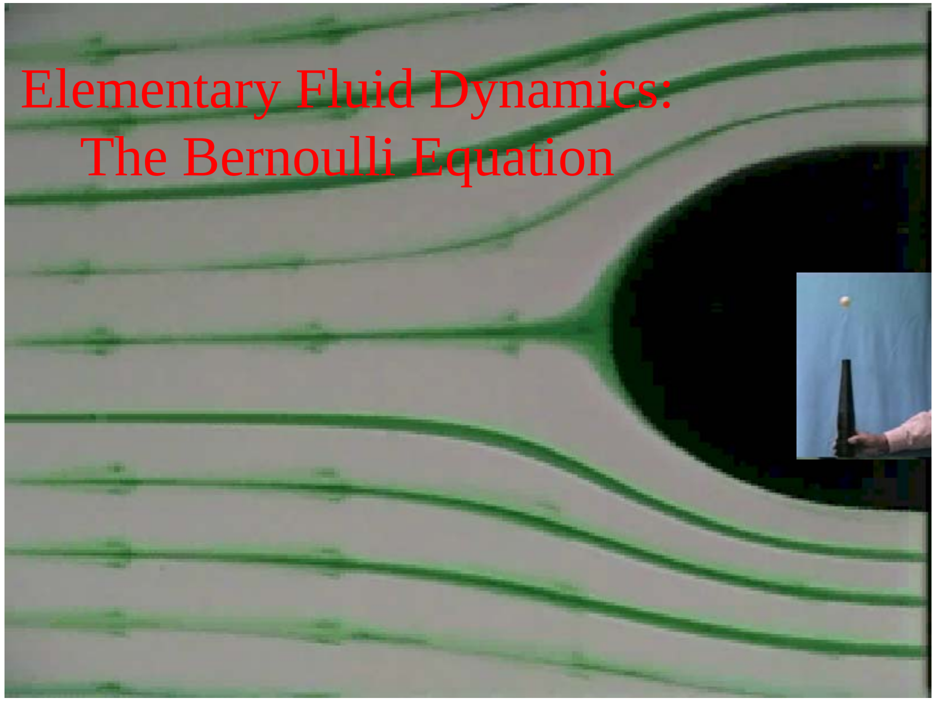
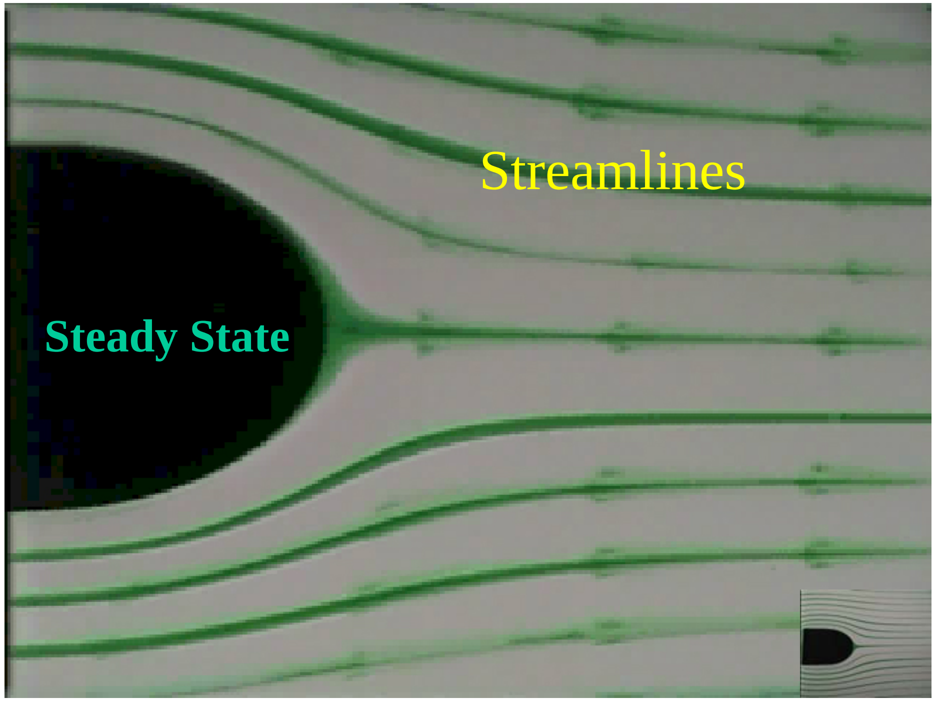




Bernoulli Eq. Along a Streamline

F*=*ma

$$-\nabla p = \rho \mathbf{a} + \gamma \hat{\mathbf{k}}$$ (eqn 2.2)

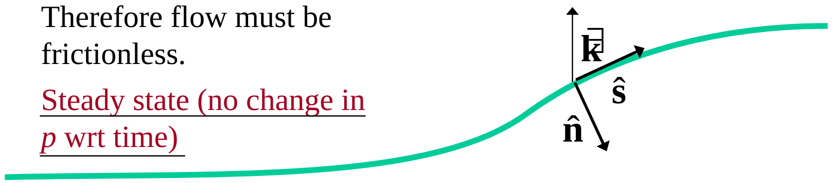
$$-\frac{\partial p}{\partial s} = \rho a_s + \gamma \frac{dz}{ds}$$

Separate acceleration due to gravity. Coordinate system may be in any orientation!

Component of *g* in *s* direction

Note: No shear forces! Therefore flow must be frictionless.

Steady state (no change in *p* wrt time)



Bernoulli Eq. Along a Streamline

$$-\frac{\partial p}{\partial s} = \rho a_s + \gamma \frac{dz}{ds}$$

chain rule

$$a_s = \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} = \frac{\partial V}{\partial s}V$$

Write acceleration as derivative wrt *s*

Can we eliminate the partial derivative?

$$dp = \frac{\partial p}{\partial s}ds + \frac{\partial p}{\partial n}dn$$

0 (*n* is constant along streamline)

$dp/ds = \partial p/\partial s$ and $dV/ds = \partial V/\partial s$

$$-\frac{dp}{ds} = \rho V \frac{dV}{ds} + \gamma \frac{dz}{ds}$$

$$V\frac{dV}{ds} = \frac{1}{2}\frac{d\left(V^2\right)}{ds}$$

Integrate *F=ma* Along a Streamline

$$-\frac{dp}{ds} = \frac{1}{2}\rho\frac{d\left(V^2\right)}{ds} + \gamma\frac{dz}{ds}$$

Eliminate *ds*

$$dp + \frac{1}{2}\rho d\left(V^2\right) + \gamma dz = 0$$

Now let's integrate…
But density is a function of pressure.

$$\int\frac{dp}{\rho} + \frac{1}{2}\int d\left(V^2\right) + g\int dz = 0$$

$$\int\frac{dp}{\rho} + \frac{1}{2}V^2 + gz = C$$

If density is constant…

$$p + \frac{1}{2}\rho V^2 + \gamma z = C$$

Along a streamline

Bernoulli Equation

- Assumptions needed for Bernoulli Equation
  - Inviscid (frictionless)
  - Steady
  - Constant density (incompressible)
  - Along a streamline
- Eliminate the constant in the Bernoulli equation?
  Apply at two points along a streamline.
- Bernoulli equation does not include
  - Mechanical energy to thermal energy
  - Heat transfer

Bernoulli Equation

The Bernoulli Equation is a statement of the conservation of Mechanical Energy

$$\frac{p}{\rho} + gz + \frac{1}{2}V^2 = C$$

gz: p.e. $\frac{1}{2}V^2$: k.e.

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = C$$

$\frac{p}{\gamma} =$ Pressure head

$z =$ Elevation head

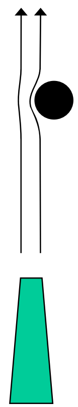
$\frac{V^2}{2g} =$ Velocity head

$\frac{p}{\gamma} + z =$ Piezometric head

$\frac{p}{\gamma} + z + \frac{V^2}{2g} =$ Total head

Ping Pong Ball

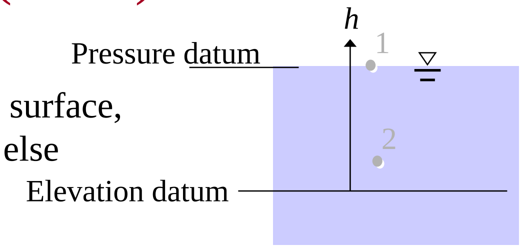
Why does the ping pong ball try to return to the center of the jet?
What forces are acting on the ball when it is not centered on the jet?



How does the ball choose the distance above the source of the jet?

Bernoulli Equation: Simple Case ($V = 0$)

- Reservoir ($V = 0$)
  - Put one point on the surface, one point anywhere else



$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = C$$

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2$$

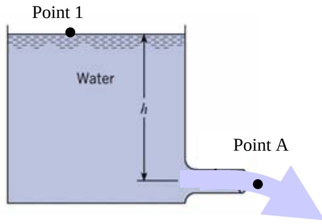
$$z_1 - z_2 = \frac{p_2}{\gamma}$$

We didn't cross any streamlines so this analysis is okay!

Same as we found using statics

Example

- **Find:** V_A
- **Solution:** Bernoulli equation



$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_A}{\gamma} + z_A + \frac{V_A^2}{2g}$$

$$\frac{0}{\gamma} + h + \frac{0}{2g} = \frac{0}{\gamma} + 0 + \frac{V_A^2}{2g}$$

$$V_A = \sqrt{2gh}$$

Bernoulli Equation: Simple Case ($p = 0$ or constant)

- What is an example of a fluid experiencing a change in elevation, but remaining at a constant pressure? Free jet

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$z_1 + \frac{V_1^2}{2g} = z_2 + \frac{V_2^2}{2g}$$

$$V_2 = \sqrt{2g(z_1 - z_2) + V_1^2}$$

Normal to the Streamlines

$$-\nabla p = \rho \mathbf{a} + \gamma \hat{\mathbf{k}}$$

$$-\frac{\partial p}{\partial n} = \rho a_n + \gamma \frac{dz}{dn}$$

Separate acceleration due to gravity. Coordinate system may be in any orientation!

Component of g in n direction

Normal to the Streamlines

$$-\frac{\partial p}{\partial n} = \rho a_n + \gamma \frac{dz}{dn}$$

$a_n = \frac{V^2}{\mathcal{R}}$ centrifugal force. $\mathcal{R}$ is local radius of curvature

n is toward the center of the radius of curvature

0 (*s* is constant normal to streamline)

$$dp = \frac{\partial p}{\partial s}ds + \frac{\partial p}{\partial n}dn \qquad \therefore dp/dn = \partial p/\partial n$$

$$dV/dn = \partial V/\partial n$$

$$-\frac{dp}{dn} = \rho \frac{V^2}{\mathcal{R}} + \gamma \frac{dz}{dn}$$

Integrate *F*=*ma* Normal to the Streamlines

$$-\frac{dp}{dn} = \rho \frac{V^2}{\mathcal{R}} + \gamma \frac{dz}{dn}$$ Multiply by *dn*

$$\int \frac{dp}{\rho} + \int \frac{V^2}{\mathcal{R}}dn + \int g dz = C$$ Integrate

$$\frac{p}{\rho} + \int \frac{V^2}{\mathcal{R}}dn + gz = C$$ If density is constant…

$$p + \rho \int \frac{V^2}{\mathcal{R}}dn + \gamma z = C$$ Normal to streamline

Pressure Change Across Streamlines

$$p + \rho \int \frac{V^2}{\mathcal{R}}dn + \gamma z = C$$

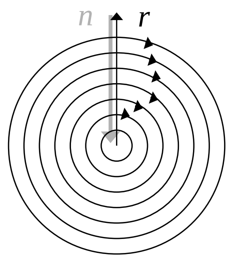
If you cross streamlines that are straight and parallel, then _$p + \gamma z = C$_ and the pressure is _hydrostatic_.

$$p - \rho C_1^2 \int r dr + \gamma z = C$$

$$p - \frac{\rho C_1^2}{2} r^2 + \gamma z = C$$

$V(r) = C_1 r$

$dn = -dr$

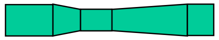


As *r* decreases *p* _decreases_

Summary

- By integrating ***F***=***ma*** along a streamline we found…
  - That energy can be converted between pressure, elevation, and velocity
  - That we can understand many simple flows by applying the Bernoulli equation
- However, the Bernoulli equation can not be applied to flows where viscosity is large or where mechanical energy is converted into thermal energy.

Bernoulli Equation Applications

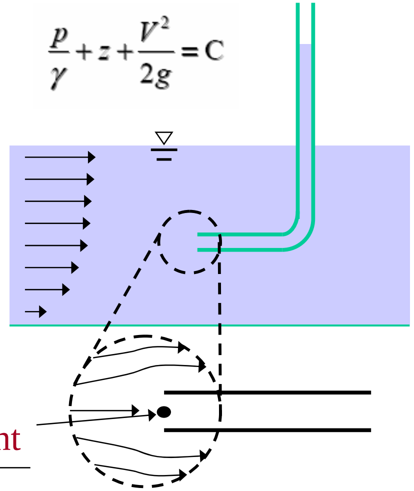
- Stagnation tube
- Pitot tube
- Free Jets
- Orifice
- Venturi 
- Sluice gate
- Sharp-crested weir

Applicable to contracting streamlines (accelerating flow).

Bernoulli Equation Application: Stagnation Tube

- What happens when the water starts flowing in the channel?
- Does the orientation of the tube matter? Yes!
- How high does the water rise in the stagnation tube?

$$\frac{p}{\gamma}+z+\frac{V^2}{2g}=C$$

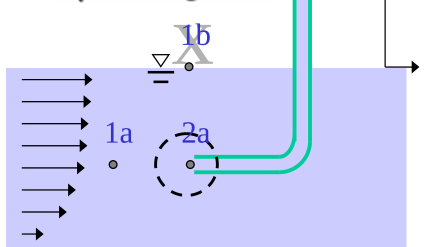


Stagnation point

Bernoulli Equation Application: Stagnation Tube

- 1a-2a
  - Same streamline
- 1b-2a
  - Crosses streamlines
- 1a-2b
  - Doesn't cross streamlines

$$\frac{p}{\gamma}+z+\frac{V^2}{2g}=C$$



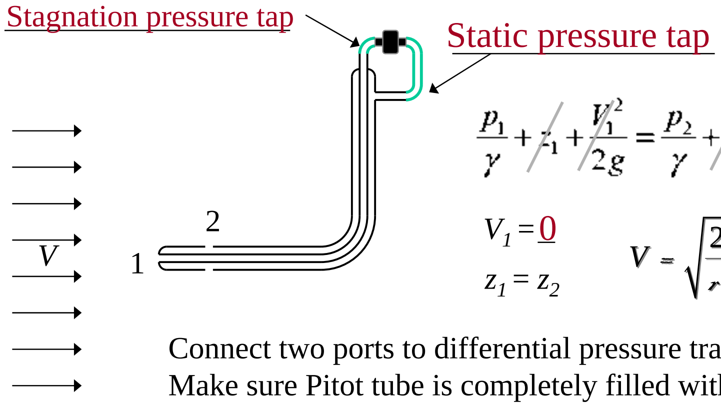
$$\frac{p_1}{\gamma}+z_1+\frac{V_1^2}{2g}=\frac{p_2}{\gamma}+z_2+\frac{V_2^2}{2g}$$

1. We can obtain V_1 if p_1 and (z_2-z_1) are known
2. z_2 is the total energy!

Pitot Tubes

- Used to measure air speed on airplanes
- Can connect a differential pressure transducer to directly measure $V^2/2g$
- Can be used to measure the flow of water in pipelines Point measurement!

Pitot Tube



$$\frac{p_1}{\gamma} + \cancel{z_1} + \cancel{\frac{V_1^2}{2g}} = \frac{p_2}{\gamma} + \cancel{z_2} + \frac{V_2^2}{2g}$$

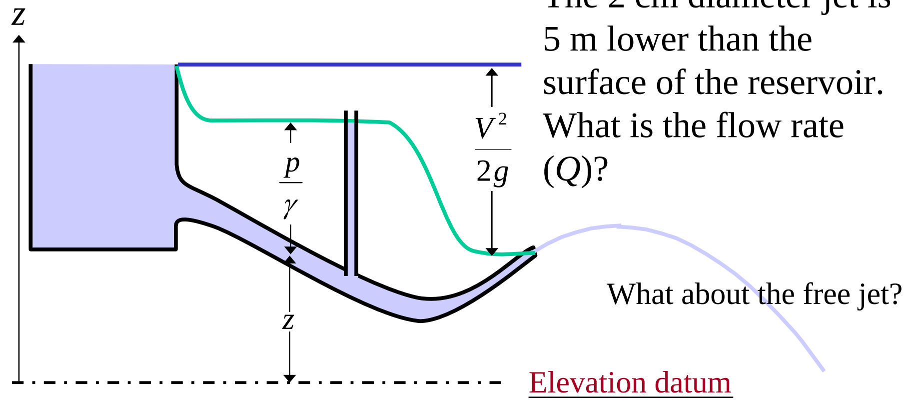
$V_1 = \underline{0}$

$z_1 = z_2$

$$V = \sqrt{\frac{2}{\rho}(p_1 - p_2)}$$

Connect two ports to differential pressure transducer.
Make sure Pitot tube is completely filled with the fluid that is being measured.
Solve for velocity as function of pressure difference

Hydraulic and Energy Grade Lines (neglecting losses for now)

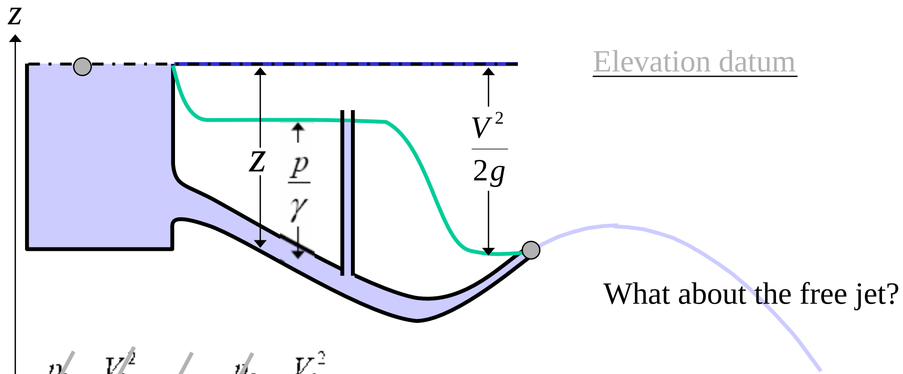


The 2 cm diameter jet is 5 m lower than the surface of the reservoir. What is the flow rate (Q)?

Pressure datum? Atmospheric pressure

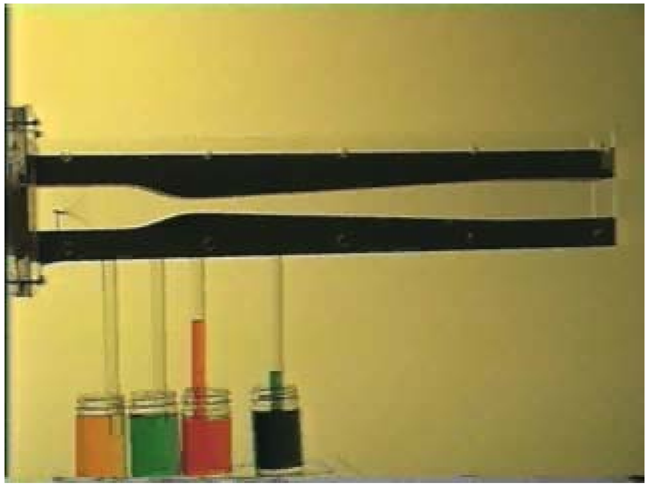
Jet Solution

The 2 cm diameter jet is 5 m lower than the surface of the reservoir. What is the flow rate (Q)?



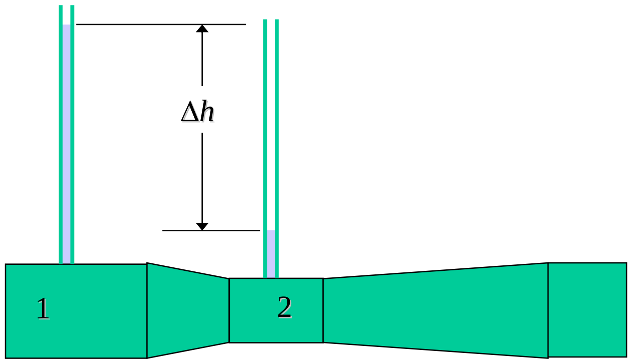
$$\cancel{\frac{p_1}{\gamma}} + \cancel{\frac{V_1^2}{2g}} + \cancel{z_1} = \cancel{\frac{p_2}{\gamma}} + \frac{V_2^2}{2g} + z_2 \qquad z_2 = -5 \text{ m}$$

Example: Venturi



Example: Venturi

Find the flow (Q) given the pressure drop between point 1 and 2 and the diameters of the two sections. You may assume the head loss is negligible.



Example: Venturi

$$\frac{p_1}{\gamma_1} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma_2} + z_2 + \frac{V_2^2}{2g}$$

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} = \frac{V_2^2}{2g}\left[1 - \left(\frac{d_2}{d_1}\right)^4\right]$$

$$V_2 = \sqrt{\frac{2g(p_1 - p_2)}{\gamma\left[1 - (d_2/d_1)^4\right]}}$$

$$Q = C_v A_2 \sqrt{\frac{2g(p_1 - p_2)}{\gamma\left[1 - (d_2/d_1)^4\right]}}$$

$$Q = VA$$

$$V_1 A_1 = V_2 A_2$$

$$V_1 \frac{\pi d_1^2}{4} = V_2 \frac{\pi d_2^2}{4}$$

$$V_1 d_1^2 = V_2 d_2^2$$

$$V_1 = V_2 \frac{d_2^2}{d_1^2}$$

Relaxed Assumptions for Bernoulli Equation

- Frictionless

  Viscous energy loss must be small

- Steady

  Or gradually varying

- Constant density (incompressible)

  Small changes in density

- Along a streamline

  Don’t cross streamlines

Exercise:

3.14, 3.49, 3.50, 3.51, 3.68
3.73, 3.78, 3.98, 3.114