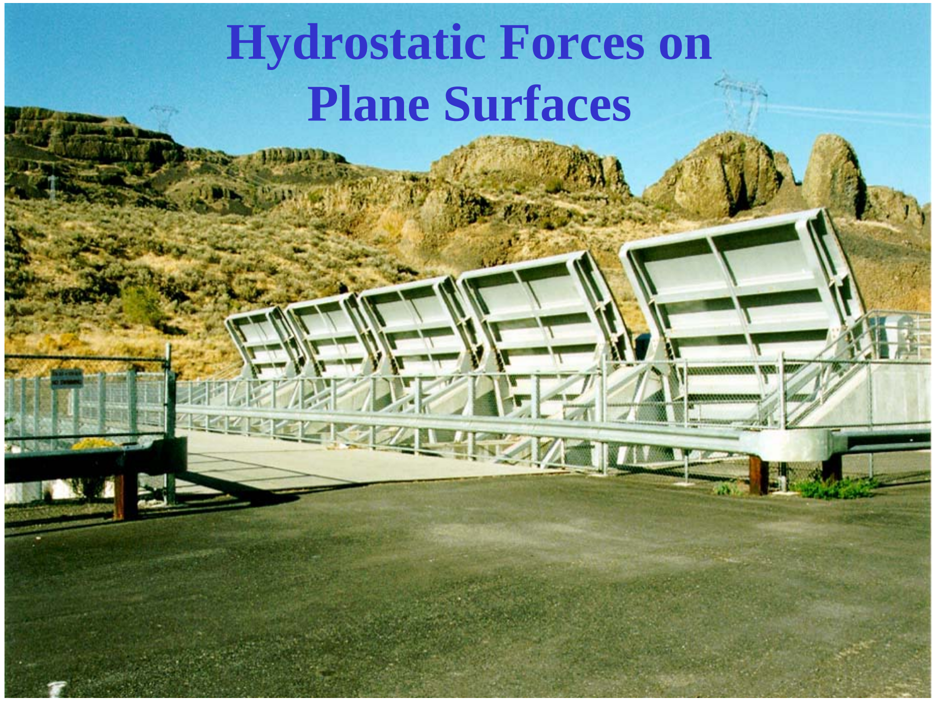
Hydrostatic Forces on Plane Surfaces



Static Surface Forces

- Forces on plane areas
- Forces on curved surfaces
- Buoyant force
- Stability of floating and submerged bodies

Forces on Plane Areas

- Two types of problems
  - Horizontal surfaces (pressure is **constant**)
  - Inclined surfaces $\frac{dp}{dz} = -\gamma$
- Two unknowns
  - **Total force**
  - **Line of action**
- Two techniques to find the line of action of the resultant force
  - Moments
  - Pressure prism

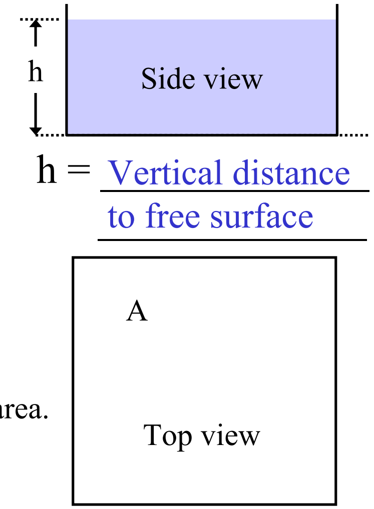
Forces on Plane Areas: Horizontal surfaces

What is the force on the bottom of this tank of water?

$$F_R = \int p dA = p \int dA = pA \qquad p = \gamma h$$

$$F_R = \gamma h A$$

F_R = weight of overlying fluid!



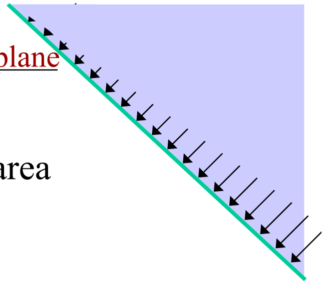
h = Vertical distance to free surface

F is normal to the surface and towards the surface if p is positive.

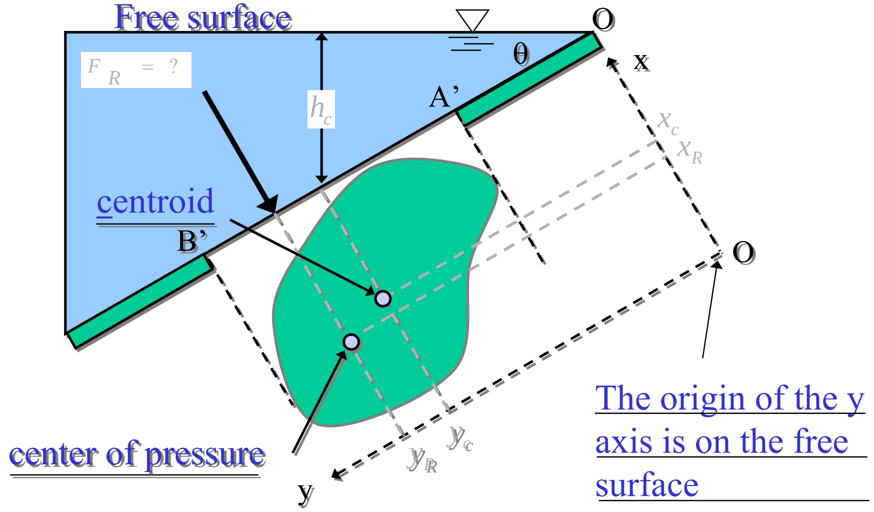
F passes through the centroid of the area.

Forces on Plane Areas: Inclined Surfaces

- Direction of force Normal to the plane
- Magnitude of force
  - integrate the pressure over the area
  - pressure is no longer constant!
- Line of action
  - Moment of the resultant force must equal the moment of the distributed pressure force



Forces on Plane Areas: Inclined Surfaces

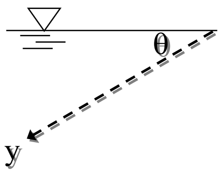


Magnitude of Force on Inclined Plane Area

$$F_R = \int p dA \qquad p = \gamma h = \gamma y \sin\theta$$

$$F_R = \gamma \sin\theta \int y dA \qquad y_c = \frac{1}{A}\int_A y dA$$

$$F_R = \gamma A y_c \sin\theta$$



$F_R = \gamma h_c A$ h_c is the vertical distance between free surface and centroid

$F_R = p_c A$ p_c is the pressure at the centroid of the area

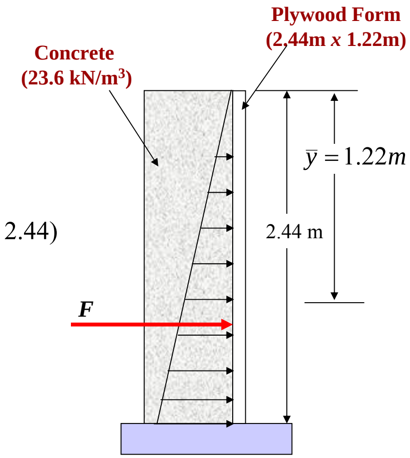
Example

$F = \bar{p}A$

$= (\gamma \bar{y} \sin\alpha) A$

$= (23{,}600 * 1.22 * 1) * (1.22 * 2.44)$

$F = 85.8 kN$



Forces on Plane Areas: Center of Pressure: x_R

- The center of pressure is not at the centroid (because pressure is increasing with depth)
  - x coordinate of center of pressure: x_R

$$x_R F_R = \int_A xp\,dA$$ Moment of resultant force = sum of moment of distributed forces

$$x_R = \frac{1}{F_R}\int_A xp\,dA \qquad F_R = y_c A\gamma\sin\theta \qquad p = \gamma y\sin\theta$$

$$x_R = \frac{1}{y_c A\gamma\sin\theta}\int_A xy\gamma\sin\theta\,dA$$

$$x_R = \frac{1}{y_c A}\int_A xy\,dA$$

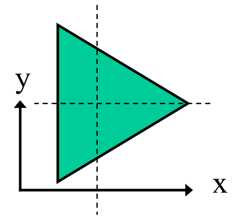
Center of Pressure: x_R

$$x_R = \frac{1}{y_c A}\int_A xy\,dA \qquad I_{xy} = \int_A xy\,dA$$ Product of inertia

$$x_R = \frac{I_{xy}}{y_c A} \qquad I_{xy} = x_c y_c A + I_{xyc}$$ Parallel axis theorem

$$x_R = \frac{x_c y_c A + I_{xyc}}{y_c A}$$

$$\boxed{x_R = \frac{I_{xyc}}{y_c A} + x_c}$$



Center of Pressure: y_R

$$y_R F_R = \int_A yp\,dA$$ Sum of the moments

$$y_R = \frac{1}{F_R}\int_A yp\,dA \qquad F_R = y_c A\gamma\sin\theta \qquad p = \gamma y\sin\theta$$

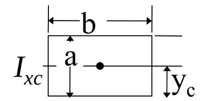
$$y_R = \frac{1}{y_c A\gamma\sin\theta}\int_A y^2\gamma\sin\theta\,dA$$

$$y_R = \frac{1}{y_c A}\int_A y^2\,dA \qquad I_x = \int_A y^2\,dA$$

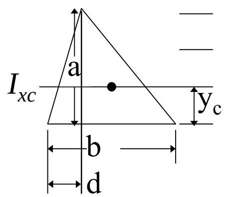
$$y_R = \frac{I_x}{y_c A} \qquad I_x = I_{xc} + y_c^2 A$$ Parallel axis theorem

$$y_R = \frac{I_{xc} + y_c^2 A}{y_c A} \qquad \boxed{y_R = \frac{I_{xc}}{y_c A} + y_c}$$

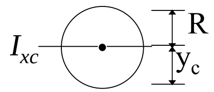
Properties of Areas



$A = ab \qquad y_c = \frac{a}{2} \qquad I_{xc} = \frac{ba^3}{12} \qquad I_{xyc} = 0$

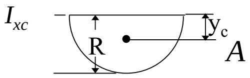


$A = \frac{ab}{2} \qquad y_c = \frac{a}{3} \qquad x_c = \frac{b+d}{3} \qquad I_{xc} = \frac{ba^3}{36} \qquad I_{xyc} = \frac{ba^2}{72}(b - 2d)$

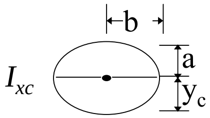


$A = \pi R^2 \qquad y_c = R \qquad I_{xc} = \frac{\pi R^4}{4} \qquad I_{xyc} = 0$

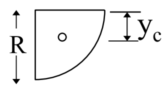
Properties of Areas



$A = \frac{\pi R^2}{2}$ $\quad y_c = \frac{4R}{3\pi}$ $\quad I_{xc} = \frac{\pi R^4}{8}$ $\quad I_{xyc} = 0$



$A = \pi ab$ $\quad y_c = a$ $\quad I_{xc} = \frac{\pi ba^3}{4}$ $\quad I_{xyc} = 0$



$A = \frac{\pi R^2}{4}$ $\quad y_c = \frac{4R}{3\pi}$ $\quad I_{xc} = \frac{\pi R^4}{16}$

Inclined Surface Findings

$$x_R = \frac{I_{xyc}}{y_c A} + x_c \qquad (I_{xyc} \to 0)$$

$$y_R = \frac{I_{xc}}{y_c A} + y_c \qquad (\frac{I_{xc}}{y_c A} > 0)$$

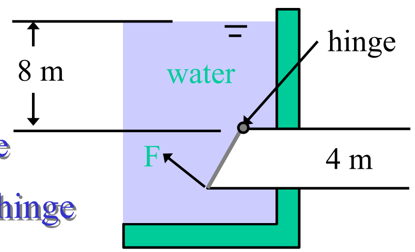
- The horizontal center of pressure and the horizontal centroid coincide when the surface has either a horizontal or vertical axis of symmetry
- The center of pressure is always below the centroid
- The vertical distance between the centroid and the center of pressure decreases as the surface is lowered deeper into the liquid (y_c increases)
- What do you do if there isn't a free surface?

Example

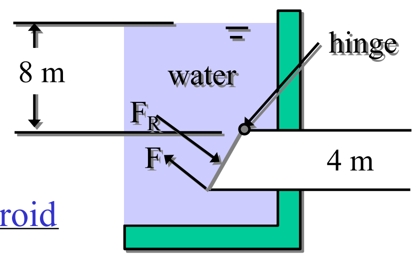
An elliptical gate covers the end of a pipe 4 m in diameter. If the gate is hinged at the top, what normal force F applied at the bottom of the gate is required to open the gate when water is 8 m deep above the top of the pipe and the pipe is open to the atmosphere on the other side? Neglect the weight of the gate.

Solution Scheme

- Magnitude of the force applied by the water
- Location of the resultant force
- Find F using moments about hinge



Magnitude of the Force



$F_R = p_c A$

$A = \pi ab$

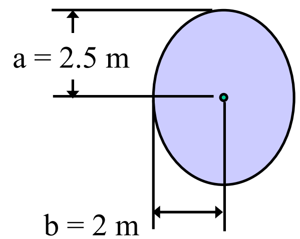
h_c = 10 m Depth to the centroid

$p_c = \gamma h_c$

$F_R = \gamma h_c \pi ab$

$$F_R = \left(9800 \frac{\text{N}}{\text{m}^3}\right)(10\text{ m})\pi(2.5\text{ m})(2\text{ m})$$

F_R = 1.54 MN



Location of Resultant Force

$$y_R = \frac{I_{xc}}{y_c A} + y_c$$

$y_c \neq h_c$

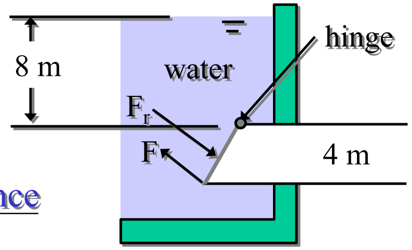
$y_c =$ 12.5 m

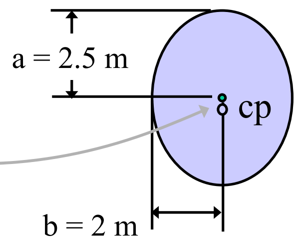
Slant distance to surface

$$y_R - y_c = \frac{\pi b a^3}{4 y_c \pi a b} \qquad I_{xc} = \frac{\pi b a^3}{4} \qquad A = \pi a b$$

$$y_R - y_c = \frac{a^2}{4 y_c} \qquad y_R - y_c = \frac{(2.5\text{ m})^2}{4(12.5\text{ m})}$$

$y_R - y_c =$ 0.125 m $\qquad x_R = x_c$





Force Required to Open Gate

How do we find the required force?

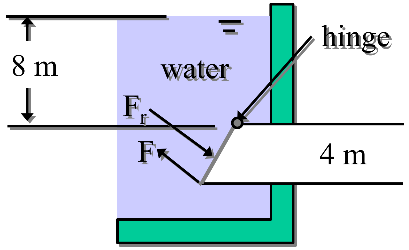
Moments about the hinge

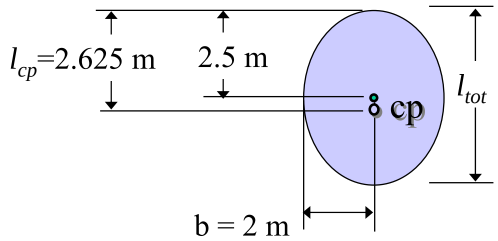
$$\sum M_{hinge} = 0 = F l_{tot} - F_R l_{cp}$$

$$F = \frac{F_R l_{cp}}{l_{tot}}$$

$$F = \frac{(1.54 \times 10^6 \text{ N})(2.625 \text{ m})}{(5 \text{ m})}$$

F = 809 kN





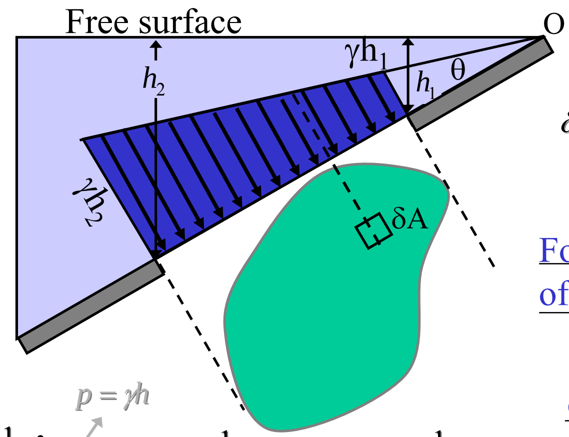
Forces on Plane Surfaces Review

- The average magnitude of the pressure force is the pressure at the centroid
- The horizontal location of the pressure force was at x_c (WHY?) The gate was symmetrical about at least one of the centroidal axes.
- The vertical location of the pressure force is below the centroid. (WHY?) Pressure increases with depth.

Forces on Plane Areas: Pressure Prism

- A simpler approach that works well for areas of constant width (rectangles)
- If the location of the resultant force is required and the area doesn't intersect the free surface, then the moment of inertia method is about as easy

Forces on Plane Areas: Pressure Prism



$$\delta F = \gamma h \delta A = \delta \forall$$

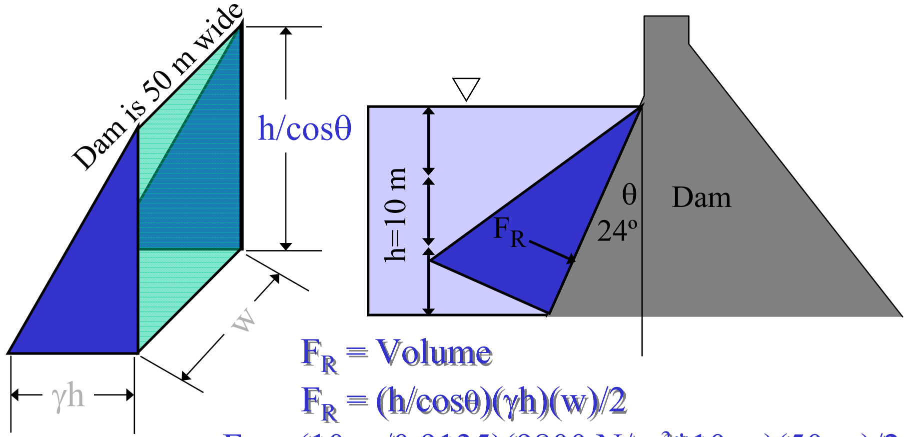
$$F = \forall$$

Force = Volume of pressure prism

$$x_R = \frac{1}{F_R}\int_A xpdA \qquad x_R = \frac{1}{\forall}\int_\forall xd\forall \qquad y_R = \frac{1}{\forall}\int_\forall yd\forall$$

Center of pressure is at centroid of pressure prism

Example : Pressure Prism



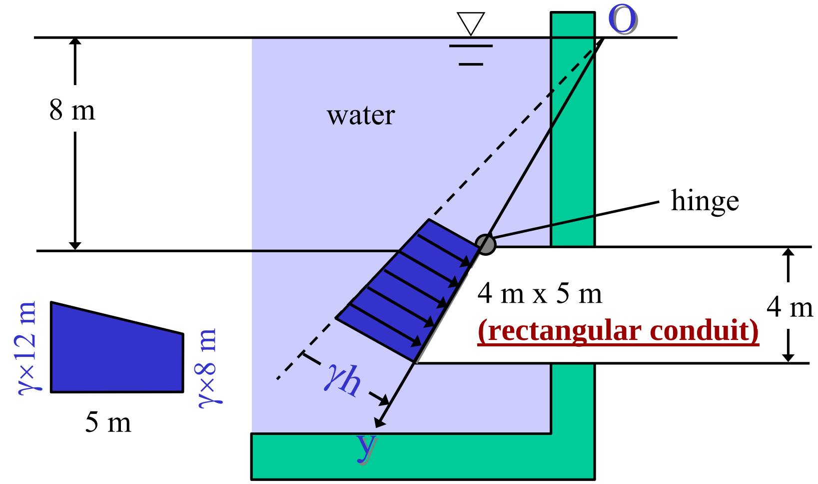
F_R = Volume

$F_R = (h/\cos\theta)(\gamma h)(w)/2$

$F_R = (10\text{ m}/0.9135)(9800\text{ N/m}^3 \cdot 10\text{ m})(50\text{ m})/2$

$F_R = 26$ MN

Example : Pressure Prism

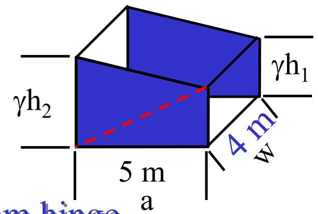


Solution : Pressure Prism

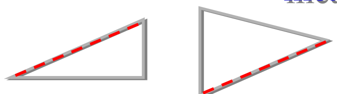
Magnitude of force

$$F_R = \forall = (9800\text{ N/m}^3)(10\text{ m})(5\text{ m})(4\text{ m}) \qquad \frac{h_1 + h_2}{2}$$

F_R = 1.96 MN



Location of resultant force measured from hinge



$$y_R F_R = \frac{a}{3}\left(\frac{aw\gamma h_1}{2}\right) + \frac{2a}{3}\left(\frac{aw\gamma h_2}{2}\right)$$

$$y_R = \frac{a^2 w\gamma}{[aw(10\text{ m})\gamma]}\left[\frac{h_1}{6} + \frac{h_2}{3}\right]$$

F_R

$$y_R = \frac{(5\text{ m})}{(10\text{ m})}\left[\frac{(8\text{ m})}{6} + \frac{(12\text{ m})}{3}\right]$$

y_R = 2.667 m

Exercise:

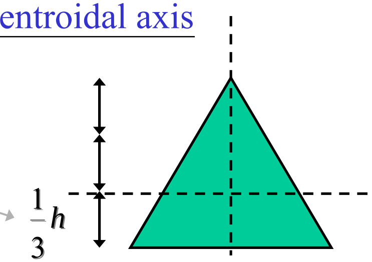
2.61, 2.67, 2.71, 2.77

First Moments

$\int_A x\,dA$ — Moment of an area A about the y axis

$x_c = \frac{1}{A}\int_A x\,dA$ — Location of centroidal axis

$y_c = \frac{1}{A}\int_A y\,dA$



For a plate of uniform thickness the intersection of the centroidal axes is also the center of gravity

Second Moments

Also called moment of inertia of the area

$$I_x = \int_A y^2 dA$$

$$I_x = I_{xc} + Ay_c^2$$

I_{xc} is the 2nd moment with respect to an axis passing through its centroid and parallel to the x axis.

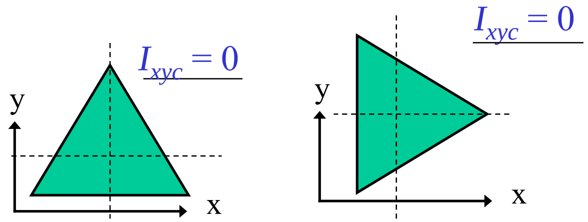
Parallel axis theorem

Product of Inertia

- A measure of the asymmetry of the area

$I_{xy} = \int_A xy\,dA$ — Product of inertia

$$I_{xy} = x_c y_c A + I_{xyc}$$



If $x = x_c$ or $y = y_c$ is an axis of symmetry then the product of inertia I_{xyc} is zero.