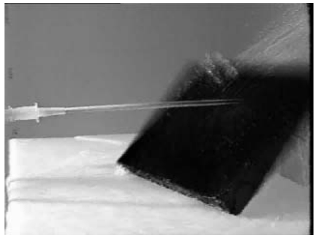
Finite Control Volume Analysis II



Energy Equation

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b\, d\forall + \int_{cs} \rho b \mathbf{V}\cdot\hat{\mathbf{n}}\, dA \quad \underline{\text{RTT}}$$

$$\frac{DE}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho e\, d\forall + \int_{cs} \rho e \mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$ What is $\boxed{DE/Dt}$ for a system?

First law of thermodynamics: The heat Q_H added to a system plus the work W done on the system equals the change in total energy E of the system.

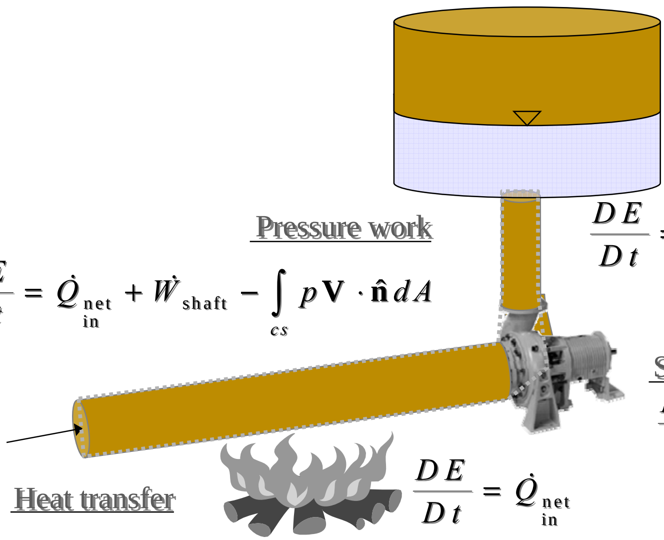
$$Q_{\substack{\text{net}\\ \text{in}}} + W_{\substack{\text{net}\\ \text{in}}} = E_2 - E_1$$

$$\dot{W}_{pr} = -\int_{cs} p\mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

$$W_{\substack{\text{net}\\ \text{in}}} = W_{pr} + W_{\text{shaft}}$$

$$\frac{DE}{Dt} = \dot{Q}_{\substack{\text{net}\\ \text{in}}} + \dot{W}_{\text{shaft}} - \int_{cs} p\mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

dE/dt for our System?



$$p = \gamma h$$

$$F = pA$$

$$\dot{W}_{pr} = -FV$$

$$\frac{DE}{Dt} = -\int_{cs} p\mathbf{V}\cdot d\mathbf{A}$$

Pressure work

$$\frac{DE}{Dt} = \dot{Q}_{\substack{\text{net}\\ \text{in}}} + \dot{W}_{\text{shaft}} - \int_{cs} p\mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

Shaft work

$$\frac{DE}{Dt} = \dot{W}_{\text{shaft}}$$

Heat transfer

$$\frac{DE}{Dt} = \dot{Q}_{\substack{\text{net}\\ \text{in}}}$$

General Energy Equation

$$\frac{DE}{Dt} = \overbrace{\dot{Q}_{\substack{\text{net}\\ \text{in}}} + \dot{W}_{\text{shaft}} - \int_{cs} p\mathbf{V}\cdot\hat{\mathbf{n}}\, dA}^{\text{1st Law of Thermo}} = \overbrace{\frac{\partial}{\partial t}\int_{cv} \rho e\, d\forall + \int_{cs} \rho e \mathbf{V}\cdot\hat{\mathbf{n}}\, dA}^{\text{RTT}}$$

$$= \dot{Q}_{\substack{\text{net}\\ \text{in}}} + \dot{W}_{\text{shaft}} = \frac{\partial}{\partial t}\int_{cv} e\rho\, d\forall + \int_{cs}\left(\frac{p}{\rho} + e\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

$$e = gz + \frac{V^2}{2} + \breve{u}$$

z

Total — Potential — Kinetic — Internal (molecular spacing and forces)

Simplify the Energy Equation

$$\cancel{\dot{Q}_{\substack{\text{net}\\\text{in}}}}^{\,q_{\substack{\text{net}\\\text{in}}}\dot{m}} + \cancel{\dot{W}_{\text{shaft}}}^{\,w_{\text{shaft}}\dot{m}} = \cancel{\frac{\partial}{\partial t}\int_{cv} e\rho\, d\forall}^{\,0} + \int_{cs}\left(\frac{p}{\rho}+e\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}\,dA$$

Steady

$$e = gz + \frac{V^2}{2} + \breve{u}$$

$$\left(q_{\substack{\text{net}\\\text{in}}} + w_{\text{shaft}}\right)\dot{m} = \int_{cs}\left(\frac{p}{\rho} + gz + \frac{V^2}{2} + \breve{u}\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}\,dA$$

$$\frac{p}{\rho} + gz = c$$

Assume...

- Hydrostatic pressure distribution at cs
- $\breve{u}$ is uniform over cs

But V is often not uniform over control surface!

Energy Equation: Kinetic Energy Term

$$\int_{cs}\left(\frac{V^2}{2}\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}\,dA = \alpha\,\frac{\rho\bar{V}^3 A}{2}$$

V = point velocity

$\bar{V}$ = average velocity over cs

If V normal to n

$$\alpha = \frac{\int_{cs}\left(\frac{V^3}{\cancel{2}}\right)\cancel{\rho}\,dA}{\frac{\cancel{\rho}\bar{V}^3 A}{\cancel{2}}}$$

$$\alpha = \frac{1}{A}\int_{cs}\left(\frac{V^3}{\bar{V}^3}\right)dA$$

α = kinetic energy correction term

α = 1 for uniform velocity

Energy Equation: steady, one-dimensional, constant density

$$\left(q_{\substack{\text{net}\\\text{in}}} + w_{\text{shaft}}\right)\dot{m} = \int_{cs}\left(\frac{p}{\rho} + gz + \frac{V^2}{2} + \breve{u}\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}\,dA$$

$$\int_{cs}\rho\mathbf{V}\cdot\hat{\mathbf{n}}\,dA = \dot{m}$$ mass flux rate

$$\left(q_{\substack{\text{net}\\\text{in}}} + w_{\text{shaft}}\right)\cancel{\dot{m}} = \left[\left(\frac{p_{out}}{\rho} + gz_{out} + \alpha\frac{V_{out}^2}{2} + \breve{u}_{out}\right) - \left(\frac{p_{in}}{\rho} + gz_{in} + \alpha\frac{V_{in}^2}{2} + \breve{u}_{in}\right)\right]\cancel{\dot{m}}$$

$$\frac{p_{in}}{\rho} + gz_{in} + \alpha_{in}\frac{V_{in}^2}{2} + \breve{u}_{in} + q_{\substack{\text{net}\\\text{in}}} + w_{\text{shaft}} = \frac{p_{out}}{\rho} + gz_{out} + \alpha_{out}\frac{V_{out}^2}{2} + \breve{u}_{out}$$

Energy Equation: steady, one-dimensional, constant density

$$\frac{p_{in}}{\rho} + gz_{in} + \alpha_{in}\frac{V_{in}^2}{2} + \breve{u}_{in} + q_{\substack{\text{net}\\\text{in}}} + w_{\text{shaft}} = \frac{p_{out}}{\rho} + gz_{out} + \alpha_{out}\frac{V_{out}^2}{2} + \breve{u}_{out}$$

divide by g

$$\underbrace{\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in}\frac{V_{in}^2}{2g} + \frac{w_{\text{shaft}}}{g} = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out}\frac{V_{out}^2}{2g}}_{\text{mechanical}} + \underbrace{\frac{\breve{u}_{out} - \breve{u}_{in} - q_{\substack{\text{net}\\\text{in}}}}{g}}_{\text{thermal}}$$

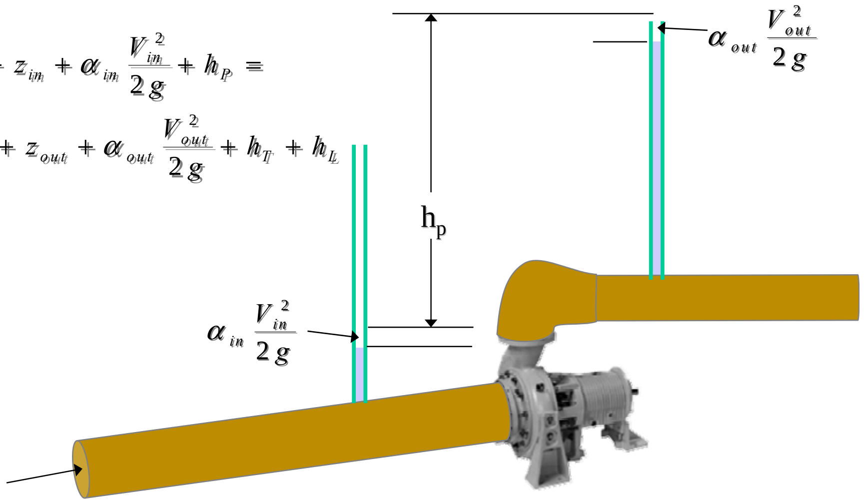
$$\frac{w_{\text{shaft}}}{g} = \boxed{h_P} - h_T \qquad \frac{\breve{u}_{out} - \breve{u}_{in} - q_{\substack{\text{net}\\\text{in}}}}{g} = h_L$$

Lost mechanical energy

$$\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in}\frac{V_{in}^2}{2g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out}\frac{V_{out}^2}{2g} + h_T + h_L$$

Pump Head

$$\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in}\frac{V_{in}^2}{2g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out}\frac{V_{out}^2}{2g} + h_T + h_L$$



Thermal Components of the Energy Equation

$$e = gz + \frac{V^2}{2} + \breve{u}$$

$$\breve{u} = c_v T \cong c_p T$$

For incompressible liquids

Water specific heat = 4184 J/(kg*K)

$$\frac{\breve{u}_{out} - \breve{u}_{in} - q_{\substack{net\\in}}}{g} = h_L$$

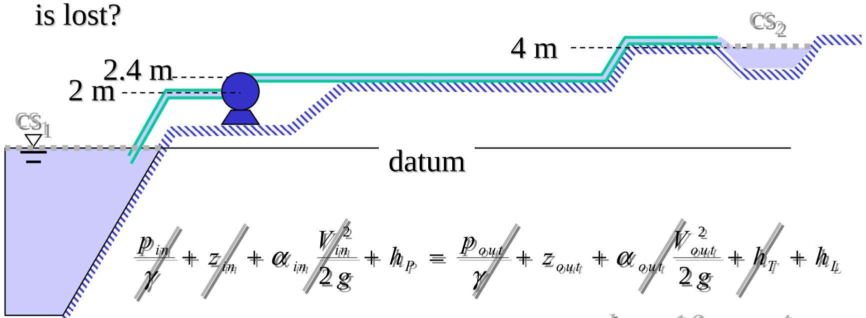
Change in temperature

$$\frac{c_p\left(\overbrace{T_{out} - T_{in}}\right) - q_{\substack{net\\in}}}{g} = h_L$$

Heat transferred to fluid

Example: Energy Equation (energy loss)

An irrigation pump lifts 50 L/s of water from a reservoir and discharges it into a farmer's irrigation channel. The pump supplies a total head of 10 m. How much **mechanical** energy is lost?

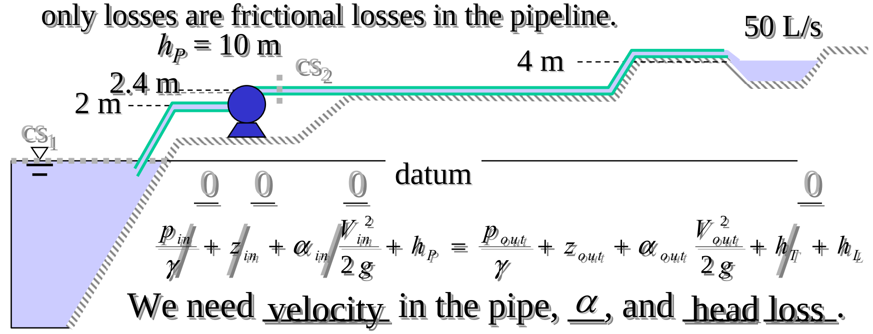


$$\cancel{\frac{p_{in}}{\gamma}} + \cancel{z_{in}} + \alpha_{in}\cancel{\frac{V_{in}^2}{2g}} + h_P = \cancel{\frac{p_{out}}{\gamma}} + z_{out} + \alpha_{out}\cancel{\frac{V_{out}^2}{2g}} + \cancel{h_T} + h_L$$

$h_p = z_{out} + h_L$ $h_L = h_p - z_{out}$ h_L = 10 m - 4 m

Example: Energy Equation (pressure at pump outlet)

The total pipe length is 50 m and is 20 cm in diameter. The pipe length to the pump is 12 m. What is the pressure in the pipe at the pump outlet? You may assume (for now) that the only losses are frictional losses in the pipeline.



$$\underset{0}{\cancel{\frac{p_{in}}{\gamma}}} + \underset{0}{\cancel{z_{in}}} + \alpha_{in}\underset{0}{\cancel{\frac{V_{in}^2}{2g}}} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out}\frac{V_{out}^2}{2g} + \underset{0}{\cancel{h_T}} + h_L$$

We need velocity in the pipe, α, and head loss.

Example: Energy Equation (pressure at pump outlet)

- How do we get the velocity in the pipe?

  $Q = VA$ $\quad A = \pi d^2/4$ $\quad V = 4Q/(\pi d^2)$

  $V = 4(0.05\ m^3/s)/[\pi \times 0.2\ m)^2] = 1.6\ m/s$
- How do we get the frictional losses?

  Expect losses to be proportional to length of the pipe

  $h_l = (6\ m)(12\ m)/(50\ m) = 1.44\ m$
- What about α?

Kinetic Energy Correction Term: α

$$\alpha = \frac{1}{A}\int_{cs}\left(\frac{V^3}{\bar{V}^3}\right)dA$$

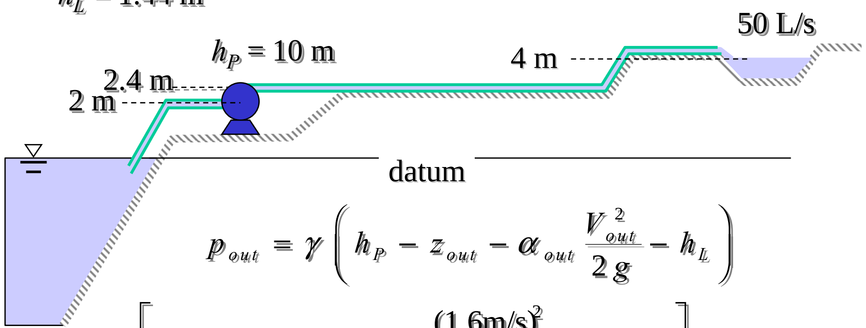
α is a function of the velocity distribution in the pipe.

- For a uniform velocity distribution α is 1
- For laminar flow α is 2
- For turbulent flow $1.01 < \alpha < 1.10$
  - Often neglected in calculations because it is so close to 1

Example: Energy Equation (pressure at pump outlet)

$V = 1.6$ m/s
$\alpha = 1.05$
$h_L = 1.44$ m

$$h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out}\frac{V_{out}^2}{2g} + h_L$$



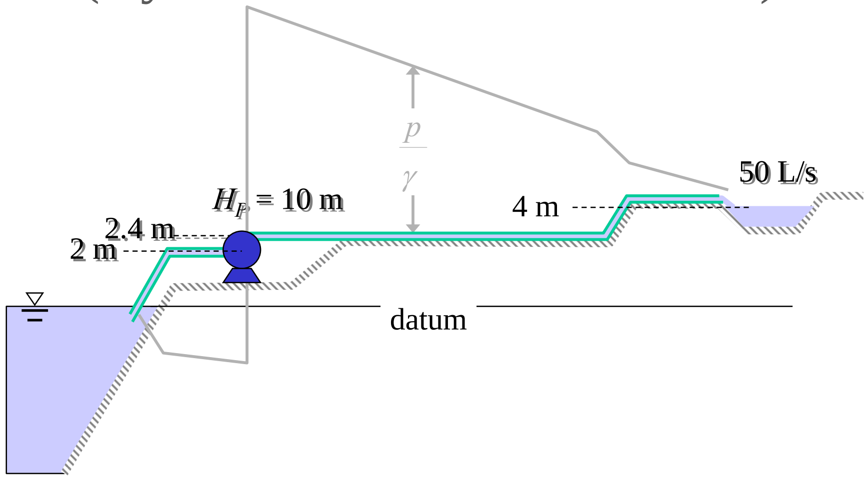
$$p_{out} = \gamma\left(h_P - z_{out} - \alpha_{out}\frac{V_{out}^2}{2g} - h_L\right)$$

$$p_2 = (9810\ N/m^3)\left[(10m) - (2.4m) - (1.05)\frac{(1.6m/s)^2}{2(9.81m/s^2)} - (1.44m)\right] = 59.1\ kPa$$

Example: Energy Equation (Hydraulic Grade Line - HGL)

- We would like to know if there are any places in the pipeline where the pressure is too high (pipe burst) or too low (water might boil - cavitation).
- Plot the pressure as piezometric head (height water would rise to in a manometer)
- How?

Example: Energy Equation (Hydraulic Grade Line - HGL)



Energy Grade Line

Hydraulic Grade Line

EGL (or TEL) and HGL

$$\text{EGL} = \frac{p}{\gamma} + z + \alpha\frac{V^2}{2g}$$

$$\text{HGL} = \underbrace{\frac{p}{\gamma} + z}_{\text{Piezometric head}}$$

Pressure head (w.r.t. reference pressure)

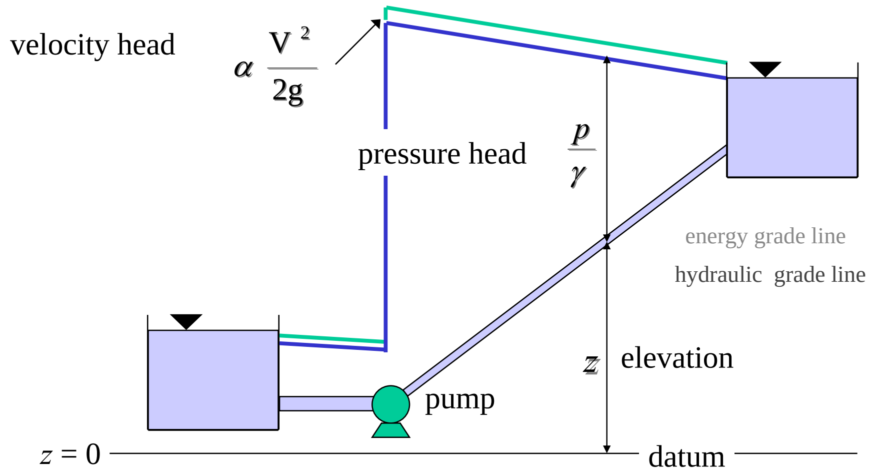
Elevation head (w.r.t. datum)

velocity head

EGL (or TEL) and HGL

- The energy grade line may never be horizontal or slope upward (in direction of flow) unless energy is added (pump)
- The decrease in total energy represents the head loss or energy dissipation per unit weight
- EGL and HGL are coincident and lie at the free surface for water at rest (reservoir)
- Whenever the HGL falls below the point in the system for which it is plotted, the local pressures are lower than the reference pressure

Example HGL and EGL



$$\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in}\frac{V_{in}^2}{2g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out}\frac{V_{out}^2}{2g} + h_T + h_L$$

Bernoulli vs. Control Volume Conservation of Energy

$$\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in}\frac{V_{in}^2}{2g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out}\frac{V_{out}^2}{2g} + h_T + h_L$$

Control surface to control surface

Has a term for frictional losses

Based on average velocity

Requires kinetic energy correction factor

Includes shaft work

$$\frac{p_1}{\gamma} + h_1 + \frac{v_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + \frac{v_2^2}{2g}$$

Point to point along streamline

No frictional losses

Power and Efficiencies

- Electrical power

$P_{electric} = \underline{IE}$

- Shaft power

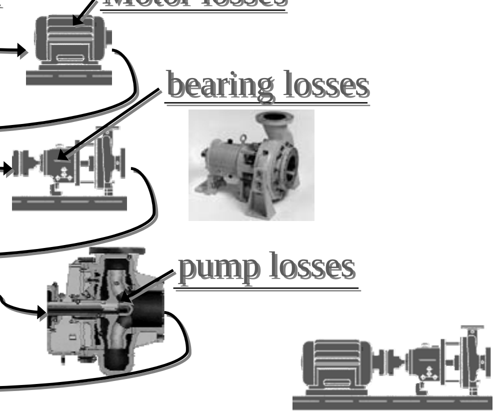
$P_{shaft} = \underline{T_{shaft}}$

- Impeller power

$P_{impeller} = \underline{T_{loss}}$

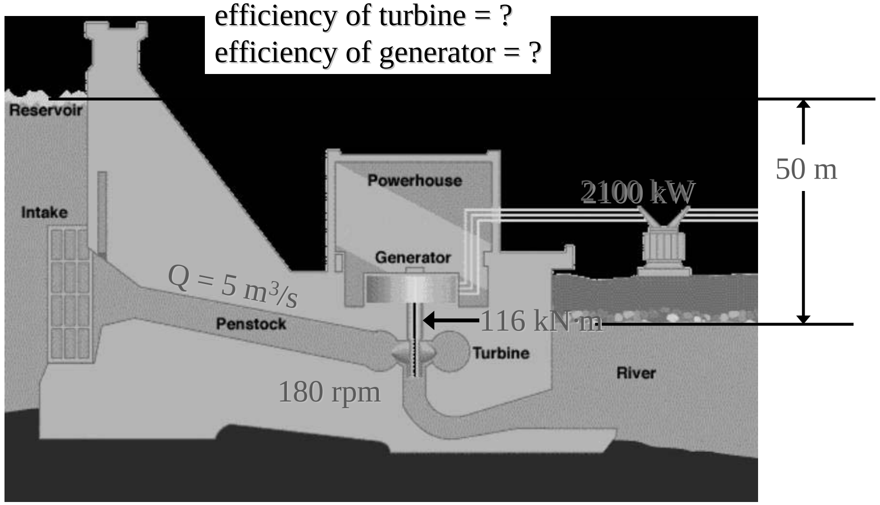
- Fluid power

$P_{water} = \underline{\gamma Q H_p}$



Example: Hydroplant

Water power = ?
total losses = ?
efficiency of turbine = ?
efficiency of generator = ?



Hydropower

$$P = \gamma Q H_p$$

$$P_{water} = \left(9806\,N/m^3\right)\left(5\,m^3/s\right)\left(50\,m\right) = 2.45\,MW$$

$$e_{total} = \frac{2.100\,MW}{2.45\,MW} = 0.857$$

$$P_{turbine} = \left(0.116\,MNm\right)\left(180\,\frac{rev}{min}\,\frac{2\pi\,rad}{rev}\,\frac{1\,min}{60\,s}\right) = 2.187\,MW$$

$$e_{turbine} = \frac{2.187\,MW}{2.45\,MW} = 0.893$$

$$e_{generator} = \frac{2.100\,MW}{2.187\,MW} = 0.96$$

$$\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in}\frac{V_{in}^2}{2g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out}\frac{V_{out}^2}{2g} + h_T + h_L$$

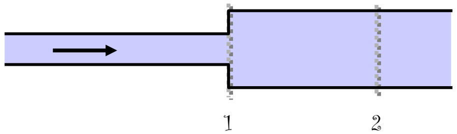
Energy Equation Review

- Control Volume equation
- Simplifications
  - steady
  - constant density
  - hydrostatic pressure distribution across control surface (cs normal to streamlines)
- Direction of flow matters (in vs. out)
- We don't know how to predict head loss

Conservation of Energy, Momentum, and Mass

- Most problems in fluids require the use of more than one conservation law to obtain a solution
- Often a simplifying assumption is required to obtain a solution
  - neglect energy losses (_to heat_) over a short distance with no flow expansion
  - neglect shear forces on the solid surface over a short distance

Head Loss due to Sudden Expansion: Conservation of Energy



$$\frac{p_{in}}{\gamma} + \cancel{z_{in}} + \cancel{\alpha_{in}}\frac{V_{in}^2}{2g} + \cancel{h_P} = \frac{p_{out}}{\gamma} + \cancel{z_{out}} + \cancel{\alpha_{out}}\frac{V_{out}^2}{2g} + \cancel{h_T} + h_L$$

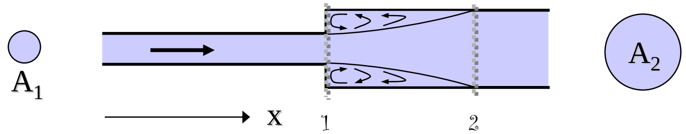
$$\frac{p_{in} - p_{out}}{\gamma} = \frac{V_{out}^2 - V_{in}^2}{2g} + h_L$$

$z_{in} = z_{out}$

$$h_L = \frac{p_{in} - p_{out}}{\gamma} + \frac{V_{in}^2 - V_{out}^2}{2g}$$

What is $p_{in} - p_{out}$?

Head Loss due to Sudden Expansion: Conservation of Momentum



$$\mathbf{M}_1 + \mathbf{M}_2 = \mathbf{W} + \mathbf{F}_{p_1} + \mathbf{F}_{p_2} + \mathbf{F}_{ss}$$

Apply in direction of flow

$$M_{1x} + M_{2x} = F_{p_{1x}} + F_{p_{2x}}$$

Neglect surface shear

$$M_{1x} = -\rho V_{in}^2 A_{in} \qquad M_{2x} = \rho V_{out}^2 A_{out}$$

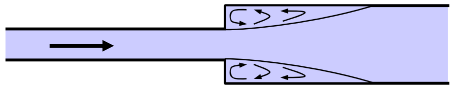
$$-\rho V_{in}^2 A_{in} + \rho V_{out}^2 A_{out} = p_{in} A_{out} - p_{out} A_{out}$$

Pressure is applied over all of section 1.
Momentum is transferred over area corresponding to upstream pipe diameter.
V_{in} is velocity upstream.

$$\frac{p_{in} - p_{out}}{\gamma} = \frac{V_{out}^2 - V_{in}^2 \frac{A_{in}}{A_{out}}}{g}$$

Divide by $(A_{out}\gamma)$

Head Loss due to Sudden Expansion



Energy $h_L = \frac{p_{in} - p_{out}}{\gamma} + \frac{V_{in}^2 - V_{out}^2}{2g}$

Mass $\frac{A_{in}}{A_{out}} = \frac{V_{out}}{V_{in}}$

Momentum $\frac{p_{in} - p_{out}}{\gamma} = \frac{V_{out}^2 - V_{in}^2 \frac{A_{in}}{A_{out}}}{g}$

$$h_L = \frac{V_{out}^2 - V_{in}^2 \frac{V_{out}}{V_{in}}}{g} + \frac{V_{in}^2 - V_{out}^2}{2g} \qquad h_L = \frac{V_{out}^2 - 2V_{in}V_{out} + V_{in}^2}{2g}$$

$$h_l = \frac{(V_{in} - V_{out})^2}{2g} \qquad h_l = \frac{V_{in}^2}{2g}\left(1 - \frac{A_{in}}{A_{out}}\right)^2 \qquad K_L = \left(1 - \frac{A_{in}}{A_{out}}\right)^2$$

Discharge into a reservoir? $K_L = 1$

$$\frac{p_{in}}{\gamma} + z_{in} + \alpha_{in}\frac{V_{in}^2}{2g} + h_P = \frac{p_{out}}{\gamma} + z_{out} + \alpha_{out}\frac{V_{out}^2}{2g} + h_T + h_L$$

Head Loss: Minor Losses

- Head (or energy) loss due to: outlets, inlets, bends, elbows, valves, pipe size changes
- Losses due to expansions are _greater_ than losses due to contractions
- Losses can be minimized by gradual transitions
- Losses are expressed in the form where K is the loss coefficient $h_L = K_L \frac{V^2}{2g}$

Summary

- Control volumes should be drawn so that the surfaces are either tangent (no flow) or normal (flow) to streamlines.
- In order to solve a problem the flow surfaces need to be at locations where all but 1 or 2 of the energy terms are known
- When possible choose a frame of reference so the flows are steady

Summary

- Control volume equation: Required to make the switch from Lagrangian to Eulerian
- Any conservative property can be evaluated using the control volume equation
  - mass, energy, momentum, concentrations of species
- Many problems require the use of several conservation laws to obtain a solution

Exercise:

5.91, 5.104, 5.106, 5.108, 5.115

Temperature Rise over Taughanock Falls

- Drop of 50 meters
- Find the temperature rise

$$\frac{c_p\left(T_{out} - T_{in}\right) - q_{\substack{net\\in}}}{g} = h_L$$

$$\Delta T = \frac{g h_L + q_{\substack{net\\in}}}{c_p}$$

$$\Delta T = \frac{\left(9.8\ \text{m/s}^2\right)\left(50\ \text{m}\right)}{\left(4184\ \dfrac{\text{J}}{\text{Kg}\cdot\text{K}}\right)}$$

$$\Delta T = 0.117\ K$$