduction

Let $0 < t \leq k \leq v$ and $\lambda > 0$ be integers, and V be a set of v elements. Each ordered k-tuple of distinct elements of V is called a *block*. In this note by an n-tuple of V, we mean an n-subset of V, which has order. A t-(v, k, λ) *directed design* (or simply t-(v, k, λ)DD) is a pair $(V, \mathcal{B})$, where V is a v-set, and $\mathcal{B}$ is a collection of blocks, for which each t-tuple of V appears in precisely λ blocks. Note that a t-tuple is said to appear in a k-tuple, if its components are contained in that block as a set, and they appear with the same order. For example the ordered 4-tuple $abcd$ contains the ordered triples abc, abd, acd and bcd.

The problem of determining the possible number of common blocks between two designs with the same parameters is studied extensively. For a recent survey on this problem see Billington [1]. Lindner and Wallis [4] and independently H.L. Fu [2] settled the spectrum of possible intersection sizes for 2-$(v, 3, 1)$DDs (transitive triple systems) for all admissible v. Mahmoodian and Soltankhah [5] solved the intersection problem for 2-$(v, 4, 1)$DDs. In this paper, we solve the intersection problem for 3-$(v, 4, 1)$DDs. The existence problem for 3-$(v, 4, 1)$DDs has been settled in [7].

The number of blocks in each 3-$(v, 4, 1)$DD is equal to $b_v = \frac{v(v-1)(v-2)}{4}$. Let $J_D(v) = \{0, 1, \ldots, b_v\} - \{b_v - 3, b_v - 1\}$, and let $I_D(v)$ denote the set of all possible integers m, such that there exist two 3-$(v, 4, 1)$DDs with exactly m common blocks. We prove the following:

For each even v, $I_D(v) = J_D(v)$.
Note that the appearance of every ordered pair in a 3-$(v,4,1)$DD is equal to $\frac{3(v-2)}{2}$, thus it is necessary that v be even.

2 Necessary Conditions

In this section, we establish necessary conditions on $I_D(v)$.

Definition. A (v,k,t) *directed trade* (or simply a (v,k,t)DT) of volume s consists of two disjoint collections T_1 and T_2, each of s blocks, such that for every t-tuple the number of blocks containing this t-tuple is the same in both T_1 and T_2. Such a *DT* is usually denoted by $T = T_1 - T_2$.

Lemma 1. For all even v, $I_D(v) \subseteq J_D(v)$.

Proof. Let D_1 and D_2 be two 3-$(v,4,1)$DDs with $|D_1 \cap D_2| = b_v - i$, $\quad 0 \le i \le b_v$. The set of different blocks in D_1 and D_2 form a $(v,4,3)$DT, T of volume i where $T = T_1 - T_2$, $T_1 = D_1 - D_2$ and $T_2 = D_2 - D_1$. There do not exist a $(v,4,3)$DT of volume 1 or 3, see [6]. Thus i can not be 1 or 3. Therefore $I_D(v) \subseteq J_D(v)$. ∎

3 Small Cases

Lemma 2. For $v = 4$: $\quad I_D(4) = \{0,1,2,4,6\}$.

Proof. Let $D = \{1230, 1032, 2013, 3012, 3210, 0231\}$ be a 3-$(4,4,1)$DD on the set $\{0,1,2,3\}$ and let α denote a permutation on the same set. For the following permutations on elements of each block of D we have:

$$\begin{array}{lll} \text{for} \quad \alpha_0 = (03) & , & |D \cap D\alpha_0| = 0 \quad ; \\ \text{for} \quad \alpha_1 = (013) & , & |D \cap D\alpha_1| = 1 \quad ; \\ \text{for} \quad \alpha_2 = (0213) & , & |D \cap D\alpha_2| = 2 \quad ; \\ \text{for} \quad \alpha_4 = (02) & , & |D \cap D\alpha_4| = 4. \end{array}$$ ∎

Let D be a t-(v,k,λ)DD which contains the collection of blocks in T_1 of a (v,k,t)DT. Then by substituting the blocks of T_2 for the blocks of T_1 in the design, we obtain a new t-(v,k,λ)DD. This method of "trade off" is used in the following lemma.

Lemma 3. For $v = 6$: $\quad I_D(6) = J_D(6)$.

Proof. Let D_1 be the following 3-$(6,4,1)$DD, on the set $\{0,1,\ldots,5\}$.

1234	1325	1420	1530	1045	2310	2415	2540	2053	2143
3450	3514	3042	3152	3201	4523	4013	4102	4251	4305
5012	5103	5204	5341	5432	0154	0235	0324	0431	0521

Now we list some small $(6,4,3)$DTs:

Directed Trade	Blocks removed	Blocks added
T	$1234, 2143$	$1243, 2134$
T'	$1234, 1325, 0235, 0324$	$1235, 1324, 0234, 0325$
T''	$4013, 4102, 5103, 5012$	$4103, 4012, 5013, 5102$

With these directed trades we may obtain following intersection numbers.

Trades used	Directed Designs	Intersection Number
T	$D_1, D_1 + T$	28
T'	$D_1, D_1 + T'$	26
T, T'	$D_1 + T, D_1 + T'$	25
T, T''	$D_1 + T, D_1 + T''$	24
T', T''	$D_1 + T', D_1 + T''$	22
T, T', T''	$D_1 + T, D_1 + T' + T''$	21

Let $D_2 = D_1 + T'$ and let α be a permutation on the set $\{0, 1, \ldots, 5\}$. Then for the following permutations on elements of each block of D_2 we have:

Permutation α	Intersection Number of D_1 and $D_2\alpha$
(012345)	0
(01235)	1
(014253)	2
(012453)	3
(013)(24)	4
(014523)	5
(01)	6
(012)(35)	7
(013452)	8
(01253)	9
(01)(23)(45)	10
(031)(254)	11
(013)(245)	12
(014)(235)	13
(01)(45)	14
(13542)	15
(024)(135)	16
(05)(34)	17
(053)(142)	18
(14)(35)	19
(035)(124)	20

And finally for D_3, the following 3-$(6,4,1)$DD, and $\alpha = (025)(143)$ we have $|D_3 \cap D_3\alpha| = 23$.

1230	1245	1354	1432	1502	2351	2543	2034	2105	3012
3045	3140	3241	3250	4512	4530	4025	4103	4201	4315
5031	5104	5240	5213	5342	0153	0214	0352	0423	0541

From these results, $I_D(6) = J_D(6)$. ∎

4 Main Theorem

Theorem. For each even v: $I_D(v) = J_D(v)$.

Proof. Hanani [3] has shown that 3-$(v, \{4,6\}, 1)$ designs exist for v even.

Case 1: $v \equiv 2, 4 \pmod 6$. In this case all the blocks in 3-$(v, \{4,6\}, 1)$ designs are of size 4. Let $s \in J_D(v)$. Then for s we have one of the following cases.

$$\begin{array}{lll}
(0) & s = 6l + 0, & 0 \le l \le b_v/6; \\
(1) & s = 6l + 1, & 0 \le l \le b_v/6 - 1; \\
(2) & s = 6l + 2, & 0 \le l \le b_v/6 - 1; \\
(3) & s = 6l + 3, & 0 \le l \le b_v/6 - 2; \\
(4) & s = 6l + 4, & 0 \le l \le b_v/6 - 1; \\
(5) & s = 6l + 5, & 0 \le l \le b_v/6 - 2.
\end{array}$$

For constructing two 3-$(v,4,1)$DDs with s common blocks, we use two copies of a 3-$(v,4,1)$ design (Steiner Quadruple System), namely S_1 and S_2. Note that each of these designs has $b_v/6$ quadruples.

In the cases (0), (1), (2), (4) (respectively), we construct D, the 3-$(4,4,1)$DD given in previous section, on (the elements of) each quadruple of S_1. This results in a 3-$(v,4,1)$DD, say E_1. And we construct the same D on each of the first l quadruples of S_2, but on the $(l+1)$-st quadruple we construct $D\alpha_i$ for $i = 0,1,2,4$ respectively, and on each of the remaining quadruples of S_2, we construct a $D\alpha_0$. This results in a 3-$(v,4,1)$DD, say E_2. Then E_1 and E_2 have the required number of blocks in common.

In the cases (3), (5) (respectively), the design E_1 is constructed as above. Also we construct D on each of the first l quadruples of S_2, but on the $(l+1)$-st quadruple we construct $D\alpha_1$ and on the $(l+2)$-nd quadruple we construct $D\alpha_{(i-1)}$ for $i = 3, 5$ respectively, and on each of the remaining quadruples of S_2, we construct a $D\alpha_0$. This results in a 3-$(v,4,1)$DD, say E_2. E_1 and E_2 give the required intersection numbers.

Case 2: $v \equiv 0 \pmod 6$. In this case the block sizes in 3-$(v, \{4,6\}, 1)$ designs are either 4 or 6. We may use these designs to construct a 3-$(v,4,1)$DD. For the blocks of size 4 we substitute a 3-$(4,4,1)$DD and for the blocks of size 6 we substitute a 3-$(6,4,1)$DD. By Lemma 2 and Lemma 3 and by a similar method as in the Case 1 we settle this case. ■

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