Particle Production during Inflation

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In this paper the theory of cosmological perturbation has been reviewed. First, the classical equation for metric perturbation has been derived. Due to gauge freedom of gravity (diffeomorphism invariant), all perturbation fields are not physical degree of freedoms that is a problem in the quantization of the theory, as the canonical commutation relation should be imposed only on dynamical variables. As a result, the gauge invariant approach which only works with gauge invariant quantities has been discussed. After the quantization, the two point correlation function of field perturbation in some inflationary scenarios has been derived and the feature of having approximately scale invariant power has been shown.

I. INTRODUCTION

Observational data suggests that the universe at early time has been homogeneous and isotropic. On the other hand, completely homogeneous universe cannot lead to the formation of structures such as galaxies and clusters because these structures have been formed through gravitational instability. In fact, parts of the universe with higher matter density have higher gravitational force and attract the matter in lower density parts and over the time they have become denser and formed the structures as we can see today. This suggests that there must be some inhomogeneities in the universe at early times which has been approved by the small anisotropy in the Cosmic Microwave Background (CMB). Indeed, these small inhomogeneities are the origin of large scale structures in the universe, but an important question is what the source of the inhomogeneities is.

Theory of inflation is the most promising candidate to describe the anisotropy of the CMB (as well as addressing other problems like flatness problem and monopole problem). In this review, the simplest model of inflation consisting of a real scalar field has been introduced. In summery, the idea of inflation is that a homogeneous real scalar field, φ, in a homogeneous isotropic space-time exists and the quantum fluctuations of the field, as a result of exponential expansion of space-time, have become the inhomogeneities at cosmological scales. Consequently, it is reasonable to divide any quantity in this scenario into two parts: its background value which is only time-dependent and the perturbation around the background value which can depend both on time and space and the perturbation are much smaller than the background value. This division into background and perturbation value will make the process of finding solution much easier by letting us to solve the linearized equation of gravity.

The flow of the article is as follows: first, the classical linearized equation of gravity with focus on gauge invariant approach has been reviewed. Second, the theory has been quantized and the ambiguity in the choice of vacuum in a time-dependent space-time which generally results in particle production has been discussed. Then, the power spectrum for some explicit examples have been computed and a general scale-invariant property of the power spectrum in the case of inflation has been shown. The last part is devoted to summary and conclusion.

II. CLASSICAL PERTURBATION THEORY

In this section, we are looking at the deviation of the space-time metric from the background metric which describes a completely spatial homogeneous and isotropic universe. As a result, we can choose the background metric to be the FRW metric [1]:

$$\begin{aligned} g^{(0)}_{00} &= a^2(\eta) \\ g^{(0)}_{ij} &= -a^2(\eta)\gamma_{ij} = -a^2(\eta)\tfrac{\delta_{ij}}{1-Kr^2} \end{aligned} \tag{1}$$

Where a is the scale factor, η is the conformal time, superscript $^{(0)}$ refers to background value, γ_{ij} is the 3-dimensional metric of hypersurface $\eta = constant$ and $K = 0, +1, -1$ for flat, closed and open universe respectively.

The full metric is a sum of the background metric and perturbation metric $\delta g_{\mu\nu}$. In general, it can be shown that the fields in $\delta g_{\mu\nu}$ can be decomposed into three different categories of scalar perturbation, vector perturbation and tensor perturbation, depending on their transformation properties with respect to spatial coordinate transformation and, more importantly in the linear approximation, the evolution of each category is independent of the other ones. Additionally, since only the scalar perturbation shows gravitational instability and results in structure [1], this part of metric perturbation is of interest in this review.

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The general form of the (scalar) perturbation metric is

$$\delta g_{\mu\nu} = a^2(\eta) \left[\begin{array}{cc} 2\phi & -B_{;i} \\ -B_{;i} & 2(\psi\gamma_{ij} - E_{;ij}) \end{array} \right] \quad (2)$$

Where $;i$ is the covariant derivative with respect to γ_{ij} (Appendix A).

A. Gauge Invariant Variables

Although the scalar part of the metric perturbation consists of four scalar fields (ϕ, ψ, B and E), they do not describe four physical degrees of freedom and are not completely independent. Indeed, any infinitesimal coordinate transformation (although describing the same space-time) would change the perturbation quantities. Consider a general scalar quantity $A(p)$ which p is a point on space-time manifold and two coordinate representations x^α and $\tilde{x^\alpha}$. Then,

$$\begin{aligned} \delta A(p) &= A(x^\mu(p)) - A^{(0)}(x^\mu(p)) \\ \widetilde{\delta A(p)} &= \tilde{A}(\tilde{x}^\mu(p)) - A^{(0)}(\tilde{x}^\mu(p)) \\ \tilde{x}^\mu &= x^\mu + \epsilon^\mu \end{aligned}$$

Where ϵ^μ is an infinitesimal four-vector field (describing infinitesimal coordinate transformation), $A^{(0)}(x^\mu(p))$ ($A^{(0)}(\tilde{x}^\mu(p))$) is the background value of A at x^μ ($\tilde{x}^\mu$) and $\delta A(p)$ ($\widetilde{\delta A(p)}$) is the perturbation of A around the background value $A^{(0)}(x^\mu(p))$ ($A^{(0)}(\tilde{x}^\mu(p))$). As $\tilde{A}(\tilde{x}^\mu(p))$ and $A(x^\mu(p))$ are the value of quantity A at point p, they are equal and the change in the perturbation quantity as a result of coordinate transformation is

$$\widetilde{\delta A(p)} - \delta A(p) = A^{(0)}(x^\mu) - A^{(0)}(\tilde{x}^\mu) \quad (3)$$

On the other hand, the general infinitesimal coordinate transformation ϵ^μ which changes only the scalar metric perturbation (2) has the following form

$$\epsilon^\mu = (\epsilon^0, \gamma^{ij}\epsilon_{;j}) \quad (4)$$

Where ϵ^0 and ϵ are two scalar functions. Under this infinitesimal transformation (ignoring the second order term) the scalar fields ϕ, ψ, E and B transform in the following way (Appendix A)

$$\begin{aligned} \tilde{\phi} &= \phi - \tfrac{a'}{a}\epsilon^0 - \epsilon^{0\prime} \\ \tilde{\psi} &= \psi + \tfrac{a'}{a}\epsilon^0 \\ \tilde{B} &= B + \epsilon^0 - \epsilon' \\ \tilde{E} &= E - \epsilon \end{aligned} \quad (5)$$

Where $'$ denotes the derivative with respect to conformal time η. As there are two scalar variables in gauge transformation, we can construct two gauge-invariant quantities Φ and Ψ from ϕ, ψ, E and B as follows [1]

$$\Phi = \phi + \tfrac{1}{a}[(B - E')a]' \quad (6)$$

$$\Psi = \psi - \tfrac{a'}{a}(B - E') \quad (7)$$

Another important quantity is the scalar field φ. Perturbation on the top of homogeneous scalar field $\varphi_0(\eta)$ transforms, as a result of coordinate transformation, using (3,4)

$$\widetilde{\delta\varphi} = \delta\varphi - \varphi_0^{'}(\eta)\epsilon^0 \quad (8)$$

So we can construct a gauge-invariant field perturbation as follow

$$\delta\varphi^{(gi)} = \delta\varphi + \varphi_0^{'}(\eta)(B - E') \quad (9)$$

III. PERTURBED ACTION

As we have studied the perturbation of metric and field, we are ready to expand the action in terms of the perturbation fields. Obviously, the zero order term in the action corresponds to homogeneous equation of motion for background field φ. In addition, the first order term vanishes as we expect from the action principal, So we need to expand the action to second order in the field perturbations.

The action consists of two parts, the action of gravity and the action of scalar field

$$S = S_{gr} + S_m \quad (10)$$

$$S_{gr} = \int \mathcal{L}_{gr}\sqrt{-g}d^4x = -\tfrac{1}{16\pi G}\int R\sqrt{-g}d^4x \quad (11)$$

$$S_m = \int (\tfrac{1}{2}\varphi_{,\mu}\varphi^{,\mu} - V(\varphi))\sqrt{-g}d^4x \quad (12)$$

A. Background Equation

In the zero order of perturbation, the metric will be FRW metric and scalar field φ will be homogeneous. Then (for $K = 0$)

$$\begin{aligned} ds^2 &= a^2(\eta)(d\eta^2 - d\vec{x}^2) \\ \sqrt{-g} &= a^4 \\ \mathcal{L}_m &= \tfrac{1}{2a^2}((\varphi')^2 - V(\varphi)) \end{aligned} \quad (13)$$

So the Euler-Lagrange equation for the scalar field is as follows

$$\varphi'' + 2\mathcal{H}\varphi' + a^2 V_{,\varphi} = 0 \quad (14)$$

Which $\mathcal{H} = \frac{a'}{a}$.

B. Perturbation of GR Action

The calculations for finding the perturbation of GR action ,although, is straightforward but is rather lengthy. Here we only point out the process of calculation in [2]. Interested reader can find the detailed calculation in part 10.1 of [2]. First, the line element can be written in the following ADM form

$$ds^2 = (N^2 - N_i N^i)d\eta^2 - 2N_i dx^i d\eta - \gamma_{ij}dx^i dx^j$$

comparing this line element with the one of the metric (equations (1,2))

$$N_i = a^2 B_{,i} \tag{15}$$

$$N = a(1+\phi-\tfrac{\phi^2}{2}+\tfrac{1}{2}B_{,i}B_{,i}) \tag{16}$$

$$\gamma_{ij} = a^2(1-2\psi)\delta_{ij}+2a^2E_{,ij} \tag{17}$$

Substituting equations(15-17) into (11), the perturbation of the GR action (up to a total derivative term) is

$\delta S_{gr} = \frac{1}{16\pi G}\int a^2[-6\psi^{'2}-12\mathcal{H}(\phi+\psi)\psi' - 9\mathcal{H}^2(\psi+\phi)^2-2\psi_{,i}(2\phi_{,i}-\psi_{,i})-4\mathcal{H}(\phi+\psi)(B-E')_{,ii} + 4\mathcal{H}\psi'E_{,ii}-4\psi'(B-E')_{,ii}-4\mathcal{H}\psi_{,i}B_{,i}+6\mathcal{H}^2(\phi+\psi)E_{,ii} - 4\mathcal{H}E_{,ii}(B-E')_{,jj}+4\mathcal{H}E_{,ii}B_{,jj}+3\mathcal{H}^2E^2_{,ii}+3\mathcal{H}^2B_{,i}B_{,i}]$(18)

C. Perturbation of Real Scalar Field Action

Another term in the action (10) is the action of real scalar field

$$S_m = \int\sqrt{-g}(\frac{1}{2}\varphi_{,\mu}\varphi^{,\mu}-V(\varphi))d^4x$$

substituting $\varphi=\varphi_0+\delta\varphi$ and $g_{\mu\nu}$ (1,2) into (12) and keeping the terms up to second order in the field perturbations, and summing with δS_{gr}, we can find the following expression (up to a total derivative) for δS

$\delta S = \frac{1}{6l^2}\int a^2[-6\psi^{'2}-12H\phi\psi'-2\psi_{,i}(2\phi_{,i}-\psi_{,i}) - 2(H'+2H^2)\phi^2+3l^2(\delta\varphi^{'2}-\delta\varphi_{,i}\delta\varphi_{,i}-a^2V_{,\varphi\varphi}\delta\varphi^2)+6l^2(\varphi_0'(\phi+3\psi)'\delta\varphi-2a^2V_{,\varphi}\phi\delta\varphi)+4(B-E')_{,ii}(\frac{3}{2}l^2\varphi_0'\delta\varphi-\psi'-H\phi)]$(19)

Where $l=\sqrt{\frac{8\pi G}{3}}$. As we can see in (19), the time derivative of $B-E'$ does not appear in the action. So the variation of the action with respect to this term leads to a constraint equation

$$\psi'+\mathcal{H}\phi = \frac{3}{2}l^2\varphi_0'\delta\varphi \tag{20}$$

Defining a gauge-invariant variable

$$v = a(\delta\varphi+\frac{\varphi_0'}{\mathcal{H}}\psi) = a(\delta\varphi^{(gi)}+\frac{\varphi_0'}{\mathcal{H}}\Psi) \tag{21}$$

and substituting (20,21) in (19), the action perturbation can be written as following simple form

$$\delta S = \tfrac{1}{2}\int(v^{'2}-v_{,i}v_{,i}+\tfrac{z''}{z}v^2) \tag{22}$$

$$z = \tfrac{a\varphi_0'}{\mathcal{H}} \tag{23}$$

Which is the action of a real scalar field with time dependent mass $m^2=-\frac{z''}{z}$. Variation of the action (22) results in the following Euler-Lagrange equation

$$v''-\nabla^2 v-\frac{z''}{z}v = 0 \tag{24}$$

Once we know the action for the perturbation field v, we can quantize the theory.

IV. QUANTIZATION

Equation (22) is the action of a real scalar field with a time-dependent mass. In order to quantize the theory, first we need to find the canonical momentum conjugate to v

$$\pi = \frac{\partial L}{\partial v'} = v'$$

Then, the fields v and π become operators ($\hat{v}$ and $\hat{\pi}$) with the following same time commutation relation

$$[\hat{v}(\vec{x},\eta),\hat{\pi}(\vec{y},\eta)] = i\delta^{(3)}(\vec{x},\vec{y}) \tag{25}$$

$$[\hat{v}(\vec{x},\eta),\hat{v}(\vec{y},\eta)] = [\hat{\pi}(\vec{x},\eta),\hat{\pi}(\vec{y},\eta)] = 0 \tag{26}$$

The operator $\hat{v}$ is a Heisenberg operator, which means it evolves with time according to

$$\hat{v}''-\nabla^2\hat{v}-\frac{z''}{z}\hat{v} = 0 \tag{27}$$

Generally, the operator $\hat{v}$ can be expanded in terms of the solutions of the classical equation of motion of field v. Here, the solutions of (24) are $e^{i\vec{k}.\vec{x}}v_k(\eta)$ which $v_k(\eta)$ satisfies the following

$$v_k''(\eta)+E_k^2(\eta)v_k(\eta) = 0, E_k^2(\eta) = k^2-\frac{z''}{z} \tag{28}$$

The operator $\hat{v}$ can be expanded in terms of these solutions as

$$\hat{v} = \frac{1}{\sqrt{2}}\int d^3\vec{k}(e^{i\vec{k}.\vec{x}}v_k^*(\eta)\hat{a}_{\vec{k}}+e^{-i\vec{k}.\vec{x}}v_k(\eta)\hat{a}^\dagger_{\vec{k}} \tag{29}$$

A. Bogolyubov Transformation

If we set the normalization factor of $v_k(\eta)$ such that

$$W(v_k(\eta),v_k^*(\eta)) = v_k'v_k^*-v_kv_k^{*'} = 2i \tag{30}$$

then, the commutation relations (25,26) result in the following commutation relation

$$[\hat{a}_{\vec{k}},\hat{a}_{\vec{k}'}] = [\hat{a}^\dagger_{\vec{k}},\hat{a}^\dagger_{\vec{k}'}] = 0 \tag{31}$$

$$[\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k}-\vec{k}') \tag{32}$$

So $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ satisfy the commutation relation of annihilation and creation operator respectively and vacuum state can be defined as the state which is annihilated by all annihilation operator

$$\hat{a}_{\vec{k}}|0_{(a)}\rangle = 0 \tag{33}$$

Once we have defined the vacuum state, particle states can be defined by acting $\hat{a}^\dagger_{\vec{k}}$ on the vacuum. However, there is no unique vacuum state by this definition. As (28) is a linear equation, any linear combination of independent solutions is another solution. So $\tilde{v}_k(\eta)$ and $\tilde{v}^*_k(\eta)$ defined as

$$\tilde{v}_k(\eta) = \alpha_k v_k(\eta) + \beta_k v^*_k(\eta) \tag{34}$$

are also solutions of (28) and we can expand $\hat{v}$ in terms of these solutions

$$\hat{v} = \frac{1}{\sqrt{2}} \int d^3\vec{k}(e^{i\vec{k}.\vec{x}}\tilde{v}^*_k(\eta)\hat{b}_{\vec{k}} + e^{-i\vec{k}.\vec{x}}\tilde{v}_k(\eta)\hat{b}^\dagger_{\vec{k}}) \tag{35}$$

If we normalize $\tilde{v}_k(\eta)$ such that

$$W(\tilde{v}_k(\eta), \tilde{v}^*_k(\eta)) = 2i \tag{36}$$

Which means

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{37}$$

then

$$[\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}'}] = [\hat{b}^\dagger_{\vec{k}}, \hat{b}^\dagger_{\vec{k}'}] = 0 \tag{38}$$

$$[\hat{b}_{\vec{k}}, \hat{b}^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k} - \vec{k}') \tag{39}$$

So we can define the vacuum state using $\hat{b}_{\vec{k}}$'s

$$\hat{b}_{\vec{k}}|0_{(b)}\rangle = 0 \tag{40}$$

And, obviously, the vacuum state $|0_{(b)}\rangle$ differs from $|0_{(a)}\rangle$. To see this, from equations (29,34,35) we can obtain the relation between $\hat{b}_{\vec{k}}$ and $\hat{a}_{\vec{k}}$

$$\hat{a}_{\vec{k}} = \alpha_{\vec{k}}\hat{b}_{\vec{k}} + \beta^*_{\vec{k}}\hat{b}^\dagger_{\vec{k}} \tag{41}$$

which means that the vacuum state defined by mode functions v_k is different from the vacuum state defined by $\tilde{v}_k$.

B. On Initial Condition

In the case of Minkowski space-time (no time dependency), the field can be expanded in terms of positive frequency mode functions

$$v_k(\eta) \sim e^{i\omega_k\eta}, \omega^2_k = k^2 + m^2 \tag{42}$$

And their coefficients in the mode expansion are the correct annihilation operators for defining the vacuum state. In fact, a more elaborate definition of being positive-frequency for a mode function at time η_0 is

$$v_k(\eta_0) = \frac{1}{\sqrt{E(\eta_0)}}, v'_k(\eta_0) = i\sqrt{E(\eta_0)} \tag{43}$$

Hence at any time, we can define the positive frequency modes and consequently the vacuum state. When equation of motion contains time dependent mass, a mode which is positive frequency at time η_0, in general will not be a positive frequency mode at a different time. Indeed, it means that the vacuum state of the field changes in time. As a result, starting from the vacuum state at time η_0, at later times this state will contain particles.

However, the definition of positive frequency modes (43) are only applicable to modes with positive E_k. In fact, the definition of vacuum state in general space-time is not well-established. In inflation period

$$\frac{z''}{z} = \frac{a''}{a} > 0 \tag{44}$$

So E_k for a long-wavelength mode can be negative and the vacuum state cannot be defined completely. However, for $k \to \infty$, we expect that the effect of the space time curvature becomes negligible and the definition of positive frequency mode for flat space time becomes valid. So we can use the following initial conditions to define the vacuum at time η_i [2]

$$v_k(\eta_i) = k^{-\frac{1}{2}}M(k\eta_i) \;\; v'_k(\eta_i) = ik^{\frac{1}{2}}N(k\eta_i) \tag{45}$$

with the following asymptotic behavior and normalization (using (30)) for N and M

$$NM^* + N^*M = 2 \tag{46}$$

$$M(k\eta_i)| \to 1, |N(k\eta_i)| \to 1, k\eta_i \to \infty \tag{47}$$

Fortunately, for the purpose of calculating the power spectrum (will be defined in the next section) up to first order in k, these general conditions will be sufficient.

V. CALCULATING THE POWER SPECTRUM

As we have mentioned before, the inflationary scenario can describe the primordial inhomogeneities in the universe which can be seen in the CMB. A good measure for the anisotropy in the primordial fluctuations is the two point correlation function of metric perturbation Φ. In order to find the two point correlation function, first we need the relation between $\hat{v}$ and $\hat{\Phi}$, as we have imposed the commutation relation on $\hat{v}$.

variation of the action (19) with respect to ϕ and ψ leads to the following equation in terms of gauge invariant variables

$$\nabla^2\Psi - 3H\Psi' - (H' + 2H^2)\Phi = \frac{3}{2}l^2(\varphi'_0\delta\varphi^{(gi)\prime} + V_{,\varphi}a^2\delta\varphi^{(gi)}) \tag{48}$$

$$\frac{1}{3}\nabla^2(\Phi - \Psi) + \Psi'' + H\Phi' + 2H\Psi' + (H' + 2H^2)\Phi = \frac{3}{2}l^2(\varphi'_0\delta\varphi^{(gi)\prime} + V_{,\varphi}a^2\delta\varphi^{(gi)}) \tag{49}$$

Equations (48,49) and definition (21) lead to the following [2]

$$\Phi = \Psi \tag{50}$$

$$\Phi'' + 2(\frac{a}{\varphi_0}')'\frac{\varphi_0'}{a}\Phi' - \nabla^2\Phi + 2\varphi_0'(H\varphi_0')\Phi = 0 \tag{51}$$

$$\nabla^2\Phi = \frac{3}{2}l^2\frac{\varphi_0'^2}{H}(\frac{v}{z})' \tag{52}$$

Substituting fields with operators in equations (50-52), using (52) we can conclude that $\hat{\Phi}$ can be expanded in terms of creation and annihilation operators

$$\hat{\Phi}(\vec{x},\eta) = \frac{1}{\sqrt{2}}\frac{\varphi_0'}{a}\int\frac{d^3k}{(2\pi)^{3/2}}[u_k^*(\eta)e^{i\vec{k}.\vec{x}}a_{\vec{k}} + u_k(\eta)e^{-i\vec{k}.\vec{x}}a_{\vec{k}}\dagger] \tag{53}$$

And (52) results in the relation between mode function v_k and u_k as follows

$$u_k(\eta) = -\frac{3}{2}l_p^2\frac{z}{k^2}(\frac{v_k}{z})' \tag{54}$$

Suppose that the universe at time η_i was at the vacuum state of the field $\hat{v}$ denoted by $|0\rangle$ defined via (45). Then the two point correlation function of metric perturbation $\hat{\Phi}$ between point $\vec{x}$ and $\vec{x}+\vec{r}$ at time η is

$$\langle 0|\hat{\Phi}(\vec{x})\hat{\Phi}(\vec{x}+\vec{r})|0\rangle \tag{55}$$

Using (53) and the fact that the state $|0\rangle$ is annihilated by all $a_{\vec{k}}$ operators

$$\langle 0|\hat{\Phi}(\vec{x})\hat{\Phi}(\vec{x}+\vec{r})|0\rangle = \int_0^\infty \frac{dk}{k}\frac{sin(kr)}{kr}|\delta_k|^2 \tag{56}$$

Which δ_k, in the above equation, is called the power spectrum of metric perturbation $\hat{\Phi}$ and []

$$|\delta_k(\eta)|^2 = \frac{1}{4\pi^2}\frac{\varphi_0'^2}{a^2}|u_k(\eta)|^2k^3 \tag{57}$$

In fact, the power spectrum is a measure of the anisotropy in the metric perturbation Φ. In order to calculate the power spectrum, we can solve (28) with initial conditions (45) at time η_i to find the mode functions v_k, then (54) gives the solution for the mode functions u_k and from definition (57), we can find an expression for the power spectrum.

A. Power Spectrum in Inflationary Universe

So far, all the calculation was general for the case of a real scalar field. However, in this section, we are going to compute the power spectrum in some special cases.

Let's assume that the space-time is de Sitter space-time

$$a(t) = e^{H_\Lambda t} = -\frac{1}{H_\Lambda\eta} \tag{58}$$

The equations in the earlier sections gives an exponentially expanding universe, if we assume that the field φ has stayed at some point on the top of the potential and does not roll over it. In this case $\varphi_0' \simeq 0$ and

$$H_\Lambda^2 = l_p^2V(\varphi) = const \tag{59}$$

Also equation (21) gives

$$v = a\delta\varphi^{(gi)} \tag{60}$$

As $\varphi_0' = 0$, from equation(53), it is obvious that the metric perturbation and its power spectrum is zero. However, this is not surprising, as we have assumed that the deviation of space-time from background metric is zero (58,59). Consequently, we are going to calculate the power spectrum for field perturbation $\delta\varphi$.

In exponentially expanding universe, (28) becomes

$$v_k'' + (k^2 - \frac{2}{\eta^2})v_k = 0 \tag{61}$$

by a change of variable

$$v_k(\eta) = \sqrt{k|\eta|}f(k|\eta|) \tag{62}$$

equation (61) reduces to Bessel equation

$$s^2\frac{df}{ds^2} + s\frac{df}{ds} + (s^2 - \frac{9}{4})f = 0, s = k|\eta| \tag{63}$$

Which has the solution

$$v_k(\eta) = \sqrt{k|\eta|}(A_kJ_{3/2}(k|\eta|) + B_kY_{3/2}(k|\eta|)) \tag{64}$$

and the normalization condition (30) imposes

$$A_kB_k^* - A_k^*B_k = \frac{i\pi}{k} \tag{65}$$

Assuming that inflation lasts forever and all modes has started from the vacuum, we expect that for $\eta \to -\infty$

$$v_k(\eta) \to \frac{1}{\sqrt{k}}e^{ik\eta} \tag{66}$$

Looking at the solution (64) in the limit of $\eta \to -\infty$ and make use of normalization condition (65), we find the following mode functions [3]

$$v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}}(J_{3/2}(k|\eta|) - iY_{3/2}(k|\eta|)) \tag{67}$$

Defining the power spectrum for the field perturbation $\delta_\varphi(k)$ as

$$\langle 0|\widehat{\delta\varphi}(\vec{x})\widehat{\delta\varphi}(\vec{x}+\vec{r})|0\rangle = \int_0^\infty \frac{dk}{k}\frac{sin(kr)}{kr}|\delta_\varphi(k)|^2 \tag{68}$$

and using (29,60)

$$\delta_\varphi(k,\eta) = \frac{\sqrt{2\pi}}{a(\eta)}k^{3/2}|v_k(\eta)| \tag{69}$$

Substituting the solution (67) into (69) and expressing the power spectrum in terms of physical wavenumber $k_{ph} = \frac{k}{a}$ we have

$$\delta_\varphi(k_{ph}) = \pi H_\Lambda (\frac{k_{ph}}{H_\Lambda})^{3/2} \sqrt{J^2_{3/2}(\frac{k_{ph}}{H_\Lambda}) + Y^2_{3/2}(\frac{k_{ph}}{H_\Lambda})} \quad (70)$$

Using the properties of Bessel function, we obtain the following asymptotic behavior

$$\delta_\varphi(k_{ph}) = \begin{cases} \sqrt{2\pi} k_{ph} & k_{ph} \gg H_\Lambda \\ \sqrt{2\pi} H_\Lambda & k_{ph} \ll H_\Lambda \end{cases} \quad (71)$$

B. Finite Duration Inflation

In a more realistic scenario, inflation only last for a finite time. Assume that the inflation starts at time η_i and ends at η_f. At time η_i, the subhorizon modes ($k|\eta_i| \gg 1$) do not feel the effect of curvature. It can be seen from (61) (which for $k|\eta| \gg 1$ becomes)

$$v''_k - k^2 v_k = 0 \quad (72)$$

It means we can assume that these modes are in the vacuum state at the beginning of inflation. However, for superhorizon modes at time η_i ($k|\eta_i| \ll 1$), we have

$$v''_k - \frac{2}{\eta^2} v_k = 0 \quad (73)$$

Which means the notion of vacuum for these modes is ambiguous and their spectrum depends on some preinflationary scenario [3].

Assuming that a subhorizon mode at η_i has started from the vacuum, the solution of equation (72) is

$$v_k(\eta) = \frac{1}{\sqrt{k}} e^{ik\eta} \quad (74)$$

for $\eta_i \leq \eta \leq -\frac{1}{k}$. This mode crosses the horizon at time $\eta_k = -\frac{1}{k}$ and then, its evolution is determined by (73). The general solution for (73) is

$$v_k(\eta) = A_k \eta^{-1} + B_k \eta^2 \quad (75)$$

Matching the solution (74) and (75) at the time of crossing η_k, we have

$$A_k \sim k^{-3/2}, B_k \sim k^{3/2} \quad (76)$$

As time passes, the second term in (75) becomes negligible and the solution up to a order one factor becomes

$$v_k(\eta) \sim k^{-3/2} \eta^{-1} \quad (77)$$

In summary, for the subhorizon modes at the beginning of inflation (η_i) which are still subhorizon at time η, equation (74) is the solution. For the subhorizon modes at the beginning of inflation which become superhorizon at time η, equation(77) is the solution.

Consequently, the power spectrum of the field φ can be computed using (69) and the solutions (74) and (77), as follows

$$\delta_\varphi(k, \eta) \sim \begin{cases} \frac{k}{a(\eta)} & k \geq -\frac{1}{\eta} \\ -\frac{1}{a\eta} & -\frac{1}{\eta} \geq k \geq -\frac{1}{\eta_i} \\ ? & k \leq -\frac{1}{\eta_i} \end{cases} \quad (78)$$

Which in terms of physical wavenumber $k_{ph} = \frac{k}{a}$ is

$$\delta_\varphi(k_{ph}, \eta) \sim \begin{cases} k_{ph} & k_{ph} \geq H_\Lambda \\ H_\Lambda & H_\Lambda \geq k_{ph} \geq H_\Lambda \frac{\eta}{\eta_i} \\ ? & k_{ph} \leq H_\Lambda \frac{\eta}{\eta_i} \end{cases} \quad (79)$$

After the end of inflation (in radiation or matter dominated era), the co-moving horizon $\frac{1}{aH}$ expands and the modes which were subhorizon at the beginning of inflation and became superhorizon during inflation, enters the horizon with scale-invariant power.

C. Time-dependent Hubble Parameter in Inflation

In a realistic model, the Hubble parameter changes slightly in time and the power spectrum is not completely scale-invariant. In this case, (28) becomes

$$v''_k + (k^2 - \frac{a''}{a}) v_k = 0 \quad (80)$$

For subhorizon modes at the beginning of inflation, we obtain

$$v_k(\eta) = \frac{1}{\sqrt{k}} e^{ik\eta}, a_i H_i \leq k \leq a(\eta) H(\eta) \quad (81)$$

Which a_i and H_i are the scale factor and Hubble parameter at the beginning of inflation. After crossing the horizon at time η_k, the evolution of the mode is determined by the following equation

$$v''_k - \frac{a''}{a} v_k = 0 \quad (82)$$

Which has the general solution as

$$v_k = A_k a + B_k a \int \frac{d\eta}{a^2} \quad (83)$$

After imposing the matching conditions between solution (81) and (83) and considering that the second term in 83) becomes negligible as time passes, we can find the following solution for the mode k after crossing the horizon

$$v_k \sim \frac{1}{\sqrt{k}} \frac{a(\eta)}{a_k} \quad (84)$$

Which a_k is the scale factor at the time of horizon crossing. Using (69), we can find the power spectrum for the subhorizon mode which crosses the horizon

$$\delta_\varphi(k,\eta) \sim \frac{1}{a(\eta)} k^{3/2}|v_k| \sim H_k, a(\eta)H(\eta) \geq k \geq a_i H_i \tag{85}$$

Which H_k is the Hubble parameter at the time of horizon crossing and $k \simeq a_k H_k$. Equation (85) shows that a specific mode enters the horizon (in radiation or matter era) with a slight dependence on the scale of the mode. Also, it is obvious that equations (80-85) give equations in the previous section provided that $a(\eta) = -\frac{1}{H_\Lambda \eta}$.

D. Back-reaction of the Scalar Field

So far, in all examples, we have neglected the effect of the scalar field on metric. However, equations (28,54,57) are completely general and we have considered the back-reaction of the scalar field on geometry. The process of finding the power spectrum in this situation is similar to previous cases and here we only express the result from [2]. The power spectrum (57) for a general potential $V(\varphi)$ is

$$\delta_k(\eta) \simeq \frac{l_p}{4\pi} \begin{cases} \frac{V_{,\varphi}}{V^{1/2}} & k_{ph} > H(\eta) \\ (\frac{V^{3/2}}{V_{,\varphi}})_{\eta_k} \frac{V^2_{,\varphi}}{V^2} & H(\eta) > k_{ph} > H_i a_i/a(\eta) \end{cases} \tag{86}$$

Which η_k is the time of horizon crossing. For the special case of $V(\varphi) = \frac{1}{2}m^2\varphi^2$ (86) becomes

$$\delta_k(\eta) \simeq \frac{\sqrt{2}}{4\pi}\frac{m}{m_p} \begin{cases} 1 & k_{ph} > H(\eta) \\ 1 + \frac{ln(\lambda_{ph}H(\eta))}{ln(\frac{a_r}{a(\eta)})} & H(\eta) > k_{ph} > H_i a_i/a(\eta) \end{cases}$$

Which $m_p = 1/l_p$ is the Planck mass, λ_{ph} is the physical wavelength and the a_r is the scale factor at the end of inflation. In this scenario, the modes enter the horizon at late times with the power depend on their scale logarithmically.

VI. SUMMARY AND CONCLUSION

As it was shown, the vacuum state in time-dependent space-times generally changes in time which leads to particle production. Here, we computed the power spectrum of field fluctuation in some more or less realistic scenarios and in all cases approximately scale invariant power spectrum has been derived.

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APPENDIX A: SCALAR, VECTOR AND TENSOR PERTURBATION OF METRIC

The metric perturbation can be decomposed into three classes, scalar, vector and tensor part based on the symmetry properties of the background metric (FRW metric has spatial rotation and translation symmetry).

δg_{00} acts like a scalar under spatial rotation, so we can express it in terms of a 3-scalar field ϕ as following

$$\delta g_{00} = 2a^2\phi$$

δg_{0i} can be expressed as a sum of a spatial gradient of a scalar field and a 3-divergenceless vector field (divergence of a 3-vector transforms like a scalar, so we can put it in the gradient of a scalar part) as follow

$$\delta g_{0i} = a^2(-B_{;i} + S_i)$$
$$S^i_{;i} = 0$$

Similarly, δg_{ij} can be decomposed into a scalar times 3-metric, gradient of a scalar, derivative of a divergenceless 3-vector and traceless divergenceless tensorial part

$$\delta g_{ij} = a^2(2\psi\omega_{ij} - 2E_{;ij} + F_{i;j} + F_{j;i} + h_{ij})$$
$$F^i_{;i} = 0$$
$$h^i_i = 0, h^i_{j;i} = 0$$

So we have four scalar perturbation ϕ, ψ, E and B, two vector perturbation S_i and F_i (having four independent components) and one tensor perturbation h_{ij} (having two independent component).

For an infinitesimal coordinate transformation (4), the metric in $\tilde{x}$ coordinate is

$$\begin{aligned} \tilde{g}_{\mu\nu}(\tilde{x}^\alpha) &= \frac{\partial x^{\mu'}}{\partial \tilde{x}^\mu}\frac{\partial x^{\nu'}}{\partial \tilde{x}^\nu} g_{\mu'\nu'}(x^\alpha) \simeq \\ & g^{(0)}_{\mu\nu}(x^\alpha) + \delta g_{\mu\nu} - g_{\mu\nu'}\epsilon^{\nu'}_{,\nu} - g_{\mu'\nu}\epsilon^{\mu'}_{,\mu} \end{aligned} \tag{A1}$$

We can split the metric in $\tilde{x}$ coordinate into background and perturbation part as

$$\tilde{g}_{\mu\nu}(\tilde{x}^\alpha) = g^{(0)}_{\mu\nu}(\tilde{x}^\alpha) + \widetilde{\delta} g_{\mu\nu} \tag{A2}$$

Using the fact that

$$g^{(0)}_{\mu\nu}(\tilde{x}^\alpha) \simeq g^{(0)}_{\mu\nu}(x^\alpha) + g^{(0)}_{\mu\nu,\beta}\epsilon^\beta \tag{A3}$$

and comparing equations (A1) and (A2), the metric perturbation in $\tilde{x}$ coordinate can be expressed in terms of δg and ϵ^μ

$$\widetilde{\delta} g_{\mu\nu} = \delta g_{\mu\nu} - g^{(0)}_{\mu\nu,\beta}\epsilon^\beta - g^{(0)}_{\beta\nu}\epsilon^\beta_{,\mu} - g^{(0)}_{\mu\beta}\epsilon^\beta_{,\nu} \tag{A4}$$

Equation (A4) together with equations (1) and (2) results in (5).

[1] Physical Foundations of Cosmology, by Viatcheslav Mukhanov, Cambridge University Press, November 2005. ISBN-10: 0521563984. ISBN-13: 9780521563987.

[2] V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger, Theory of cosmological perturbations, Physics Reports, Volume 215, Issues 5-6, June 1992, Pages 203-333

[3] Introduction to Quantum Effects in Gravity, by Viatcheslav Mukhanov and Sergei Winitzki, Cambridge University Press, . ISBN-10: . ISBN-13