- For each $1 \leq i \leq n$, the function f_i is a ζ_i-excessive (resp. ζ-deficient) function (with respect to $\overline{\Delta}$) on X.
- For each $1 \leq i \leq n$, the subset Q_i is a nonnegative (resp. nonpositive) bipolar part of f_i.

♠

Theorem 3. *Consider a graph G and let $\Gamma = (\zeta_1, \zeta_2, \ldots, \zeta_n)$. If $F = (f_1, f_2, \ldots, f_n)$ along with $\mathcal{Q} = (Q_1, Q_2, \ldots, Q_n)$ is a compatible transverse set of functions for $\overline{\Delta}$, then*

$$2\, \overline{\zeta}_n \;\geq\; \iota_n(G)^2.$$

Proof. Let $0 \neq g_i \stackrel{\text{def}}{=} f_i|_{Q_i}$. Then,

$$\begin{aligned}\overline{\zeta}_n &\geq \frac{1}{n} \sum_{i=1}^{n} \frac{\|\overline{\nabla} g_i\|^2_{2,\overline{\phi}}}{\|g_i\|^2_{2,\pi}} \geq \frac{1}{2n} \sum_{i=1}^{n} \frac{\|\vec{\nabla} g_i^2\|^2_{1,\phi}}{\|g_i^2\|^2_{1,\pi}} \\ &\geq \frac{1}{2} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{\|\vec{\nabla} g_i^2\|_{1,\phi}}{\|g_i^2\|_{1,\pi}} \right)^2 \geq \frac{1}{2}\, \iota_n(G)^2,\end{aligned}$$

where the first and the second inequalities follow from Lemma 6 and 4, respectively, and the third one is a direct application of Cauchy-Schwarz inequality. ■

It ought to be noted that Theorems 2 and 3 together, can be considered as a *generalized* Cheeger inequality. In what follows we deduce a special case where one may get an explicit inequality for the mean spectrum.

Theorem 4. *Consider a kernel K on a base graph G. Let $F = (f_2, f_3, \ldots, f_{n+1})$ be a list of eigenfunctions of $\overline{\Delta}$ for the list of eigenvalues $\Gamma = (\lambda_2, \lambda_3, \ldots, \lambda_{n+1})$, respectively, such that along with $\mathcal{Q} = (Q_2, Q_3, \ldots, Q_{n+1})$ form a compatible transverse set of functions for $\overline{\Delta}$. Then,*

$$\overline{\lambda}_n \leq \iota_n(G) \leq \sqrt{\frac{2(n+1)}{n}\, \overline{\lambda}_{n+1}}. \tag{7}$$

Moreover, we would like to add that following the same scenario described for the mean version, one may define the nth max-isoperimetric constant as

$$\varsigma_n(K, \pi) \stackrel{\text{def}}{=} \min_{\{Q_i\}_1^n \in \mathcal{D}_n(G)} \left(\max_{1 \leq i \leq n} \frac{\vec{\partial}(Q_i)}{\pi(Q_i)} \right).$$

It is noteworthy that all of the previous mentioned results such as the Federer-Fleming theorem can also be verified for this version with appropriate modifications. For instance, we may state a more standard Cheeger inequality for the max-isoperimetric constant ς_n using Theorems 2 and 3 and their counterparts, along with Courant-Fischer variational theorem as follows.

Theorem 5. *For a given graph G, let f be an eigenfunction of $\overline{\Delta}$ corresponding to the nth eigenvalue λ_n. Also, let $(Q_1, Q_2, \ldots, Q_n)$ be a list of n disjoint nonempty subsets of $V(G)$ such that for every $1 \leq i \leq n$ we have $f|_{Q_i} \neq 0$ and each Q_i is a nonnegative or nonpositive bipolar part of f. Then,*

$$\frac{\lambda_n}{2} \leq \varsigma_n(G) \leq \sqrt{2\, \lambda_n}, \quad \text{and} \quad \overline{\lambda}_n \leq \iota_n(G) \leq \sqrt{2\, \overline{\lambda}_n}. \tag{8}$$

Also, as a corollary of Theorem 5 by considering the fact that always the second eigenvalue has an eigenfunction with two nodal domains, we obtain the classical Cheeger inequality as,

$$\frac{\lambda_2}{2} \le \iota_2(G) \le \varsigma_2(G) \le \sqrt{2\,\lambda_2}. \tag{9}$$

It also must be emphasized that a direct use of eigenvalues and eigenfunctions (not necessarily tuned with repetition) in Theorem 3 will definitely make a deviation from sharpness which can be easily verified by a comparison to the classical Cheeger inequality (Inequality (9)). Note that the classical Cheeger inequality is far from being sharp by a recent result of Montenegro and Tetali [39].
To provide some examples let us recall the following result.

Theorem B. [4, 5] *Let K be a kernel on a tree T and let f_n be an eigenfunction of $\overline{\Delta}$ with eigenvalue λ_n which does not vanish on any vertex. Then λ_n is simple and f_n has exactly n strong nodal domains.*

Therefore, a generalized Cheeger inequality is valid for any Markov chain on a tree T with a nowherezero eigenfunction f_n of an eigenvalue λ_n, i.e.

$$\min(\overline{\lambda}_n, \frac{\lambda_n}{2}) \le \iota_n(T) \le \varsigma_n(T) \le \sqrt{2\,\lambda_n}.$$

For more on the extensive literature of Markov chains on trees the interested reader is referred to [5, 37] and references therein.
On the other hand, it is quite interesting that even for the case of trees we do not know enough about the behavior of parameters discussed in this article, and as Example 4 shows one encounters nongeometric trees in very small cases. Hence, we believe that the following problem can be considered to be a nice starting point for the study of supergeometric graphs.

Problem 1. *Characterize the class of supergeometric trees.*

5.3 Algorithmic considerations

In this section we touch on some algorithmic aspects of the isoperimetry problem and we study its relationships to some well-known concepts as the k-means algorithm and the normalized cuts method. This section is mainly influenced by the seminal contribution of J. Malik and J. Shi [43] (also see [21]) that was brought to our attention after the presentation of the first two authors' article on the isoperimetric spectrum of graphs [19].
Following our notations in Section 3, for a set X, $\mathcal{D}_n(X)$ stands for the set of all n-sets $\{Q_i\}_1^n$, where Q_i's are nonempty disjoint subsets of X. Also $\mathcal{P}_n(X) \subseteq \mathcal{D}_n(X)$ consists of all n-partitions of X.

Definition 6. Given a function $f \in \mathcal{F}^d(X)$ and a weight function $\omega : X \to \mathbb{R}^+ - \{0\}$, for every $1 \le n \le |X|$, the cost function $\mathcal{C}_n^{f,\omega} : \mathcal{D}_n(X) \to \mathbb{R}^+$ is defined as follows

$$\mathcal{M}_n^{f,\omega}(\{Q_i\}_1^n) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \sum_{u \in Q_i} \omega(u) \|f(u) - \mathbf{m}_i\|^2, \text{ where } \mathbf{m}_i \stackrel{\text{def}}{=} \frac{\sum_{u \in Q_i} \omega(u) f(u)}{\sum_{u \in Q_i} \omega(u)},$$

and

$$\mathcal{C}_n^{f,\omega}(\{Q_i\}_1^n) \stackrel{\text{def}}{=} \mathcal{M}_n^{f,\omega}(\{Q_i\}_1^n) + \sum_{u \in Q^*} \omega(u) \|f(u)\|^2,$$