Special relativity

Problem Set 6

January 2, 2026

Problem 1

Lagrangian Formalism and Its Limitation in Special Relativity

I. In classical mechanics, the dynamics of a system with generalized coordinates $q^i(t)$ is described by the Lagrangian

$$L(q^i, \dot{q}^i, t),$$

and the action

$$S = \int L\, dt.$$

Show that requiring the action to be stationary under variations of $q^i(t)$ leads to the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0.$$

II. For a free particle of mass m in classical mechanics, the Lagrangian is

$$L_{\rm cl} = \frac{1}{2}mv^2.$$

Show explicitly that this Lagrangian is *not Lorentz invariant*.

III. Explain why, in Special Relativity, the action (rather than the Lagrangian itself) must be Lorentz invariant, and why this motivates the search for a new relativistic Lagrangian.

Problem 2

Derivation of the Relativistic Free-Particle Lagrangian

I. In Special Relativity, the proper time $d\tau$ along a particle worldline is defined as

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, dt.$$

Show that $d\tau$ is invariant under Lorentz transformations.

II. Using dimensional analysis and Lorentz invariance, argue that the simplest action for a free relativistic particle must be proportional to the proper time, Find α:

$$S = \int \alpha \, d\tau.$$

III. Express the action in terms of the coordinate time t and identify the corresponding Lagrangian $L(v)$.
Show that the resulting Lagrangian is:

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}.$$

IV. Compute the canonical momentum

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$$

and show that it reduces to the relativistic momentum

$$\mathbf{p} = \gamma m \mathbf{v}, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

V. Expand the Lagrangian for $v \ll c$ and show that it reproduces the classical free-particle Lagrangian up to an additive constant.

Problem 3

Problem: Energy–Momentum Tensor in Special Relativity

Consider a field $\phi(x)$ in flat spacetime with the Lagrangian density $\mathscr{L}(\phi, \partial_\mu \phi)$ in Special Relativity. The canonical energy–momentum tensor is defined as

$$T^{\mu\nu}_{\text{canonical}} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathscr{L}, \tag{1}$$

I. It is possible to add a divergence-free term $\partial_\lambda X^{\lambda\mu\nu}$ (with $X^{\lambda\mu\nu} = -X^{\mu\lambda\nu}$) to $T^{\mu\nu}_{\text{canonical}}$ to obtain a symmetric energy–momentum tensor:

$$T^{\mu\nu}_{\text{sym}} = T^{\mu\nu}_{\text{canonical}} + \partial_\lambda X^{\lambda\mu\nu}. \tag{2}$$

Explain why this procedure does not change the conservation law $\partial_\mu T^{\mu\nu} = 0$.

II. For a free scalar field with Lagrangian density

$$\mathscr{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2}m^2\phi^2, \tag{3}$$

compute the symmetric energy–momentum tensor $T^{\mu\nu}_{\text{sym}}$.

III. Classical electromagnetism (with no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right),$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Furthermore, Maxwell's equations can be derived as the Euler–Lagrange equations corresponding to this action, where the components $A_\mu(x)$ are treated as the dynamical variables. The equations then appear in their standard form upon making the appropriate identifications

$$E^i = -F^{0i} \qquad \text{and} \qquad \epsilon^{ijk}B^k = -F^{ij}.$$

Construct the energy-momentum tensor for this theory. What is the physical meaning of each component of $T^{\mu\nu}_{\text{sym}}$? In particular, identify which components represent energy density, momentum density, and stress.

Problem 4

From the class lecture, identify any part that you find ambiguous or unclear. You may optionally include a suggested answer or comment, but it is sufficient to submit a single, clearly stated question.

When the people no longer fear death, how can they be threatened with death?
Laozi, Dao De Jing