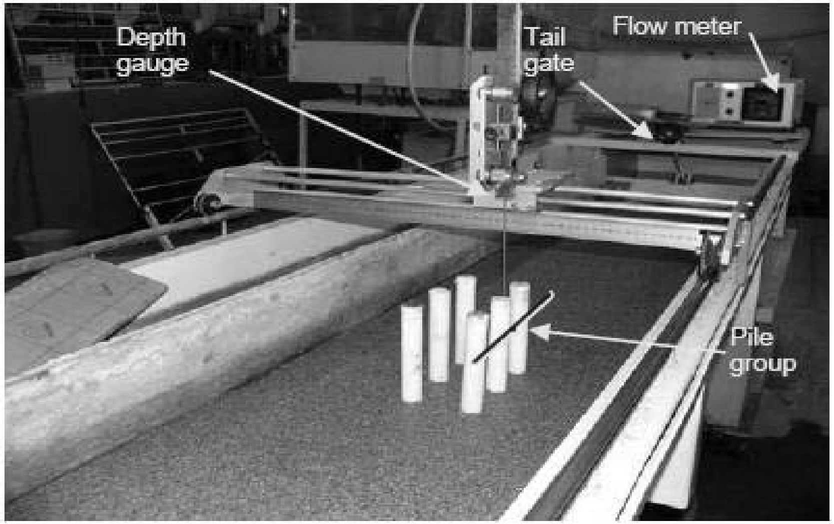
Dimensional Analysis and Similarity



Dimensional Analysis

To this point, we have concentrated on **analytical methods** of solution for fluids problems. However, analytical methods are not always satisfactory due to:

(1) limitations due to simplifications required in the analysis,

(2) complexity and/or expense of a detailed analysis.

The most common alternative is to use experimental test & verification procedures. However, without planning and organization, experimental procedures can :

(a) be time consuming,

(b) lack direction,

(c) be expensive.

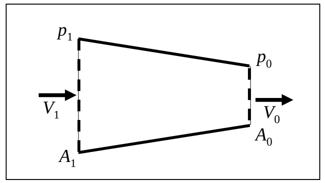
Dimensional Analysis

Dimensional analysis provides a procedure that will typically reduce both the time and expense of experimental work necessary to experimentally represent a desired set of conditions and geometry.

It also provides a means of "normalizing" the final results for a range of test conditions. A normalized (non-dimensional) set of results for one test condition can be used to predict the performance at different but fluid dynamically similar conditions (including even a different fluid).

Dimensional Analysis

- Want to study pressure drop as function of velocity (V_1) and diameter (d_o)
- Carry out numerous experiments with different values of V_1 and d_o and plot the data



$$p_1 + \rho\frac{V_1^2}{2} = p_0 + \rho\frac{V_0^2}{2}$$

$$\Delta p = \frac{\rho}{2}\left(V_0^2 - V_1^2\right)$$

$$\Delta p = \frac{\rho}{2}V_1^2\left(\left(\frac{d_1}{d_0}\right)^4 - 1\right)$$

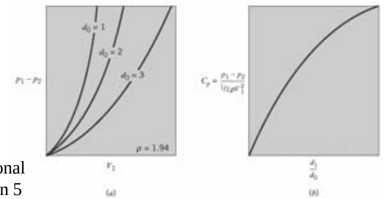
5 parameters: $\Delta p, \rho, V_1, d_1, d_o$

$$\frac{\Delta p}{\frac{\rho}{2}V_1^2} = \left(\frac{d_1}{d_0}\right)^4 - 1$$

2 dimensionless parameters: $\Delta p/(\rho V^2/2)$, (d_1/d_o)

$$C_p = f\left(\frac{d_1}{d_0}\right)$$

Much easier to establish functional relations with 2 parameters, than 5



The Buckingham Pi Theorem

- "in a physical problem including ***n*** quantities in which there are ***m*** dimensions, the quantities can be arranged into ***n-m*** independent dimensionless parameters"

- We reduce the number of parameters we need to vary to characterize the problem!

Buckingham Pi Theorem

Force F on a body immersed in a flowing fluid depends on: L, V, ρ, and μ

$$F = f(L, V, \rho, \mu)$$

$n = 5$ No. of dimensional parameters
$j = 3$ No. of dimensions
$k = n - j = 2$ No. of dimensionless parameters

F	L	V	ρ	μ
MLT^{-2}	L	LT^{-1}	ML^{-3}	$ML^{-1}T^{-1}$

Select "repeating" variables: L, V, and ρ
Combine these with the rest of the variables: F & μ

$$\pi_1 = \mu(L^a V^b \rho^c)$$
$$M^0 L^0 T^0 = (ML^{-1}T^{-1})(L)^a (LT^{-1})^b (ML^{-3})^c$$
$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$
$$L: \quad 0 = -1 + a + b - 3c \quad \Rightarrow \quad a = -1$$
$$T: \quad 0 = -1 - b \quad \Rightarrow \quad b = -1$$

$$\pi_1 = \frac{\mu}{LV\rho} \quad or \quad \pi_1 = \Re = \frac{\rho VL}{\mu}$$

Reynolds number

Buckingham Pi Theorem

F	L	V	ρ	μ
MLT^{-2}	L	LT^{-1}	ML^{-3}	$ML^{-1}T^{-1}$

$$\pi_2 = F(L^a V^b \rho^c)$$
$$M^0 L^0 T^0 = (MLT^{-2})(L)^a (LT^{-1})^b (ML^{-3})^c$$
$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$
$$L: \quad 0 = 1 + a + b - 3c \quad \Rightarrow \quad a = -2$$
$$T: \quad 0 = -2 - b \quad \Rightarrow \quad b = -2$$

$$\pi_2 = \frac{F}{L^2 V^2 \rho} \quad and \quad \pi_2 = f(\pi_1)$$

$$\frac{F}{\rho V^2 L^2} = f(\Re)$$

Dimensionless force is a function of the Reynolds number

Buckingham Pi Theorem

1. List all n variables involved in the problem
   - Typically: all variables required to describe the problem geometry (D) or define fluid properties (ρ, μ) and to indicate external effects (dp/dx)
2. Express each variables in terms of MLT dimensions (j)
3. Determine the required number of dimensionless parameters ($n - j$)
4. Select a number of repeating variables = number of dimensions
   - All reference dimensions must be included in this set and each must be dimensionalls independent of the others
5. Form a dimensionless parameter by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an unknown exponent
6. Repeat for each nonrepeating variable
7. Express result as a relationship among the dimensionless parameters

Example

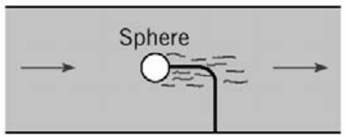
- **Find:** Drag force on rough sphere is function of D, ρ, μ, V and k. Express in form:

$$\pi_3 = f(\pi_1, \pi_2)$$

F_D	D	ρ	μ	V	k
MLT^{-2}	L	ML^{-3}	$ML^{-1}T^{-1}$	LT^{-1}	L

$n = 6$ No. of dimensional parameters
$j = 3$ No. of dimensions
$k = n - j = 3$ No. of dimensionless parameters

Select "repeating" variables: D, V, and ρ
Combine these with nonrepeating variables: F, μ & k



$$\pi_1 = \mu(D^a V^b \rho^c)$$
$$M^0 L^0 T^0 = (ML^{-1}T^{-1})(L)^a (LT^{-1})^b (ML^{-3})^c$$
$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$
$$L: \quad 0 = -1 + a + b - 3c \quad \Rightarrow \quad a = -1$$
$$T: \quad 0 = -1 - b \quad \Rightarrow \quad b = -1$$

$$\pi_1 = \frac{\mu}{DV\rho} \quad or \quad \pi_1 = \Re = \frac{\rho VD}{\mu}$$

Example

F_D	D	ρ	μ	V	k
MLT^{-2}	L	ML^{-3}	$ML^{-1}T^{-1}$	LT^{-1}	L

Select "repeating" variables: D, V, and ρ
Combine these with nonrepeating variables: F, μ & k

$$\pi_2 = k(D^a V^b \rho^c)$$
$$M^0 L^0 T^0 = (L)(L)^a (LT^{-1})^b (ML^{-3})^c$$
$$M: \quad 0 = c \quad \Rightarrow \quad c = 0$$
$$L: \quad 0 = 1 + a + b - 3c \quad \Rightarrow \quad a = -1$$
$$T: \quad 0 = -b \quad \Rightarrow \quad b = 0$$

$$\pi_2 = \frac{k}{D}$$

$$\pi_3 = F_D(D^a V^b \rho^c)$$
$$M^0 L^0 T^0 = (MLT^{-2})(L)^a (LT^{-1})^b (ML^{-3})^c$$
$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$
$$L: \quad 0 = 1 + a + b - 3c \quad \Rightarrow \quad a = -2$$
$$T: \quad 0 = -2 - b \quad \Rightarrow \quad b = -2$$

$$\pi_3 = \frac{F_D}{\rho V^2 D^2}$$

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho VD}{\mu}, \frac{k}{D}\right)$$

Common Dimensionless No's.

- Reynolds Number (inertial to viscous forces)
  - Important in all fluid flow problems

$$\Re = \frac{\rho Vd}{\mu}$$

- Froude Number (inertial to gravitational forces)
  - Important in problems with a free surface

$$F = \frac{V}{\sqrt{gh}}$$

- Euler Number (pressure to inertial forces)
  - Important in problems with pressure differences

$$C_p = \frac{\Delta p}{\rho V^2}$$

- Mach Number (inertial to elastic forces)
  - Important in problems with compressibility effects

$$M = \frac{V}{\sqrt{E/\rho}} = \frac{V}{c}$$

- Weber Number (inertial to surface tension forces)
  - Important in problems with surface tension effects

$$W = \frac{\rho LV^2}{\sigma}$$

Parameter	Definition	Qualitative ratio of effects	Importance
Reynolds number	$R_E = \frac{\rho UL}{\mu}$	$\frac{\text{Inertia}}{\text{Viscosity}}$	Always
Mach number	$MA = \frac{U}{A}$	$\frac{\text{Flow speed}}{\text{Sound speed}}$	Compressible flow
Froude number	$Fr = \frac{U^2}{gL}$	$\frac{\text{Inertia}}{\text{Gravity}}$	Free-surface flow
Weber number	$W_e = \frac{\rho U^2 L}{\gamma}$	$\frac{\text{Inertia}}{\text{Surface tension}}$	Free-surface flow
Cavitation number (Euler number)	$Ca = \frac{p - p_v}{\rho U^2}$	$\frac{\text{Pressure}}{\text{Inertia}}$	Cavitation
Prandtl number	$Pr = \frac{C_p \mu}{k}$	$\frac{\text{Dissipation}}{\text{Conduction}}$	Heat convection
Eckert number	$Ec = \frac{U^2}{c_p T_o}$	$\frac{\text{Kinetic energy}}{\text{Enthalpy}}$	Dissipation
Specific-heat ratio	$\gamma = \frac{c_p}{c_v}$	$\frac{\text{Enthalpy}}{\text{Internal energy}}$	Compressible flow
Strouhal number	$St = \frac{\omega L}{U}$	$\frac{\text{Oscillation}}{\text{Mean speed}}$	Oscillating flow
Roughness ratio	$\frac{\varepsilon}{L}$	$\frac{\text{Wall roughness}}{\text{Body length}}$	Turbulent, rough walls
Grashof number	$Gr = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Temperature ratio	$\frac{T_w}{T_o}$	$\frac{\text{Wall temperature}}{\text{Stream temperature}}$	Heat transfer
Pressure coefficient	$C_p = \frac{p - p_\infty}{1/2 \rho U^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Aerodynamics, hydrodynamics
Lift coefficient	$C_L = \frac{L}{1/2 \rho U^2 A}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$	Aerodynamics hydrodynamics
Drag coefficient	$C_D = \frac{D}{1/2 \rho U^2 A}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics

Similarity

- Similarity
  - Predict prototype behavior from model results
  - Models resemble prototype, but are
    - Different size (usually smaller) and may operate in
    - Different fluid and under
    - Different conditions
  - Problem described in terms of dimensionless parameters which may apply to the model or the prototype

$$\pi_1 = f(\pi_2, \pi_3, ..., \pi_n)$$

  - Suppose it describes the prototype

$$\pi_{1p} = f(\pi_{2p}, \pi_{3p}, ..., \pi_{np})$$

  - A similar relationship can be written for a model of the prototype

$$\pi_{1m} = f(\pi_{2m}, \pi_{3m}, ..., \pi_{nm})$$

Similarity

- If the model is designed & operated under conditions that

$$\left.\begin{array}{l} \pi_{2m} = \pi_{2p} \\ \pi_{3m} = \pi_{3p} \\ ... \\ \pi_{nm} = \pi_{np} \end{array}\right\} \text{Similarity requirements or modeling laws}$$

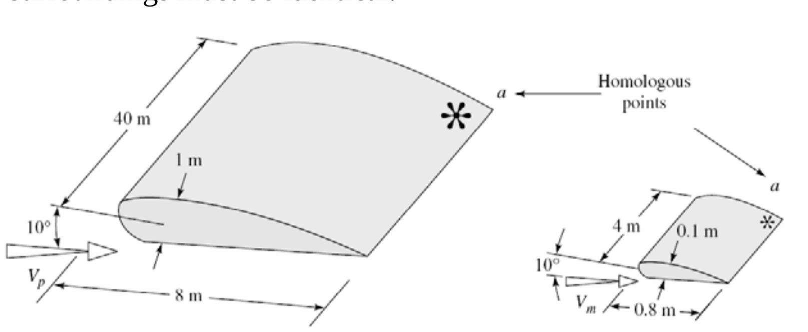
- then $\pi_{1m} = \pi_{1p}$ → Dependent variable for prototype will be the same as in the model

Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype.

But this is easier said than done!

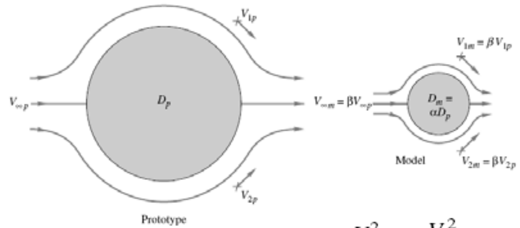
Similarity conditions

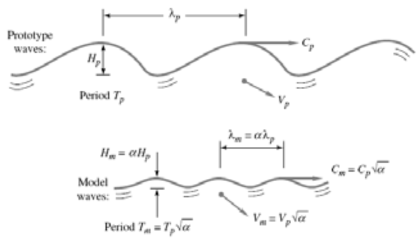
- Instead of complete similarity, the engineering literature speaks of particular types of similarity, the most common being **geometric, kinematic, and dynamic.**
- A model and prototype are *geometrically similar* if and only if all body dimensions in all three coordinates have the same linear-scale ratio.
- All angles are preserved in geometric similarity. All flow directions are preserved. The orientations of model and prototype with respect to the surroundings must be identical.



Similarity conditions

- **Kinematic similarity** requires that the model and prototype have the same length-scale ratio and the same time-scale ratio. The result is that the velocity-scale ratio will be the same for both.
- The motions of two systems are kinematically similar if homologous particles lie at homologous points at homol





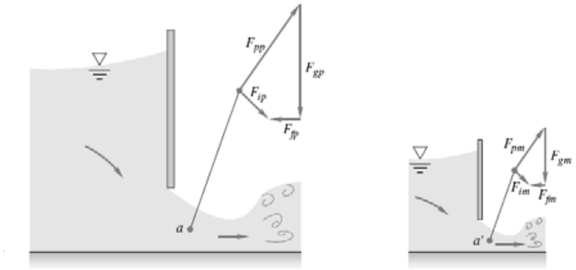
$$L_m = \alpha L_p \qquad \mathrm{Fr}_m = \frac{V_m^2}{gL_m} = \frac{V_p^2}{gL_p} = \mathrm{Fr}_p$$

$$\frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{1/2} = \sqrt{\alpha}$$

$$\frac{T_m}{T_p} = \frac{L_m/V_m}{L_p/V_p} = \sqrt{\alpha}$$

Similarity conditions

- ***Dynamic similarity*** exists when the model and the prototype have the same lengthscale ratio, time-scale ratio, and force-scale (or mass-scale) ratio.
- The dynamic-similarity laws ensure that each of these forces will be in the same ratio and have equivalent directions between model and prototype. The force polygons at homologous points have exactly the same shape.
- Kinematic similarity is also ensured by these model laws.



Example

- Consider predicting the drag on a thin rectangular plate (w*h) placed normal to the flow.
- Drag is a function of: *w, h, μ, ρ, V*

$$F_D = f(w, h, \mu, \rho, V)$$

- Dimensional analysis shows:
- And this applies BOTH to a model and a prototype

$$\pi_1 = f(\pi_2, \pi_3)$$

$$\frac{F_D}{w^2 \rho V^2} = f(\frac{w}{h}, \frac{\rho V w}{\mu})$$

- We can design a model to predict the drag on a prototype.
- Model will have:

$$\pi_{1m} = f(\pi_{2m}, \pi_{3m})$$

$$\frac{F_{Dm}}{w_m^2 \rho_m V_m^2} = f(\frac{w_m}{h_m}, \frac{\rho_m V_m w_m}{\mu_m})$$

- And the prototype will have:

$$\pi_{1p} = f(\pi_{2p}, \pi_{3p})$$

$$\frac{F_{Dp}}{w_p^2 \rho_p V_p^2} = f(\frac{w_p}{h_p}, \frac{\rho_p V_p w_p}{\mu_p})$$

Similarity conditions

Example

- **Similarity conditions**

Geometric similarity

$$\pi_{2m} = \pi_{2p} \quad \frac{w_m}{h_m} = \frac{w_p}{h_p} \quad \Rightarrow \quad w_m = \frac{h_m}{h_p} w_p$$

Gives us the size of the model

Dynamic similarity

$$\pi_{3m} = \pi_{3p} \quad \frac{\rho_m V_m w_m}{\mu_m} = \frac{\rho_p V_p w_p}{\mu_p} \quad \Rightarrow \quad V_m = \frac{\mu_m}{\mu_p} \frac{\rho_p}{\rho_m} \frac{w_p}{w_m} V_p$$

Then

Gives us the velocity in the model

$$\pi_{1m} = \pi_{1p} \quad \frac{F_{Dm}}{w_m^2 \rho_m V_m^2} = \frac{F_{Dp}}{w_p^2 \rho_p V_p^2} \quad \Rightarrow \quad F_{Dp} = \left(\frac{w_p}{w_m}\right)^2 \frac{\rho_p}{\rho_m} \left(\frac{V_p}{V_m}\right)^2 F_{Dm}$$

Example

- Given: Submarine moving below surface in sea water (ρ=1015 kg/m^3, $\nu=\mu/\rho$=1.4x10-6 m^2/s). Model is 1/20-th scale in fresh water (20^{o}C).
- Find: Speed of water in the testdynamic similarity and the ratio of drag force on model to that on prototype.
- Solution: Reynolds number is significant parameter.

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$$

$$V_m = \frac{L_p}{L_m} \frac{\nu_m}{\nu_p} V_p$$

$$= \frac{20}{1} \frac{1}{1.4} 2\, m/s$$

$$V_m = 28.6\, m/s$$

$$\frac{F_m}{\rho_m V_m^2 l_m^2} = \frac{F_p}{\rho_p V_p^2 l_p^2}$$

$$\frac{F_m}{F_p} = \frac{\rho_m V_m^2 l_m^2}{\rho_p V_p^2 l_p^2}$$

$$= \frac{1000}{1015} \left(\frac{28.6}{2}\right)^2 \left(\frac{1}{20}\right)^2$$

$$\frac{F_m}{F_p} = 0.504$$

Scaling in Open Hydraulic Structures

- Examples
  - spillways
  - channel transitions
  - weirs
- Important Forces
  - inertial forces
  - gravity: from changes in water surface elevation
  - viscous forces (often small relative to gravity forces)
- Minimum similitude requirements
  - geometric
  - Froude number

$$F = \frac{V}{\sqrt{gl}}$$

$$\cancel{R = \frac{\rho V l}{\mu}}$$

$$F = \frac{V}{\sqrt{gl}}$$

Froude similarity

$$F_m = F_p$$

$$\frac{V_m^2}{g_m L_m} = \frac{V_p^2}{g_p L_p}$$

$$\frac{V_m^2}{L_m} = \frac{V_p^2}{L_p}$$

- Froude number the same in model and prototype
- difficult to change g
- define length ratio (usually larger than 1) $L_r = \frac{L_p}{L_m}$
- velocity ratio $V_r = \sqrt{L_r}$
- time ratio $t_r = \frac{L_r}{V_r} = \sqrt{L_r}$
- discharge ratio $Q_r = V_r A_r = \sqrt{L_r}\, L_r L_r = L_r^{5/2}$
- force ratio $F_r = M_r a_r = \rho_r L_r^3 \frac{L_r}{t_r^2} = L_r^3$

Example: Spillway Model

- A 50 cm tall scale model of a proposed 50 m spillway is used to predict prototype flow conditions. If the design flood discharge over the spillway is 20,000 m^3/s, what water flow rate should be tested in the model?

$F_m = F_p$ $\quad L_r = 100$

$$Q_r = L_r^{5/2} = 100{,}000$$

$$Q_m = \frac{20{,}000\, m^3/s}{100{,}000} = 0.2\, m^3/s$$

Ship's Resistance

- Skin friction Viscosity, roughness
- Wave drag (free surface effect) gravity
- Therefore we need Reynolds and Froude similarity

$$\frac{2\mathrm{Drag}}{\rho V^2 A} = C_d = f\left(\frac{\varepsilon}{l}, \mathbf{R}, \mathbf{F}\right)$$

Reynolds and Froude Similarity?

Reynolds

$$\mathrm{R} = \frac{\rho V l}{\mu}$$

$$\frac{\cancel{\rho_m} V_m l_m}{\cancel{\mu_m}} = \frac{\cancel{\rho_p} V_p l_p}{\cancel{\mu_p}}$$

<u>Water is the only practical fluid</u>

$$V_m l_m = V_p l_p$$

$$\frac{V_p}{V_m} = \frac{l_m}{l_p}$$

$$V_r = \frac{1}{L_r}$$

Froude

$$\mathrm{F} = \frac{V}{\sqrt{gl}}$$

$$\mathrm{V_r} = \sqrt{\mathrm{L_r}}$$

$$\frac{1}{L_r} = \sqrt{\mathrm{L_r}}$$

<u>$\mathrm{L_r} = 1$</u>

Ship's Resistance

- Can't have both Reynolds and Froude similarity
- Froude hypothesis: the two forms of drag are independent
- Measure total drag on Ship
- Use analytical methods to calculate the skin friction
- Remainder is wave drag

$$\frac{2\mathrm{D_{total}}}{\rho V^2 A} = \mathrm{C}_d = f\left(\frac{\varepsilon}{D}, \mathbf{R}, \mathbf{F}\right)$$

$$\mathrm{D_{total}} = \underline{\mathrm{D_f} + \mathrm{D_w}}$$

$$\mathrm{D_f} = \frac{\rho V^2 A}{2} \underbrace{f\left(\frac{\varepsilon}{D}, \mathbf{R}\right)}_{\text{analytical}}$$

$$\mathrm{D_w} = \frac{\rho V^2 A}{2} \underbrace{f(\mathbf{F})}_{\text{empirical}}$$

<u>**Exercise:**</u>

<u>**7.8, 7.9, 7.27, 7.34, 7.41, 7.49**</u>