A CRITICAL CASE METHOD OF PROOF
IN COMBINATORIAL MATHEMATICS

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INDEX

TABLE OF CONTENTS

PREFACE

I believe that at this stage, the theory of combinatorics in general and graph theory in particular are in need of good tools to simplify the proofs of theorems and, if possible, categorize seemingly divergent proofs of the "same type" of theorems.

In a survey of the theory of "factors of a graph" (also known as "spanning subgraphs with specified valencies") I developed a method which seems very useful in proving several combinatorial theorems involving certain inequalities as necessary and sufficient conditions. This method is an outgrowth of one which was first employed by P. R. Halmos and H. E. Vaughan [HV] to prove P. Hall's theorem on system of distinct representatives. By the application of this method I have simplified the proofs of some of the standard theorems of combinatorics and graph theory; which state the necessary and sufficient conditions for the following:

I. A graph* to have a matching of deficiency d (a theorem of Berge's [B])

II. A graph* to have a 1-factor (Tutte [T1])

III. The existence of a (0,1)-matrix, with given

*Graphs in this paper are undirected and loopless.

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NOTATIONS

$E(G)$ 27

$G \times H$ 57

$h_G(S)$ 5

$q(S,T)$ 49

$V(G)$ 5

$\Delta(G)$ 56

$\chi_1(G)$ 56

$[0]$ 21

$[c]$ 21

$| \ |$ 1

$< \ >$ 5

$\lceil \ \rceil$ 52

$\square$ end of the proof for lemma

$\blacksquare$ end of the proof for theorem or corollary

[ ] reference is given inside the brackets

row and column sum vectors (Ryser [R1], Gale [G])

IV. The existence of a graph* with given degree sequence (Erdős-Gallai [EG])

V. A set of integers to be the score vector of some tournament (Landau [M])

In each of these theorems, the stated conditions take the form of a set of inequalities, whose necessity is always trivial. To prove sufficiency we use induction and distinguish between the critical case when we have equality holding in one of the conditions and the non-critical case when inequalities are strict. This splits the proof into two straightforward cases. The non-critical case can always be handled easily.

In the light of this method, we have found it very casy to prove a new result involving bipartite realizability.

"A degree sequence $(d_1,d_2,\ldots,d_n)$ is bipartite realizable if and only if (i) for some p, $1 \leq p \leq n-1$, $d_{i_1} + d_{i_2}+\ldots+ d_{i_p} = d_{i_{p+1}}+\ldots+ d_{i_n}$ and (ii) $(d_{i_1}+ p-1,d_{i_2}+p-1,\ldots,d_{i_p}+p-1,d_{i_{p+1}},\ldots,d_{i_n})$ is realizable."

Also a theorem by Beineke and Plummer [BP] which indicates:

"If an n-connected graph has a 1-factor, then it has at least n different 1-factors"

becomes a corollary of II.

W. T. Tutte [T2] gave a necessary and sufficient condition for a graph G to have an f-factor. Fulkerson, Hoffman, and McAndrew [FHM] by using linear programming, showed that the conditions can be simplified if the graph G has a certain cycle condition. We show (by a very simple calculation) that the latter is a corollary of the former.

Finally, we also found it very simple to discuss $\chi_1(G \times H)$, the edge-chromatic number of the cartesian product of two graphs, in terms of $\chi_1(G)$ and $\chi_1(H)$:

<u>Theorem</u>. If $\chi_1(G) = \max \deg G$, then $\chi_1(G \times H) = \max \deg G + \max \deg H$.

I should add that some of the theorems (I-V) already had nice proofs, but that the most important thing in our opinion is that the <u>same</u> method proves all the theorems.

I. INTRODUCTION

The terminology not defined here can be found in the standard texts. For example, the reader may look at Behzad and Chartrand [BC] for graph theory, and Ryser [R 2] for combinatorics. We try to follow the terminology of these two books.

Most of the present work (Chapters 1-5) is to develop a method to be called critical case method and apply it to prove several combinatorial theorems involving certain inequalities as necessary and sufficient conditions. This method was first employed by Halmos and Vaughan [HV] to prove P. Hall's theorem on systems of distinct representatives.

We recall this theorem, and give a proof based on [HV].

Hall's Theorem (1.1). Let $S_1, S_2, \ldots, S_n$ be subsets of a set X. Then a necessary and sufficient condition that there exist distinct elements $x_1, x_2, \ldots, x_n$ such that $x_i \in S_i$ $(i = 1, 2, \ldots, n)$, is that the union of every k sets from among the S_i contain at least k elements. In other words for any collection $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$ of S_i's we have:

$$(1.1) \qquad \left| \bigcup_{j=1}^{k} S_{i_j} \right| \geq k .$$

The set $\{x_1, x_2, \ldots, x_n\}$ is called a system of distinct representatives (<u>SDR</u>) of $S_1, S_2, \ldots, S_n$.

<u>Proof</u>: The necessity of the conditions (1.1) is trivial. For the sufficiency we proceed by induction on n. For $n = 1$ the result is trivial. If $n > 1$ we consider two cases:

<u>Critical Case</u>: For some $\mathcal{S}_1$, a collection of k sets, $1 \leq k < n$ we have equality in (1.1), say, $\mathcal{S}_1 = \{S_1, S_2, \ldots, S_k\}$. Then $|S'| = k$, where $|S'| = \bigcup_{j=1}^{k} S_j$, and by the induction hypothesis the collection $\mathcal{S}_1$ has an SDR, say $\{x_1, x_2, \ldots, x_k\}$. The collection $\mathcal{S}_2 = \{S_j - S' \mid \text{for } j = k+1, \ldots, n\}$ also satisfies (1.1). Indeed, let $\mathcal{S}_2' \subset \mathcal{S}_2$, say, $\mathcal{S}_2' = \{S_{k+1} - S', \ldots, S_{k+\ell} - S'\}$. Let $\mathcal{S}_2'' = \{S_{k+1}, \ldots, S_{k+\ell}\}$; then by (1.1)

$$|\mathcal{S}_1 \cup \mathcal{S}_2''| \leq \Big| \bigcup_{S \in \mathcal{S}_1 \cup \mathcal{S}_2''} S \Big| .$$

The left side, however, equals $|\mathcal{S}_1 \cup \mathcal{S}_2'| = k + \ell$; the right side equals

$$\Big| \bigcup_{i=1}^{k+\ell} S_i \Big| = \Big| \Big(\bigcup_{i=1}^{k} S_i \Big) \cup \Big(\bigcup_{i=k+1}^{k+\ell} (S_i - S') \Big) \Big| = k + \Big| \bigcup_{S \in \mathcal{S}_2'} S \Big| .$$

Hence

$$\left| \bigcup_{S \in \mathcal{S}_2'} S \right| \geq \ell .$$

By the induction hypothesis $\mathcal{S}_2$ has an SDR, say $\{y_i \mid i = k+1,\ldots,n\}$, and together with $\{x_1,x_2,\ldots,x_k\}$ this gives an SDR for $S_1,S_2,\ldots,S_n$.

Non-critical Case: For all k , $1 \leq k < n$, we have strict inequality in (1.1). In this case take any one of the S_i's - for example S_1 - and take any element x_1 of S_1 to be representative of S_1 ; then the collection $\{S_i - \{x_1\} \mid i = 2,\ldots,n\}$ satisfies (1.1); by the induction hypothesis it will have an SDR, which together with x_1 will give an SDR for $\{S_1,S_2,\ldots,S_n\}$. ∎

The elegant method in this proof is the inspiration to prove most of the remaining theorems in this thesis. The following result is needed in Chapter 2. We state it as a corollary of Theorem 1.1.

Corollary 1.2. Let $S_1,S_2,\ldots,S_n$ be subsets of a set X , which have an SDR. Then there exists an S_i such that each of the elements in S_i can be taken as a representative of S_i in some SDR for $\{S_1,S_2,\ldots,S_n\}$.

To prove the corollary one may include this result in the statement of the theorem and just carry it along in the proof, through the hypothesis.

Indeed, in the critical case the collection $\mathcal{S}_1$ has such a set by induction; in the non-critical case, the free choice of x_1 shows that every S_i has this property.

II. MATCHINGS AND 1-FACTORS

We will use the word graph for a finite, undirected, loopless graph without multiple edges. In other cases we will specify the kind of graph in the discussion.

Let d be a non-negative integer, and $V(G)$ the vertex set of a graph G, then G is said to possess a matching of deficiency d, if there is a subgraph of G consisting of non-adjacent edges which together cover $|V(G)| - d$ vertices of G. We state and prove a theorem of Berge's [B] with the critical case method:

Theorem 2.1. Let d be a non-negative integer and S a subset of the vertex set $V(G)$ of a graph G, and $h_G(S)$ the number of components with an odd number of vertices in the induced subgraph of G on the vertices $V(G) - S$ (this subgraph is denoted by $\langle V(G)-S \rangle$). Then G has a matching of deficiency d if and only if

$$(2.1)\begin{cases} \text{i)} & d \leq |V(G)|, \text{ and} \\ \text{ii)} & d + |V(G)| \text{ is even, and} \\ \text{iii)} & \text{for all sets } S \subseteq V(G) \text{ we have } d + |S| \geq h_G(S). \end{cases}$$

Proof: The necessity of the theorem is easily checked. To prove sufficiency we take d a fixed integer and apply induction on $n = |V(G)|$. For $n = d$, $d+2$ the theorem is obvious.

Let G be a graph with n vertices which satisfies (2.1). We may assume that every graph with fewer than n vertices, and satisfying (2.1) has a matching of deficiency d

Now we prove that G has a matching of deficiency d. In the course of the proof we give several lemmas.

Lemma 2.2. Given (2.1), for any S, both sides of (iii) have the same parities.

Proof of lemma 2.2. Let $s = d + |S| - h_G(S)$, we must show that s is even. Denote the odd components of $\langle V(G)-S \rangle$ by $O_1, O_2, \ldots, O_{h_G(S)}$ and the even components by $E_1, E_2, \ldots, E_\ell$. Then we have modulo 2:

$$s \equiv s + 2|V(G)| = (d + |V(G)| + \sum_{i=1}^{\ell} |V(E_i)| + \sum_{i=1}^{h_G(S)} |V(O_i)| - h_G(S) + 2|S| \equiv \sum_{i=1}^{h_G(S)} (|V(O_i)| - 1) \equiv 0 \;\Box$$

Now to prove the theorem we distinguish two cases:

<u>Critical Case</u>: There exists a non-empty set S such that equality holds in (iii). Let S be maximal, with $d + |S| = h_G(S)$. Denote the odd components of $\langle V(G)-S \rangle$ by $O_1, O_2, \ldots, O_{|S|+d}$. There are no even components. Indeed, if E were a non-empty even component and $x \in V(E)$, then $|S \cup \{x\}| + d = h_G(S \cup \{x\})$, contradicting the maximality of S.

We say $a_i \in S$ is <u>adjacent</u> to O_k if there is an edge connecting a_i to a vertex of O_k.

<u>Lemma 2.3</u>. For each $a_i \in S$, let P_i be a set of odd components of $\langle V(G)-S \rangle$ defined as:

$$P_i = \{O_k \mid a_i \text{ is adjacent to } O_k\} \quad i = 1, 2, \ldots, |S| .$$

Then the P_i satisfy the conditions of Hall's theorem (1.1).

<u>Proof of lemma 2.3</u>: Indeed, let $Q = \bigcup_{j=1}^{k} P_j$, $k > 0$ and suppose $|Q| < k$. Then the vertices in $S' = \{a_1, a_2, \ldots, a_k\}$, all together, are adjacent to at most $k-1$ odd components. This implies that the rest of the odd components, of which there are at least $d + |S| - (k-1)$, are just adjacent to the elements in $S - S'$. Therefore:

$$d + |S-S'| = d + |S| - k < d + |S| - (k-1) \leq h_G(S-S')$$

in contradiction to (iii). □

Lemma 2.3 implies that there is an SDR, say without loss of generality, $\{O_1, O_2, \ldots, O_{|S|}\}$ of the P_i, O_i being a representative of P_i meaning that there exists a vertex, say b_i, in O_i adjacent to a_i.

Lemma 2.4. Let $c_i \in V(O_i)$, and define

$$H_i = \langle V(O_i) - \{c_i\} \rangle \qquad i = 1, 2, \ldots, |S| + d .$$

Then each of the H_i satisfies the conditions of the theorem, with $d = 0$.

Proof of lemma 2.4: (i) and (ii) are trivial. Let $S' \subset V(H_i)$. We must prove that:

$$|S'| \geq h_{H_i}(S'). \tag{2.2}$$

But since S is maximal and since both sides of (iii) have the same parity (lemma 2.2) we have:

$$d + |S \cup S' \cup \{c_i\}| \geq h_{H_i}(S \cup S' \cup \{c_i\}) + 2$$

or

$$d + |S| + |S'| + 1 \geq |S| - 1 + d + h_{H_i}(S') + 2$$

which implies (2.2). □

Now for each i the graph $H_i \cup \overline{K}_d$ clearly satisfies conditions (2.1). ($\overline{K}_d$ is a totally disconnected graph with d vertices.) This by induction hypothesis implies that there is a matching of deficiency d for $H_i \cup \overline{K}_d$, which is a 1-factor (see definition on page 10) for H_i.

Note that in Lemma 2.4 the vertex c_i can be any vertex of H_i. So for $i = 1, 2, \ldots, |S|$, we take $c_i = b_i$, and in this manner the edges $a_i b_i$ $(i = 1, 2, \ldots, |S|)$ together with the 1-factors of H_i's $(i = 1, 2, \ldots, |S|+d)$ will cover all but d vertices (one in each of $O_{|S|+1}, \ldots, O_{|S|+d}$) of G. Therefore, it is a matching of deficiency d.

<u>Noncritical Case</u>: For each non-empty set S we have strict inequality in (iii). We take any edge ab of G and look at $H = \langle V(G) - \{a,b\} \rangle$. H satisfies the conditions of the theorem.

To see (i) if $d > |V(H)|$, then $d = |V(G)|$, which is trivial. As to (ii) obviously, $d + |V(H)| = d + |V(G)| - 2$ is even. Finally, to show (iii), take any set $T \subset V(H)$.

Then

$$d + |T \cup \{a,b\}| > h_G(T \cup \{a,b\})$$

and since both sides of (iii) have the same parity (Lemma 2.2) we have:

$$d + |T \cup \{a,b\}| \geq h_G(T \cup \{a,b\}) + 2$$

so

$$d + |T| + 2 \geq h_H(T) + 2 \ ,$$

which is (iii). Therefore, by induction, H (and consequently G) has a matching of deficiency d . ∎

Definition: If a matching for a graph G covers all of the vertices, then it is called a 1-factor (or perfect matching).

If in the theorem 2.1 we let $d = 0$, then we will get a necessary and sufficient condition for a graph to have a 1-factor. This gives the well-known 1-factor theorem due to Tutte. We will give a separate proof here for this theorem, for one may be interested just in the 1-factor theorem and its proof, but not in matchings.

Theorem 2.5 [T1]. Let S and $h_G(S)$ be as in Theorem 2.1.

Theorem 2.5 [T1] Let $h_G(S)$ be the number of odd components of subgraph induced on $V(G)-S$. Then the graph G has a 1-factor if and only if for all $S \subset V(G)$ the inequality:

$$|S| \geq h_G(S) \qquad (2.3)$$

holds.

Proof: The necessity is trivial. To prove sufficiency we apply induction on $n = |V(G)|$. For graphs with 1 or 2 vertices the theorem is trivial. Let G be a graph with n vertices, which satisfies (2.3). We may assume that every graph with fewer than n vertices, and satisfying (2.3), has a 1-factor. We now prove that G has a 1-factor.

Inequalities (2.3) imply that $h_G(\emptyset) = 0$; so G has only even components; G therefore has an even number of vertices, and $|S|$ and $h_G(S)$ have the same parities. If $S = \{a\}$, then (2.3) implies that $h_G(\{a\}) = 1$, so equality holds in (2.3) in this case. Let S be maximal with $|S| = h_G(S)$. Then $|S| \geq 1$. For such an S , denote the odd components of $\langle V(G)-S\rangle$ by $O_1, O_2, \ldots, O_{|S|}$. There are no even components. Indeed, if E were a non-empty even component and $x \in V(E)$, then

$$|S \cup \{x\}| = h_G(S \cup \{x\}),$$

contradicting the maximality of S .

Lemma 2.6. There is a 1-1 correspondence between the elements of S and the odd components O_i's, such that if a_j corresponds to O_k , then a_j is adjacent to O_k (i.e., there is an edge connecting a_j to a vertex of O_k).

Proof of lemma 2.6: Let $P_1, P_2, \ldots, P_{|S|}$ be sets defined as follows:

$$P_i = \{x \mid x \in S \text{ and } x \text{ is adjacent to } O_i\} \qquad (i = 1, 2, \ldots, |S|).$$

Then the P_i satisfy P. Hall's conditions (Theorem 1.1). Indeed, let $S' = \bigcup_{j=1}^{k} P_j$, $k > 0$, and suppose $|S'| < k$, then $\langle V(G) - S' \rangle$ contains at least k odd components, namely $O_1, O_2, \ldots, O_k$; hence $h_G(S') \geq k > |S'|$ in contradiction to (2.3). Therefore there is an SDR for $\{P_1, P_2, \ldots, P_{|S|}\}$; these are the a_i's of lemma 2.6. □

Without loss of generality we can assume that a_i $(\in S)$ is adjacent to O_i . Let b_i be any point in $V(O_i)$ adjacent to a_i . Define the graphs

$$H_i = \langle V(O_i) - \{b_i\} \rangle .$$

We show that each H_i admits a 1-factor $(i = 1,2,\ldots,|S|)$. Add to them the edges $(a_i b_i)$ $(i = 1,2,\ldots,|S|)$, then we have a 1-factor for G.

Lemma 2.7. Each of the H_i satisfies (2.3).

Proof of lemma 2.7: Let $S' \subset V(H_i)$; take $S^* = S \cup S' \cup \{b_i\}$, since S is maximal and since both sides of (2.3) have the same parity we have

$$|S^*| \geq h_G(S^*) + 2 ,$$

or

$$|S| + |S'| + 1 \geq |S| - 1 + h_{H_i}(S') + 2 ,$$

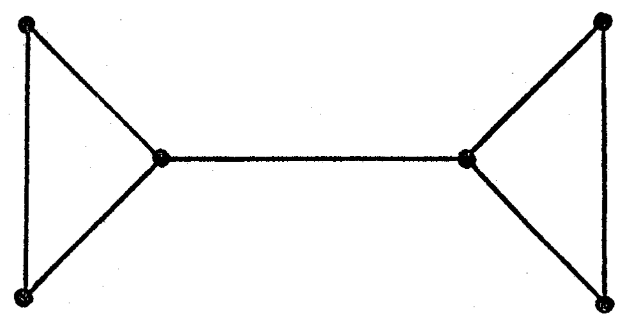
which implies $|S'| \geq h_{H_i}(S')$. □■

Hall's Theorem (1.1) gives a necessary and sufficient condition for a bipartite graph to have a 1-factor. (For, let $A = \{S_1, S_2, \ldots, S_n\}$ be one set of vertices and $B = \bigcup_{i=1}^{n} S_i$ another set. Join a vertex $x \in B$ to a vertex $S_i \in A$ if and only if $x \in S_i$.)

A vertex v in a graph G having a 1-factor is called totally covered if each edge of G incident to v belongs to some 1-factor of G. Now Corollary 1.2 implies that:

Corollary 2.8. If a bipartite graph G has a 1-factor, then G has at least one (therefore by symmetry at least two) totally covered vertex.

The above corollary is not true for an arbitrary graph. For example,



has a unique 1-factor; therefore no totally covered vertex. However, if G is an n-connected graph, then it has at least n totally covered vertices. [See [Z] and also [L2].] A refinement of this statement is the following corollary which was first proved by Beineke and Plummer [BP]; we show that it is an immediate corollary to Tutte's theorem (2.5).

Corollary 2.9: If G is an n-connected graph with a 1-factor, then it has at least n different 1-factors.

Proof: Since G has a 1-factor, it satisfies the conditions (2.3). Applying the result of corollary 1.2 to the P_i's in lemma 2.6, there exists a P_i (without loss of generality, say P_1) such that each of the elements in P_1 can be taken as a representative of P_1 in some SDR of the P_i's. Thus we have at least $|P_1|$ system of distinct representatives for the P_i's. As we noticed in the proof of the theorem 2.5, each SDR of the P_i's gives a 1-factor for G. Removing P_1 from $V(G)$ will disconnect the graph G, since G is n-connected, $|P_1| \geq n$, which implies that we have at least n SDR, consequently at least n 1-factors. ∎

At the last stages of writing the author found that Anderson [A] (and woodal [W, p.237] resp.) have used a similar method to prove Theorem 2.5 (and Theorem 2.1 resp.). However, our proofs are shorter.

III. INTEGER VALUED MATRICES WITH GIVEN ROW AND COLUMN SUM VECTORS

Let A be an $(m \times n)$ matrix with integer entries. Let the sum of row i of A be denoted by r_i and let the sum of column j of A be denoted by s_j. We call the vector

$R = (r_1, r_2, \ldots, r_m)$ the <u>row sum vector</u> and the vector

$S = (s_1, s_2, \ldots, s_n)$ the <u>column sum vector</u> of A.

If Δ denotes the sum of all entries in A, then it is clear that

$$\Delta = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j .$$

In 1957, Ryser [R 1] and Gale [G] independently found a theorem giving necessary and sufficient conditions for a pair of vectors (R, S) of non-negative components, to be row and column sum vectors of a matrix A with entries equal to 0 or 1 (called <u>(0,1)-matrix</u>). We state a more general theorem concerning the existence of a matrix with non-negative entries each less than or equal to a constant c, and we prove it by the critical case method. The Gale-Ryser theorem will follow if we take $c = 1$.

Theorem 3.1. Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be two vectors whose components are non-negative integers. Let c be a positive integer. Then there exists a matrix A with non-negative integer entries not exceeding c, and with row sum vector R and column sum vector S if and only if

i) $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j = \Delta$, and

ii) for any two sets $I \subset M = \{1,2,\ldots,m\}$ and $J \subset N = \{1,2,\ldots,n\}$, we have

$$\sum_{i \in I} r_i + \sum_{j \in J} s_j \leq \Delta + c|I||J| . \tag{3.1}$$

Proof: The necessity of the conditions is trivial. Indeed if

$$A = [a_{ij}]_{i \in M, j \in N} ,$$

with the properties of the theorem, exists, then a permutation of the rows and the columns yields the matrix

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \tag{3.2}$$

where $A_{11} = [a_{ij}]_{i \in I, j \in J}$, etc. Now the inequalities in

(3.1) are just the interpretation of both sides in terms of row and column sums in some of the A_{ij}'s.

For the sufficiency we apply induction on m and n. For $m = 1$, $n = 1$ the theorem is trivial. Assume that the result holds for each pair of sequences with $m - 1$ and $n - 1$ components or less, and let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be two sequences satisfying the hypothesis of the theorem.

For $(I = \emptyset, J = N)$ or $(I = M, J = \emptyset)$ equality holds in (3.1); we call these trivial cases. We have two cases:

<u>Critical Case</u>: There exist I and J (non-trivial) such that equality holds in (3.1), i.e.,

$$\sum_{i \in I} r_i + \sum_{j \in J} s_j = \Delta + c|I||J| . \tag{3.3}$$

Then if A exists we must have:

$$\begin{aligned} a_{ij} &= c \quad i \in I , j \in J \text{ and} \\ &= 0 \quad i \in M-I , j \in N-J . \end{aligned}$$

Hence to find the matrix A we must find two smaller matrices A_{12} and A_{21}. Accordingly, define the vector R' with components r_i' $(i \in I)$ and the vector S' with components s_j' $(j \in N-J)$ such that

$$(3.4) \qquad \begin{cases} r_i' = r_i - c|J| & i \in I \\ s_j' = s_j & j \in N-J \end{cases} .$$

Also, similarly define R" and S" such that

$$(3.5) \qquad \begin{cases} r_i'' = r_i \quad \text{if} & i \in M-I \\ s_j'' = s_j - c|I| & j \in J \end{cases} .$$

We show that each pair of (R',S') and (R",S") satisfy the conditions of the theorem:

Lemma 3.2. R' and S' satisfy the conditions of the theorem.

Proof of lemma 3.2: First note that $r_k' \geq 0$, $k \in I$. For, let $k \in I$, then (3.1) implies that

$$\sum_{i \in I-\{k\}} r_i + \sum_{j \in J} s_j \leq \Delta + c(|I|-1)|J|$$

and this together with (3.3) implies that $r_k \geq c|J|$, hence $r_k' \geq 0$.

Now to show (i) by (3.4) we have

$$\Delta' = \sum_{i \in I} r_i' = \sum_{i \in I} (r_i - c|J|) = \sum_{i \in I} r_i - c|I||J|$$

which is, by (3.3), equal to

$$\Delta - \sum_{j\in J} s_j = \sum_{j\in N-J} s_j = \sum_{j\in N-J} s'_j$$

and thus (i) holds for R' and S' .

Let $I' \subset I$ and $J' \subset N - J$ to show

$$\text{(3.6)} \qquad \sum_{i\in I'} r'_i + \sum_{j\in J'} s'_j \leq \Delta' + c|I'||J'| \ .$$

We write (3.1) for the sets I' and $J\cup J'$:

$$\sum_{i\in I'} r_i + \sum_{j\in J\cup J'} s_j \leq \Delta + c|I'||J\cup J'| \ .$$

By (3.4) the left side can be expanded:

$$[\sum_{i\in I'} r'_i + c|I'||J|]+[\sum_{j\in J} s_j + \sum_{j\in J'} s'_j]$$

$$\leq \Delta + c|I'||J| + c|I'||J'|$$

or

$$\sum_{i\in I'} r'_i + \sum_{j\in J'} s'_j \leq \Delta - \sum_{j\in J} s_j + c|I'||J'|$$

and since $\Delta' = \Delta - \sum_{j\in J} s_j$, the last inequality implies (3.6).□

<u>Lemma 3.3</u>. R" and S" satisfy the conditions of the theorem.

<u>Proof of Lemma 3.3</u>: The proof of this lemma is similar to

the one of lemma 3.2. In this case we have to prove that for any two sets $I'' \subset M-I$ and $J'' \subset J$ we have

$$\sum_{i \in I''} r_i'' + \sum_{j \in J''} s_j'' \leq \Delta'' + c|I''||J''|.$$

This follows from the inequality of (3.1) for the sets $I \cup I''$ and J''. □

Now by lemma 3.2 (lemma 3.3, resp.) and by the induction hypothesis there exists a matrix A' (A'' resp.) with the row sum R' (R'' resp.) and with the column sum S' (S'' resp.). Hence

$$A = \left[\begin{array}{c|c} [c] & A' \\ \hline A'' & [0] \end{array}\right]$$

where $[0]$ (and $[c]$) are matrices of all zeros (all c's resp.) of the appropriate sizes.

Non-critical Case: For all I, J (non-trivial) we have strict inequality in (3.1). Let I and J be such that

$$s = \Delta + c|I||J| - \sum_{i \in I} r_i - \sum_{j \in J} s_j$$

is minimal (note that $s > 0$ is the difference of both sides in (3.1)). Let r_1 and s_1, without loss of generality, be such that

$$r_1 = \max_{i \in M} r_i \qquad \text{and} \qquad s_1 = \max_{j \in N} s_j \; .$$

Then obviously $1 \in I$ and also $1 \in J$. Define

$$\bar{R} = (r_1+1, r_2, \ldots, r_m) \qquad \text{and} \qquad \bar{S} = (s_1+1, s_2, \ldots, s_n) \; .$$

In this case $\bar{R}$ and $\bar{S}$ satisfy the conditions of the theorem, and the minimum difference of both sides of (3.1) for the components of $\bar{R}$ and $\bar{S}$ is 1 less than s. By induction on s, there exists a matrix $\bar{A}$ with the row and column sum vectors $\bar{R}$ and $\bar{S}$ (resp.). Then by using $\bar{A}$ we can construct a matrix A with row and column sum vectors R and S (resp.). Indeed, let $\bar{A} = [\bar{a}_{ij}]$; if $\bar{a}_{11} \neq 0$, then by changing it to $\bar{a}_{11} - 1$ we get A; if $\bar{a}_{11} = 0$ then choose column j such that $\bar{a}_{1j} \neq 0$. The maximality of s_1 implies that there exists a row i with $\bar{a}_{ij} < \bar{a}_{i1}$. By changing the elements of the square submatrix

$$\begin{bmatrix} \bar{a}_{11} & \bar{a}_{1j} \\ \bar{a}_{i1} & \bar{a}_{ij} \end{bmatrix}$$

to

$$\begin{bmatrix} 1 & \bar{a}_{1j} - 1 \\ \bar{a}_{i1} - 1 & \bar{a}_{ij} + 1 \end{bmatrix}$$

we get a matrix (with non-negative entries not exceeding c) and row and column sum vectors still equal to $\bar{R}$ and $\bar{S}$ (resp.) but here the $(1,1)$ entry is nonzero. $\blacksquare$

If we take $c = 1$ in the theorem 3.1 we will have the condition (i) and

$$\sum_{i \in I} r_i + \sum_{j \in J} s_j \leq \Delta + |I|\,|J| \tag{3.7}$$

as the necessary and sufficient conditions for the existence of a <u>(0,1)-matrix</u> with row and column sum vector R and S.

The conditions for the existence of a $(0,1)$-matrix, due to Gale and Ryser, have a different form. Let $R = (r_1,\dots,r_m)$ be a vector of non-negative integers; form the <u>maximal</u> $m \times n$ matrix $\bar{A}$ which in row i has ones in the first r_i columns $(i = 1,2,\dots,m)$; all other entries are zero. Let $\bar{S} = (\bar{s}_1,\bar{s}_2,\dots,\bar{s}_n)$ be the column sum vector of $\bar{A}$. Clearly $\bar{s}_1 \geq \bar{s}_2 \geq \dots \geq \bar{s}_n$. Note that

$$\bar{s}_i = |\{j \mid r_j \geq i\}| .$$

<u>Theorem 3.4</u> ([Ryser 1] and [Gale 1]). Let $R = (r_1, r_2, \ldots, r_m)$, $S = (s_1, s_2, \ldots, s_n)$ be two vectors of non-negative integers with $s_1 \geq s_2 \geq \cdots \geq s_n$, and $\bar{A}$ the maximal matrix with row sum vector R and $\bar{S}$ its column sum vector. Then there exists a $(0,1)$-matrix A with row sum vector R and column sum vector S if and only if S is <u>dominated</u> (or majorized) by $\bar{S}$, i.e., if

$$(3.8) \qquad s_1 + s_2 + \cdots + s_i \leq \bar{s}_1 + \bar{s}_2 + \cdots + \bar{s}_i$$

for $i = 1, 2, \ldots, n-1$, while equality holds in (3.8) for $i = n$.

Ryser's proof of this theorem is based on constructing A from $\bar{A}$ by shifting 1's in the rows of $\bar{A}$. But Gale proves the theorem by interpreting it as a flow problem. We show directly that (3.7) and (3.8) are equivalent.

Assuming (3.7) to prove (3.8) take $J = \{1, 2, \ldots, i\}$ and $I = \{j \mid r_j \geq i\}$. Then the right side of (3.8) can be reduced:

$$\bar{s}_1 + \bar{s}_2 + \cdots + \bar{s}_i = \sum_{k=1}^{i} |\{j \mid r_j \geq k\}| = \sum_{j=1}^{m} \min(i, r_j)$$

$$= \sum_{j \in M-I} r_j + |I| i = \Delta - \sum_{i \in I} r_i + |I||J| .$$

As (3.1) states that this is $\geq s_1 + s_2 + \ldots + s_i$, (3.8) is proved.

Conversely, assume (3.8), and let I, J be non-trivial such that

$$\delta = \Delta + |I||J| - \sum_{i \in I} r_i - \sum_{j \in J} s_j$$

is minimal. We must show $\delta \geq 0$. The minimality implies $r_i \geq |J|$ for $i \in I$ and $r_i \leq |J|$ for $i \in M - I$. Otherwise δ could be reduced in value by increasing or reducing I by one element r_i. Similarly, $s_j \geq |I|$ for $j \in J$; $s_j \leq |I|$ for $j \in N - J$.

Let $i = |J|$, then

$$s_1 + s_2 + \ldots + s_i = \sum_{j \in J} s_j ,$$

while

$$\bar{s}_1 + \bar{s}_2 + \ldots + \bar{s}_i = \sum_{k=1}^{i} |\{j | r_j \geq k\}| = \sum_{k=1}^{m} \min(i, r_k)$$

$$= \sum_{k \in M-I} r_k + |I||J| = \Delta - \sum_{k \in I} r_k + |I||J| .$$

Hence, (3.8) implies $\delta \geq 0$.

The Gale-Ryser conditions for the general theorem (i.e., c not necessarily equal to 1) will become

$$\sum_{i=1}^{m} \min(kc, r_i) \geq \sum_{j=1}^{k} s_j \qquad (3.9)$$

for $k = 1,2,\ldots,n-1$, while equality holds in (3.9) for $k = n$.

The conditions (3.9) (which are equivalent to (3.1)) for $m = n$ are also the necessary and sufficient conditions for the existence of a directed graph of n vertices $v_1, v_2, \ldots, v_n$, in which the indegree (out degree, resp.) of the vertex v_i is r_i (s_i , resp.), and c_{ij} , the number of edges from v_i to v_j , is at most c (loops allowed).

The question of existence of a (0,1)-matrix with given row and colum sum vectors is closely related to that of bipartite graphs. We will discuss this at the end of the following chapter where we also present a new set of conditions for the existence of a bipartite graph with the given degree sequence, which in turn will lead to a corollary concerning the existence of a (0,1)-matrix with given row and column sum vectors!

IV. GRAPHICAL DEGREE SEQUENCES

This chapter is divided into three sections. In section 4.1 we discuss the Erdős-Gallai theorem, and in section 4.2 other necessary and sufficient conditions for realizability of a sequence are discussed. In section 4.3 we present a new result on the bipartite realizability of a finite sequence.

§4.1. Erdős-Gallai Theorem.

Let G be a graph with n vertices. The degree of a vertex v_i, denoted by d_i, is the number of edges incident with v_i. Then the non-negative sequence $\{d_i\} = \{d_1, d_2, \ldots, d_n\}$ is called the degree sequence of G. Obviously $\sum_{i=1}^{n} d_i = 2|E(G)|$ where $E(G)$ is the set of edges of G.

Definition: Let $\{d_i\} = \{d_1, d_2, \ldots, d_n\}$ be a sequence of non-negative integers. $\{d_i\}$ is called graphical (or realizable) if there exists a graph G whose degree sequence is $\{d_i\}$. Such a G is called a configuration of $\{d_i\}$.

Erdős and Gallai[EG] found a theorem giving necessary and sufficient conditions for a sequence to be graphical. We prove this theorem with the critical case method. The following inequalities are different from the ones that Erdős and Gallai had given. Later in this chapter we discuss the relation between these and their conditions.

<u>Theorem 4.1</u>. Let $\{d_i\} = \{d_1, d_2, \ldots, d_n\}$ be a sequence of non-negative integers such that $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $\{d_i\}$ is graphical if and only if

(i) $\sum_{i=1}^{n} d_i$ is even, and

(ii) For each pair of integers r, ℓ such that

$1 \leq r \leq \ell \leq n$,

$$\sum_{i=1}^{r} d_i \leq r(\ell-1) + \sum_{i=\ell+1}^{n} d_i . \tag{4.1}$$

<u>Proof</u>: The necessity of the conditions is trivial. Indeed, for a given graph G, look at A, the adjacency matrix of G i.e., $A = [a_{ij}]$ where $a_{ij} = 1$ if the vertex v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. Then A is symmetric and has zeros on the diagonal.

$$\begin{aligned} \sum_{i=1}^{r} d_i &= \sum_{j=1}^{n} \sum_{i=1}^{r} a_{ij} = \sum_{j=1}^{\ell} \sum_{i=1}^{r} a_{ij} + \sum_{j=\ell+1}^{n} \sum_{i=1}^{r} a_{ij} \\ &\leq \sum_{j=1}^{\ell} \sum_{i=1}^{r} (1-\delta_{ij}) + \sum_{j=\ell+1}^{n} d_j = r(\ell-1) + \sum_{j=\ell+1}^{n} d_j \end{aligned} \tag{4.2}$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

The proof of the sufficiency is by induction on n . Clearly the result holds for sequences with $n = 1$ or 2 . Assume that it holds for sequences with $n-1$ elements or less, and let $\{d_i\} = \{d_1, d_2, \ldots, d_n\}$ be a sequence satisfying the hypothesis of the theorem. We consider two cases:

<u>Critical Case</u>: There exist r and ℓ , non-trivial, for which equality holds in (4.1) i.e.,

$$(4.3) \qquad \sum_{i=1}^{r} d_i = r(\ell-1) + \sum_{i=\ell+1}^{n} d_i \; .$$

In this case if $\{d_i\}$ is graphical with a configuration G , then the adjacency matrix of G looks like this:

$$(4.4) \quad A = \begin{array}{c} \\ 1 \\ 2 \\ \cdot \\ \cdot \\ r \\ r+1 \\ \cdot \\ \cdot \\ \cdot \\ \ell \\ \ell+1 \\ \vdots \\ n \end{array} \left[\begin{array}{ccc|ccc|c} \multicolumn{3}{c}{1\ 2 \ldots r} & \multicolumn{3}{c}{r+1 \ldots \ell} & \ell+1 \ldots n \\ 0 & 1 \ldots & 1 & 1 & \ldots & 1 & \\ 1 & 0 \ldots & 1 & 1 & \ldots & 1 & C \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ 1 & 1 \ldots & 0 & 1 & \ldots & 1 & \\ \hline 1 & 1 \ldots & 1 & & & & \\ 1 & 1 \ldots & 1 & & & & \\ \cdot & \cdot & & & B & & 0 \\ \cdot & \cdot & & & & & \\ 1 & 1 \ldots & 1 & & & & \\ \hline & & & & & & \\ & C^T & & & 0 & & 0 \\ & & & & & & \end{array} \right]$$

(C^T is the transpose of C). This gives us the following idea to proceed through the proof of the theorem with several lemmas. Define the sequence $\{b_i\}_{i=r+1,\dots,\ell}$ by

$$b_i = d_i - r \qquad i = r+1,\dots,\ell \,.$$

Lemma 4.2. The sequence $\{b_i\}$ satisfies the conditions of the theorem.

Proof of lemma 4.2: Clearly, $\ell > r$, as there is nothing to prove if $\ell = r$.

i) By adding $\sum_{i=r+1}^{n} d_i$ to both sides of (4.3) we see that

$$\sum_{i=r+1}^{\ell} d_i \equiv r(\ell-1) \pmod 2 .$$

Hence,

$$\sum_{i=r+1}^{\ell} b_i = \sum_{i=r+1}^{\ell} d_i-(\ell-r)r \equiv r(\ell-1)-(\ell-r)r = r(r-1) \equiv 0 \pmod 2 .$$

ii) We must show that for any r' , ℓ' with $r < r' \leq \ell' \leq \ell$

$$\sum_{i=r+1}^{r'} b_i \leq (r'-r)(\ell'-r-1) + \sum_{i=\ell'+1}^{\ell} b_i \,. \tag{4.5}$$

By writing (4.1) for r' and ℓ' , and using the equality in (4.3) we find

$$\sum_{i=r+1}^{r'} d_i \le r'(\ell'-1) - r(\ell-1) + \sum_{i=\ell'+1}^{\ell} d_i$$

or

$$\sum_{i=r+1}^{r'} b_i + (r'-r)r \le r'(\ell'-1) - r(\ell-1) + \sum_{i=\ell'+1}^{\ell} b_i + (\ell-\ell')r \ ,$$

which implies (4.5) by simple algebra.

Note that $b_i \ge 0$, $i = r+1,\ldots,\ell$. Since $\{b_i\}$ is decreasing we need to show that $b_\ell \ge 0$ or $d_\ell \ge r$. Write (4.3) as

$$\sum_{i=1}^{r} d_i - \{r[(\ell-1)-1] + \sum_{i=\ell}^{n} d_i\} = r - d_\ell \ ;$$

by (4.1) the left side is ≤ 0 .□

Now define k_i and s_j such that:

$$k_i = d_i - \ell + 1 \qquad\qquad i = 1,2,\ldots,r$$

$$s_j = d_j \qquad\qquad j = \ell+1,\ldots,n \ .$$

Lemma 4.3. $K = (k_1,k_2,\ldots,k_r)$ and $S = (s_{\ell+1},\ldots,s_n)$

satisfy the conditions of Theorem 3.1.

Proof of lemma 4.3: To show (i) we have

$$\Delta = \sum_{i=1}^{r} k_i = \sum_{i=1}^{r} (d_i-\ell+1) = \sum_{i=1}^{r} d_i - r(\ell-1)$$

which by (4.3) is equal to

$$\sum_{i=\ell+1}^{n} d_i = \sum_{j=\ell+1}^{n} s_j .$$

To show (ii), since k_i and s_j are decreasing, we just need to verify (3.1) for $I = \{1,2,\ldots,r'\}$ and $J = \{\ell+1,\ldots,\ell'\}$, where $r' \leq r$ and $\ell < \ell' \leq n$:

$$(4.6) \qquad \sum_{i=1}^{r'} k_i + \sum_{j=\ell+1}^{\ell'} s_j \leq \sum_{j=\ell+1}^{n} s_j + r'(\ell'-\ell).$$

But (4.1) for r' and ℓ' implies that

$$\sum_{i=1}^{r'} (k_i+\ell-1) \leq r'(\ell'-1) + \sum_{j=\ell'+1}^{n} s_j$$

$$\sum_{i=1}^{r'} k_i + r'(\ell-1) \leq r'(\ell'-1)+\left(\sum_{j=\ell+1}^{n} s_j - \sum_{j=\ell+1}^{\ell'} s_j\right)$$

which implies (4.6) immediately.

Note that $k_i \geq 0$ and $s_j \geq 0$. For (4.3) implies that

$$\sum_{i=1}^{r-1} d_i - \{ \sum_{i=\ell+1}^{n} d_i + (r-1)(\ell-1)\} = -d_r + \ell - 1 .$$

By (4.1) the left side is ≤ 0. □

Now, by the induction hypothesis, lemma 4.2 implies that there is a graph G_1 with adjacency matrix B, whose degree sequence is $\{b_i\}$ and lemma 4.3 implies that there exists a (0,1)-matrix C with row sum vector $(k_1, k_2, \ldots, k_r)$ and column sum vector $(s_{\ell+1}, \ldots, s_n)$. Then the matrix A in (4.4) will be the adjacency matrix of a graph G with the degree sequence $\{d_1, d_2, \ldots, d_n\}$.

Non-critical Case: For all r and $\ell \geq r$ we have strict inequality in (4.1).

Let s be the deficit

$$s = \min_{r,\ell} [r(\ell-1) + \sum_{i=\ell+1}^{n} d_i - \sum_{i=1}^{r} d_i]$$

and let it be achieved for $r = r_0$, $\ell = \ell_0$; assume r_0 is maximal; for this r_0, assume ℓ_0 is minimal. Distinguish two cases:

Case $s > 1$: Then $\{d_1+2, d_2, \ldots, d_n\}$ satisfies (4.1) and its deficit is $s-2$. By induction on s, this sequence is graphical. Let G' be a configuration and let v_i, v_j be two vertices connected to v_1 but not to each other. This exists because $d_i, d_j < d_1+2$. Then G is obtained from G' by dropping the edges $(1,i)$ and $(1,j)$ and inserting (i,j).

Case $s = 1$: Then $\ell_0 > r_0$. For if $\ell_0 = r_0$, then s is even. The extremal conditions imply that

$$(\ell_0-1) - d_{r_0+1} > 0 \,, \quad -(\ell_0-1) + d_{r_0} \geq 0$$

so

$$d_{r_0} > d_{r_0+1} \,.$$

Now $\{d_1+1, d_2, \ldots, d_{r_0}, d_{r_0+1}+1, d_{r_0+2}, \ldots, d_n\}$ is decreasing, satisfies (4.1) and is critical; hence is graphical. Let G' be a configuration. We claim that v_1 is connected to v_{r_0+1} in G'; G is obtained by dropping this edge from G'.

To prove our claim, suppose not; then a new graph G'' could be formed by joining v_1 and v_{r_0+1}. However, the sequence $\{d_1+2, d_2, \ldots, d_{r_0}, d_{r_0+1}+2, d_{r_0+2}, \ldots, d_n\}$ ($d_{r_0+1}+2$ may have to move to the left to give a decreasing sequence) is no longer graphical, for the minimum deficit is negative: take $r = r_0$, $\ell = \ell_0$. ∎

§4.2. Other N. & S. Conditions for Realizability.

The original theorem of Erdős-Gallai had the following inequalities as necessary and sufficient conditions for theorem 4.1 (of course besides the condition $\sum_{i=1}^{n} d_i \equiv 0 \pmod 2$): for each integer r , $1 \le r \le n-1$,

$$(4.7) \qquad \sum_{i=1}^{r} d_i \le r(r-1) + \sum_{i=r+1}^{n} \min\{r, d_i\} .$$

The conditions (4.1) were also found by D. R. Fulkerson et.al.[FHM]. We will discuss this paper later in chapter 6. To see directly that (4.1) and (4.7) are equivalent for given r , let L be the smallest integer such that $L \ge r$ and $d_{L+1} < r$. Then the right side of (4.7) equals

$$r(r-1) + \sum_{r+1}^{L} r + \sum_{i=L+1}^{n} d_i = r(L-1) + \sum_{i=L+1}^{n} d_i .$$

This shows that (4.1) implies (4.7). Conversely, for every $\ell \ge r$ we have

$$\sum_{i=1}^{r} d_i \leq r(r-1)+\sum_{i=r+1}^{n} \min(r,d_i) \leq r(r-1)+\sum_{i=r+1}^{\ell} r+\sum_{i=\ell+1}^{n} d_i$$

$$= r(r-1)+r(\ell-r)+\sum_{i=\ell+1}^{n} d_i$$

which implies (4.1).

We can also show that the Erdős-Gallai conditions (4.7) are equivalent to:

for any partition (S,T,U) of the set $\{1,2,\ldots,n\}$ we have

$$\sum_{i\in T} d_i \leq \sum_{j\in S} d_j + |T|(|T|+|U|-1). \tag{4.8}$$

Actually we show that (4.8) is equivalent to (4.1). Indeed, in (4.8) let $T = \{1,2,\ldots,r\}$ and $U = \{r+1,\ldots,\ell\}$; this shows that (4.8) implies (4.1). Conversely, since d_i is decreasing (by applying (4.1)) we have

$$\sum_{i\in T} d_i \leq \sum_{i=1}^{|T|} d_i \leq \sum_{j=|T|+|U|+1}^{n} d_j + |T|(|T|+|U|-1)$$

$$\leq \sum_{j\in S} d_j + |T|(|T|+|U|-1)$$

which implies (4.8).

Note that in (4.8) the sequence $\{d_i\}$ does not need to be decreasing. Sometimes the inequalities of (4.8) are easier to apply than those of (4.7). For example, let G be a graph with degree sequence $\{d_i\}$. Then $\bar{G}$, the complement of G, has degree sequence $\{n-1-d_i\}$. Therefore:

Corollary 4.4. The sequence $\{d_i\}$ satisfies (4.8) if and only if $\{n-1-d_i\}$ does.

But we prove this corollary directly.

Proof: The conditions (4.8) for $\{n-1-d_i\}$ are:

for any partition (S,T,U) of $\{1,2,\ldots,n\}$ we have

$$(4.9)\qquad \sum_{i\in T}[(n-1)-d_i] \leq -\sum_{j\in S}[(n-1)-d_j] + |T|(|T|+|U|-1).$$

But this can be written as

$$(n-1)|T| - \sum_{i\in T} d_i \leq -\sum_{j\in S} d_j + (n-1)|S| + |T|(n-|S|-1)$$

or

$$\sum_{j\in S} d_j \leq \sum_{i\in T} d_i + |S|(n-1-|T|) = \sum_{i\in T} d_i + |S|(|S|+|U|-1).$$

This shows that any inequality of (4.9) with a partition (S,T,U) will lead to an inequality of (4.8) with the partition (T,S,U) and vice versa. ∎

A proof of corollary 4.4 using the conditions (4.7) is not straightforward.

§4.3. <u>Bipartite Realizable Sequences</u>.

One may ask the following question: What are the conditions for a sequence $\{d_i\}$ to be the degree sequence of a bipartite graph? One necessary condition is that there exists a set $I \subset N = \{1,2,\ldots,n\}$ such that $\sum_{i\in I} d_i = \sum_{i\in N-I} d_i$. With subscripts renumbered this condition is: for some integer k

$$\sum_{i=1}^{k} d_i = \sum_{i=k+1}^{n} d_i .$$

Then the question will be the same as in chapter 3, i.e., What are the conditions for the sequences $(d_1,d_2,\ldots,d_k)$ and $(d_{k+1},\ldots,d_n)$ to be the row (and column resp.) sum vector of a $(0,1)$-matrix A ? Indeed, if A exists then

$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$$

will be the adjacency matrix of a bipartite graph with degree sequence $\{d_i\}$.

A. J. Hoffman has found some conditions for a more general question [H]. Applying his conditions to this case we get exactly the conditions (3.7) of Chapter 3.

Now we state and prove a recent result concerning this question for which the critical case method gave inspiration for an easy proof.

Theorem 4.5. Let $\{d_i\} = \{d_1, d_2, \ldots, d_n\}$ be a sequence of non-negative integers (not necessarily decreasing). Then there is a bipartite graph with degree sequence $\{d_i\}$ if and only if there exists a set $I \subset N = \{1, 2, \ldots, n\}$ with say k elements, such that

i) $\sum_{i \in I} d_i = \sum_{j \in N-I} d_j$, and

ii) the sequence $\{e_i\}$ defined as

$$\begin{cases} e_i = d_i + k - 1 & i \in I \\ e_i = d_i & i \in N-I \end{cases}$$

is graphical.

Indeed, any graph configuration of $\{e_i\}$ will lead to a bipartite graph with degree sequence $\{d_i\}$ and vice versa.

<u>Proof</u>: Assume that there is a bipartite graph with adjacency matrix

$$\begin{bmatrix} 0 & A_1 \\ A_1^T & 0 \end{bmatrix}$$

whose degree sequence is $\{d_i\}$, where A_1 is $(k \times n-k)$ matrix. Then take

$$A^* = \begin{bmatrix} J^* & A_1 \\ A_1^T & 0 \end{bmatrix}$$

where J^* is a $(k \times k)$ matrix, whose entries are all 1 except the diagonal elements which are all 0 . A^* represents the adjacency matrix of a graph with degree sequence $\{e_i\}$. This shows the necessity of the conditions.

To prove sufficiency, let G be a graph with degree sequence $\{e_i\}$. Then, by the definition of $\{e_i\}$ and condition (i)

$$\sum_{i \in I} e_i = \sum_{i \in I} (d_i + k - 1) = \sum_{i \in I} d_i + k(k-1) = \sum_{i \in N-I} d_i + k(k-1) .$$

Therefore,

$$\text{(4.10)} \qquad \sum_{i \in I} e_i = \sum_{i \in N-I} e_i + k(k-1) .$$

But this indicates that in $B = [b_{ij}]$, the adjacency matrix of G, we must have

$$\begin{cases} b_{ij} = 1 & \text{for } i \text{ and } j \in I\,,\ i \neq j \\ b_{ij} = 0 & \text{for } i \text{ and } j \in N-I \end{cases}$$

(i.e., the matrix B looks like A^*). Indeed, if we have $b_{ij} = 0$ for some i and j ($i \neq j$; i , $j \in I$) or $b_{rs} = 1$ for some r and s ($r \neq s$; r , $s \in N-I$), then in either case we have

$$\sum_{i\in I} e_i = \sum_{\substack{i\in I\\ j\in I}} b_{ij} + \sum_{\substack{i\in I\\ j\in N-I}} b_{ij} < k(k-1) + \sum_{i\in N-I} e_i$$

which is in contradiction with (4.10). Now in B by changing all of the elements b_{ij} (i , $j \in I$) to zero we get

$$\begin{bmatrix} 0 & B_1 \\ B_1^T & 0 \end{bmatrix}$$

which is an adjacency matrix of a bipartite graph with degree sequence $\{d_j\}$. ∎

V. TOURNAMENTS

In this chapter we state and prove a theorem on tournaments. We define a <u>tournament</u> T_n, to consist of n vertices $v_1, v_2, \ldots, v_n$ such that each pair of distinct vertices v_i and v_j is joined by one and only one of the oriented arcs $v_i v_j$ or $v_j v_i$. If $v_i v_j$ is in T_n, then we say v_i <u>dominates</u> v_j. For a discussion on tournaments we refer the reader to a book by John W. Moon [M].

The <u>score</u> of a vertex v_i is the number s_i of vertices that v_i dominates. The <u>score vector</u> of T_n is the ordered n-tuple $(s_1, s_2, \ldots, s_n)$. The following theorem can be found in [M, page 61] with a proof due to Ryser. But we will prove it by the critical case method.

<u>Theorem 5.1</u>. A set of integers $(s_1, s_2, \ldots, s_n)$, where $s_1 \leq s_2 \leq \cdots \leq s_n$, is the score vector of some tournament T_n if and only if

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} \tag{5.1}$$

for $k = 1, 2, \ldots, n-1$ with equality holding when $k = n$.

<u>Proof</u>: Any k vertices of a tournament are joined by $\binom{k}{2}$ arcs, by definition. Consequently, the sum of scores of any k vertices of a tournament must be at least $\binom{k}{2}$. This

shows the necessity of (5.1)

The sufficiency of (5.1) when $n = 1$ is obvious. The proof for the general case will be by induction. Assume that the result holds for sequences with $n-1$ elements or less, and let $(s_1, s_2, \ldots, s_n)$ be a sequence satisfying (5.1).

We consider two cases:

<u>Critical Case</u>: There exists k , $1 \leq k \leq n-1$ such that

$$\sum_{i=1}^{k} s_i = \binom{k}{2} \tag{5.2}$$

(i.e., equality holds in 5.1).

(i) $(s_1, s_2, \ldots, s_k)$ satisfies (5.1) and by the induction hypothesis there exists a tournament T_k with the score vector $(s_1, s_2, \ldots, s_k)$. Let the adjacency matrix of T_k be M_1 .

(ii) Define

$$s'_j = s_j - k \ , \quad j = k+1, \ldots, n \ .$$

Clearly $s'_j \geq 0$. Indeed, (5.1) implies

$$\sum_{i=1}^{k+1} s_i \geq \binom{k+1}{2} \tag{5.3}$$

and by subtracting (5.2) from (5.3) we get

$$s_{k+1} \geq \frac{(k+1)k}{2} - \frac{(k-1)k}{2} = k .$$

Therefore, since s_j is increasing, $s_j' \geq 0$ $(j = k+1,\ldots,n)$. Now we prove the following lemma.

Lemma 5.2. The s_j's defined in (ii) satisfy (5.1).

Proof of lemma 5.2: We must show that:

$$\sum_{j=k+1}^{r} s_j' \geq \binom{r-k}{2}, \quad r = k+1,\ldots,n-1 \tag{5.4}$$

and

$$\sum_{j=k+1}^{n} s_j' = \binom{n-k}{2}. \tag{5.5}$$

For (5.5) we subtract from $\sum_{i=1}^{n} s_i = \binom{n}{2}$ the equality (5.2):

$$\sum_{i=k+1}^{n} s_i = \binom{n}{2} - \binom{k}{2}$$

but also

$$\sum_{i=k+1}^{n} s_i = \sum_{i=k+1}^{n} (s_i'+k) = \sum_{i=k+1}^{n} s_i' + k(n-k)$$

which implies (5.5).

To show (5.4) we write

$$\sum_{i=1}^{r} s_i \geq \binom{r}{2} \quad \text{for} \quad r > k .$$

Again, subtract (5.2). Then we find

$$\sum_{i=k+1}^{r} s_i \geq \binom{r}{2} - \binom{k}{2} = (r-k)k + \binom{r-k}{2}$$

which implies (5.4) immediately. □

Now by the induction hypothesis, Lemma 5.2 implies that there exists a tournament, say T_{n-k}, with score vector s_j' and adjacency matrix M_2.

Let A be a matrix such that

$$A = \begin{bmatrix} M_1 & [0] \\ [1] & M_2 \end{bmatrix}$$

where $[0]$ (and $[1]$) are matrices of all zeros (all ones, respectively) of the appropriate sizes. Then A is the adjacency matrix of a tournament with score vector $(s_1, s_2, \ldots, s_n)$.

Non-critical Case: For all $k \neq n$,

(5.6) $$\sum_{i=1}^{k} s_i > \binom{k}{2}.$$

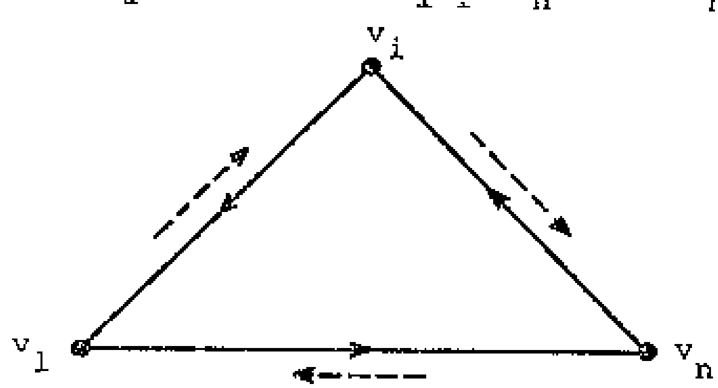
Then $s_1 \geq 1$ and $s_n < n-1$. Let

$$s = \min_{1 \leq k \leq n-1} \left(\sum_{i=1}^{k} s_i - \binom{k}{2} \right).$$

Then $s > 0$. Among all sequences $\{s_i\}$ satisfying (5.6), we take the one with the smallest s. We define a new sequence, namely

(5.7) $$(s_1-1, s_2, \ldots, s_{n-1}, s_n+1).$$

This sequence satisfies (5.1) and has a smaller s; therefore, it is a score vector of some tournament T_n. If, in T_n, v_1 dominates v_n (i.e., $v_1 v_n \in T_n$) then, since $s_1 \leq s_n$, there exists v_i such that $v_i v_1 \in T_n$ and $v_n v_i \in T_n$:



If we change the direction of the arcs v_1v_n, v_nv_i and v_iv_1 in T_n we get another tournament T_n' with the same score vector as that of T_n. But here $v_nv_1 \in T_n'$. Therefore without loss of generality we can assume that $v_nv_1 \in T_n$. The tournament with score vector $(s_1, s_2, \ldots, s_n)$ can be constructed from T_n just by changing the direction of v_nv_1. ∎

Here the author wishes to thank Professor Saul Stahl who mentioned Theorem 5.1 as a candidate to prove with the critical case method.

This chapter ends a series of theorems which are proved by the critical case method. We should also add that Roy B. Levow [L1] proved a theorem of Hardy, Littlewood, and Polya relating vector majorization and doubly stochastic matrices, using this same method.

VI. f-FACTORS

Let G be a graph with multiple edges. Let f be a function from the vertex set $V(G)$ of G to the non-negative integers. We define an f-factor of G as a spanning subgraph F of G such that the degree (valence) of each vertex x in F is $f(x)$. The question of the necessary and sufficient condition for $f(x)$, such that G contains an f-factor is answered by Tutte in an excellent article [T 3] which is a short proof of a theorem he discovered earlier [T 2]. Tutte recently has further developed his work on this subject [T 4]. This is a very important theorem because it generalizes several others. For example, if we take for G the complete graph k_n on n vertices, then it will give us the Erdős-Gallai theorem (4.1). If we take $f(x) = 1$ for all vertices x in G, then Tutte's 1-factor theorem (2.5) follows from this theorem. The theorem of Berge's (2.1) is also a result of this theorem. Tutte has shown that his conditions will produce the conditions of these theorems in each case. The f-factor theorem of Tutte with a slight modification in notations and formulation is

Theorem 6.1. Let G be a graph with multiple edges, and let f be a non-negative function defined on $V(G)$. G contains an f-factor if and only if for every partition (S,T,U) of vertices of G, we have

$$(6.1) \qquad \sum_{a\in T} f(a) \leq \sum_{a\in S} f(a) + \sum_{\substack{a\in T \\ b\in T\cup U}} c_{ab} - q(S,T)$$

where c_{ab} is the number of edges joining a to b, and $q(S,T)$ is the number of components C of $\langle U\rangle$ (the induced subgraph of G on the vertices U) such that (we write $a \in C$ instead of $a \in V(C)$)

$$(6.2) \qquad B(C,T) = \sum_{a\in C} f(a) - \sum_{\substack{a\in C \\ b\in T}} c_{ab}$$

is odd.

In [T 3], Tutte proves this theorem by constructing a larger graph G', from the graph G. Then G' has a 1-factor if and only if G has an f-factor. Then the conditions for G' having a 1-factor will lead to (6.1). We illustrate a proof for necessity of the conditions (6.1) in order to understand the quantity $q(S,T)$.

<u>Necessity of the conditions (6.1)</u>. Let F be an f-factor of G, and define the matrix $[f_{ab}]$, such that f_{ab} is the number of edges joining a to b in the subgraph F. Obviously, $f_{ab} \leq c_{ab}$ for all a and b. Therefore,

$$\sum_{a \in T} f(a) = \sum_{\substack{a \in T \\ b \in V(G)}} f_{ab} \leq \sum_{\substack{a \in T \\ b \in S}} f_{ab} + \sum_{\substack{a \in T \\ b \in T \cup U}} c_{ab} .$$

Hence,

$$\sum_{a \in T} f(a) \leq \sum_{a \in S} f(a) + \sum_{\substack{a \in T \\ b \in T \cup U}} c_{ab} . \tag{6.3}$$

Now, let C be a component of $\langle U \rangle$. We expand $B(C,T) \pmod 2$

$$\sum_{a \in C} f(a) + \sum_{\substack{a \in C \\ b \in T}} c_{ab} \overset{(\mathrm{mod}\ 2)}{\equiv} \sum_{\substack{a \in C \\ b \in T}} (c_{ab} - f_{ab}) + \sum_{\substack{a \in C \\ b \in S}} f_{ab} = Q(C) ;$$

the "missing" summand is

$$\sum_{\substack{a \in C \\ b \in U}} f_{ab} = \sum_{\substack{a \in C \\ b \in C}} f_{ab} \equiv 0 \pmod 2$$

because C is a component of the induced subgraph of F on the vertices U.

Now we redefine $q(S,T)$ to be the number of components C for which $Q(C)$ is odd. But if $Q(C)$ is odd for some C, then either for some $a \in C$ and $b \in T$, the quantity $(c_{ab} - f_{ab})$ is odd, or for some $a \in C$ and $b \in S$, f_{ab} is odd. But in the first case there is at least one edge in G and in the second case there is at least one edge in F which contribute 1 to the right side of (6.3) but not to the left side. Therefore, if we subtract $q(S,T)$, i.e., the number of such components, from the right side of (6.3), we still will have inequality. ∎

Define $\delta(S,T)$ to be the difference of both sides in (6.1), i.e.,

$$\delta(S,T) = \sum_{a \in S} f(a) - \sum_{a \in T} f(a) + \sum_{\substack{a \in T \\ b \in T \cup U}} c_{ab} - q(S,T).$$

Substituting from (6.2)

$$\delta(S,T) = \sum_{a \in S} f(a) - \sum_{a \in T} f(a) + \sum_{\substack{a \in T \\ b \in T}} c_{ab} + $$
$$+ \sum_{C \subset U} \Big(-B(C,T) + \sum_{a \in C} f(a)\Big) - q(S,T)$$
$$= \sum_{a \in S \cup U} f(a) - \sum_{a \in T} f(a) + \sum_{a \in T} c_{ab} - \sum_{C \subset U} B(C,T) - q(S,T)$$

or

$$(6.4)\quad \delta(S,T) = \sum_{a\in V(G)} f(a) - 2\sum_{a\in T} f(a) + \sum_{\substack{a\in T\\ b\in T}} c_{ab} - \sum_{C\subset U} 2\left\lceil \frac{B(C,T)}{2} \right\rceil$$

where $\lceil x \rceil$ = minimal integer $\geq x$.

Note that if (6.1) holds, then by taking $T = S = \emptyset$ we get $q(S,T) = 0$; hence $B(C,\emptyset) = \sum_{a\in C} f(a)$ is even for every component C of G . Therefore $\sum_{a\in V(G)} f(a)$ is even. That is if (6.1) holds, then by (6.4) $\delta(S,T)$ is always even for all partitions (S,T,U).

As the quantity $q(S,T)$ in (6.1) is hard to calculate, it is interesting to note that the $q(S,T)$ may be deleted from (6.1) if G has the <u>odd-cycle property</u>; i.e., if G has the property that any two of its odd (simple) cycles either have a common vertex, or there exists a pair of vertices, one from each cycle, which are joined by an edge. This was proved by Fulkerson, et.al. [FHM] using integer programming techniques.

We give a new proof of this fact:

<u>Corollary 6.2</u>. Assume that G satisfies the odd-cycle condition. Then G has an f-factor if and only if

i) $\sum_{a \in V(G)} f(a)$ is even, and

ii) for every partition (S,T,U) of $V(G)$

$$(6.3) \qquad \sum_{a \in T} f(a) \leq \sum_{a \in S} f(a) + \sum_{\substack{a \in T \\ b \in T \cup U}} c_{ab} .$$

Proof: We showed the necessity of the condition in the proof of the necessity of (6.1). To prove sufficiency, we show that if G satisfies the hypothesis, then there exists a partition (S,T,U) for which $\delta(S,T)$ is minimal and $q(S,T) \leq 1$. If $\sum_{a \in V(G)} f(a)$ is even, then (6.4) implies that $\delta(S,T)$ is even; hence (6.1) is satisfied.

Let (S,T,U) be any partition of $V(G)$ for which $\delta(S,T)$ is minimal. Then at most one of the components of $\langle U \rangle$ can have any odd cycles; all the other components are bipartite graphs. Let C be one such; $V(C) = C_1 \cup C_2$, where $\langle C_1 \rangle$ and $\langle C_2 \rangle$ are totally disconnected subgraphs.

Let C' be any component of $\langle U \rangle$, $C' \neq C$; then

$$B(C', T \cup C_1) - B(C',T) = -\sum_{\substack{a \in C' \\ b \in C_1}} c_{ab} = 0 .$$

Hence,

$$\delta(S\cup C_2, T\cup C_1) - \delta(S,T) =$$

$$-2\sum_{a\in C_1} f(a) + \sum_{\substack{a\in C_1\\ b\in T}} c_{ab} + \sum_{\substack{a\in T\\ b\in C_1}} c_{ab} + \sum_{a,b\in C_1} c_{ab} + 2\lceil \tfrac{1}{2} B(C,T)\rceil .$$

Since C_1 is totally disconnected, $\sum_{a,b\in C_1} c_{ab} = 0$. A similar relation holds with C_1 replaced by C_2 . Adding those two we find

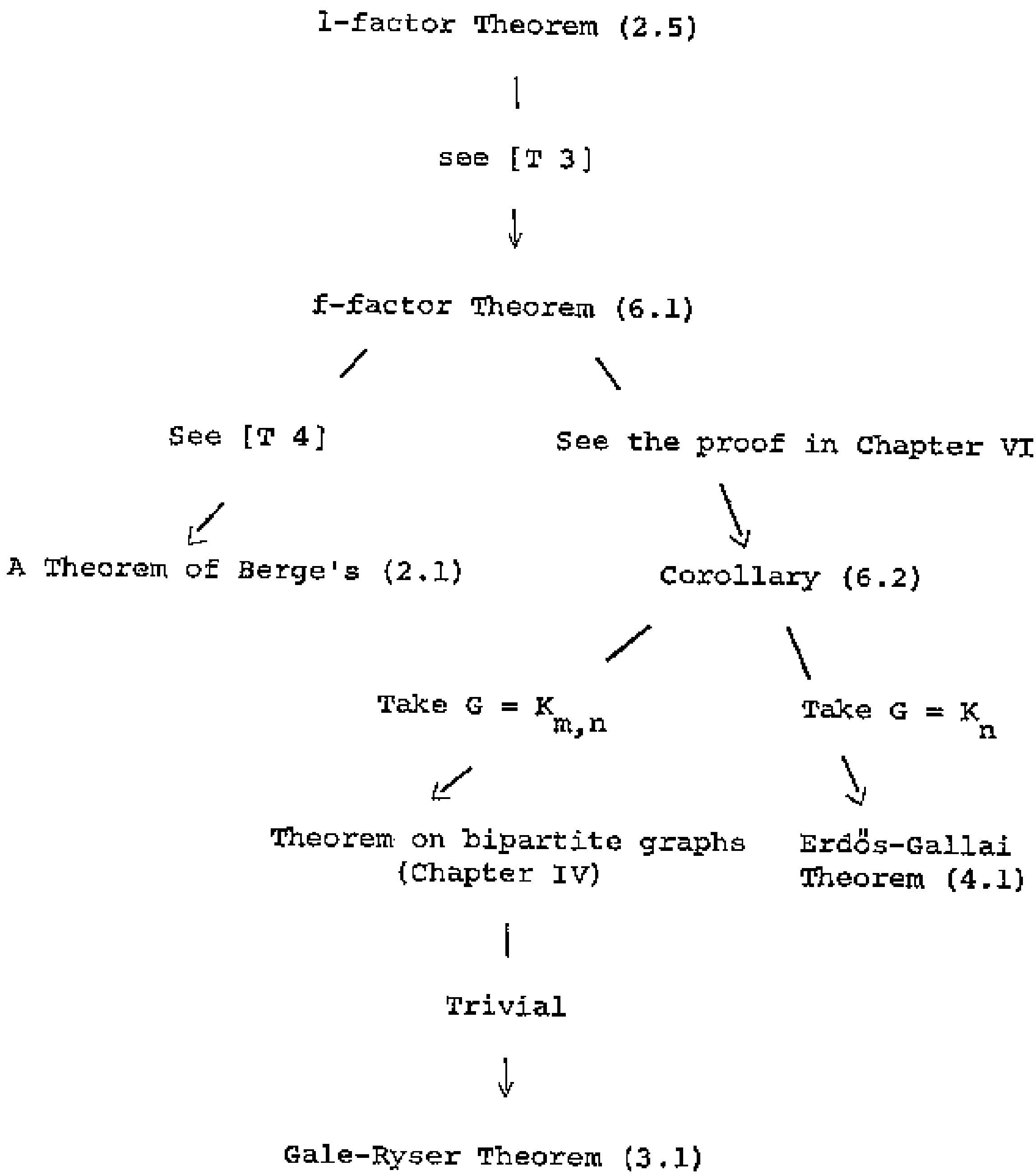
$$[\delta(S\cup C_2, T\cup C_1) - \delta(S,T)] + [\delta(S\cup C_1, T\cup C_2) - \delta(S,T)] =$$

$$= -2B(C,T) + 4\lceil \tfrac{1}{2} B(C,T)\rceil .$$

The right side is 0 if $B(C,T)$ is even, and 2 if $B(C,T)$ is odd. As all δ's are even, either $\delta(S\cup C_1, T\cup C_2)$ or $\delta(S\cup C_2, T\cup C_1)$ equals $\delta(S,T)$, i.e., is also minimal.

In this manner all bipartite components of <U> can be removed, leaving a partition (S^*,T^*,U^*) in which U^* has at most one component. Hence $q(S^*,T^*) \leq 1$, while $\delta(S^*,T^*)$ is minimal. ∎

The interested reader will find a logical sequence of theorems on factors by starting with the 1-factor theorem in chapter 2 and then following this diagram:


1-factor Theorem (2.5)
see [T 3]
f-factor Theorem (6.1)
See [T 4]
See the proof in Chapter VI
A Theorem of Berge's (2.1)
Corollary (6.2)
Take $G = K_{m,n}$
Take $G = K_n$
Theorem on bipartite graphs (Chapter IV)
Erdős-Gallai Theorem (4.1)
Trivial
Gale-Ryser Theorem (3.1)

VII. 1-FACTORABILITY AND CARTESIAN PRODUCTS OF GRAPHS

A graph (without loops or multiple edges) is 1-factorable if it can be expressed as a union of edge-disjoint 1-factors. There is no known characterization of 1-factorable graphs. In this section first we discuss the relation between 1-factorability and edge-colorability of a graph and then investigate the 1-factorability of Cartesian product of graphs.

An assignment of n colors to the edges of a nonempty graph G so that adjacent edges are colored differently is an edge-coloring of G. The minimum n for which a graph G is n-edge-colorable is its edge-chromatic number $\chi_1(G)$. By a theorem of Vizing (see [Ore] p. 245) the edge-chromatic number $\chi_1(G)$ of a graph G is bounded by

$$\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1$$

where $\Delta(G)$ is the maximum degree of G. Now, it is clear that G is 1-factorable if and only if G is a regular graph of degree $\Delta(G)$ and $\chi_1(G) = \Delta(G)$. Hence any theorem concerning the 1-factorability of regular graphs has as an immediate corollary a result concerning edge-colorability, which is useful since there is also no known characterization of those graphs which are $\Delta(G)$-edge-colorable.

<u>Definition</u>: The <u>cartesian product</u> (or product) of two graphs G and H, denoted by $G \times H$, is defined by:

$$V(G\times H) = V(G) \times V(H);$$

$$E(G\times H) = \{[(u,v),(u',v')] \mid u = u' \text{ and } vv' \in E(H) \text{ or } v = v' \text{ and } uu' \in E(G)\}.$$

<u>Theorem 7.1</u>. If $\chi_1(G) = \Delta(G)$, then $\chi_1(G\times H) = \Delta(G) + \Delta(H)$.

<u>Proof</u>: $G\times H$, which is isomorphic with $H\times G$, contains $|V(H)|$ disjoint "horizontal" copies $G_1, G_2, \ldots, G_{|V(H)|}$ of G and $|V(G)|$ disjoint "vertical" copies $H_1, H_2, \ldots, H_{|V(G)|}$ of H. A horizontal copy G_i and a vertical copy H_j have only one vertex (u_i, v_j) in common.

By the above theorem of Vizing we have

$$\Delta(G\times H) \leq \chi_1(G\times H) \leq \Delta(G\times H) + 1 .$$

But, $\Delta(G\times H) = \Delta(G) + \Delta(H)$. Therefore it is enough to show that $\chi_1(G\times H) \leq \Delta(G) + \Delta(H)$.

To see this, color the edges of each horizontal copy properly with colors $\{1, 2, \ldots, \Delta(G) = \chi_1(G)\}$ and each vertical copy properly with colors

$\{\Delta(G)+1, \Delta(G)+2, \ldots, \Delta(G)+\chi_1(H)\}$. If $\chi_1(H) = \Delta(H)$ then we are done. If $\chi_1(H) = \Delta(H)+1$, then take any edge $e = [(u_i,v_k),(u_j,v_k)]$ in any horizontal copy G_k, which is colored in color number 1. Each end point of e in the copies H_i or H_j is joined to at most $\Delta(H)$ vertical edges. Therefore there is at least one color missing at both ends. We color the edge e the missing color. In this manner, color 1 is removed, and we have colored $G \times H$ in just $\Delta(G)+\Delta(H)$ colors $\{2,3,\ldots,\Delta(G)+\Delta(H)+1\}$. ∎

In [BM] Behzad and the author discussed the topological invariants of $G \times H$ in terms of those of G and H. In that article (page 159), we showed that if both $\chi_1(G)$ and $\chi_1(H)$ assume the right side of the Vizing inequalities (i.e., $\chi_1(G) = \Delta(G)+1$, $\chi_1(H) = \Delta(H)+1$) then $\chi_1(G \times H)$ can assume either side of the inequalities with the proper G and H. The above theorem shows that if at least one of $\chi_1(G)$ or $\chi_1(H)$ assumes the left side of the Vizing inequalities then so does $\chi_1(G \times H)$.

Theorem 7.1 implies the following corollary which was proved by Himelwright and Williamson [HW] directly.

<u>Corollary 7.2</u>. If G is a 1-factorable graph and H is a

regular graph, then $G \times H$ is a 1-factorable graph.

Proof: The 1-factorability of G implies $\chi_1(G) = \Delta(G)$. Then by Theorem 7.1 $\chi_1(G \times H) = \Delta(G \times H)$, and since $G \times H$ is regular, it is 1-factorable.