$\hat{A} = -\frac{d^2}{dx^2}$: $\hat{A}\psi_{(x)} = a\psi_{(x)}$; $a = k^2, -k^2, 0$ $(k, k > 0)$ (a 1

$(a=k^2)$ $\Rightarrow \frac{d^2\psi}{dx^2} + k^2\psi_{(x)} = 0 \Rightarrow \psi_{(x)} = B\cos kx + C\sin kx$ (I)

$(a=-k^2)$ $\frac{d^2\psi}{dx^2} - k^2\psi(x) = 0 \Rightarrow \psi_{(x)} = Be^{-kx} + Ce^{kx}$ (II)

$(a=0)$ $\frac{d^2\psi}{dx^2} = 0 \Rightarrow \psi_{(x)} = Bx + C$ (III)

حالت (I) : $\begin{cases} x=0 \Rightarrow \psi_{(0)} = B = 0 \\ x = a \Rightarrow \psi_{(a)} = C\sin ka = 0 \rightarrow ka = n\pi \rightarrow \boxed{k = \frac{n\pi}{a}} \end{cases}$

بهنجارش : $\int_0^a \psi\psi^* dx = 1 \Rightarrow |C|^2 \int_0^a \sin^2(\frac{n\pi x}{a}) = |C|^2 \int_0^a (\frac{1-\cos\frac{2n\pi x}{a}}{2}) dx$

$\Rightarrow \boxed{C = \sqrt{2/a}} \Rightarrow \psi_{(x)} = \sqrt{2/a}\, \sin(\frac{n\pi x}{a})$

احتمال یافتن در $0 < x < a/2$ $= \int_0^{a/2} \frac{2}{a}\sin^2(\frac{n\pi x}{a}) dx = \frac{2}{a}\left[\frac{a}{4} - \frac{a}{4n\pi}\sin(n\pi)\right] = \frac{1}{2}$

حالت (II) : $x=0 \rightarrow \psi_{(0)} = B + C = 0 \rightarrow B = -C$ ✗ جواب معقول ندارد

$x = a \rightarrow \psi_{(a)} = Be^{-ka}(1 - e^{2ka}) = 0 \Rightarrow a = 0$ یا $B = 0$

حالت (III) : $x = 0 \Rightarrow C = 0$ بدون جواب معقول

$x = a \Rightarrow Ba = 0 \rightarrow B = 0$

(b 1

$$\hat{A} = \hat{x}\frac{d}{dx} + 2 \Rightarrow x\frac{d\psi}{dx} + 2\psi = 0 \Rightarrow \frac{d\psi}{\psi} = -2\frac{dx}{x}$$

$$\Rightarrow \boxed{\psi = Cx^{-2}}$$

$$[\hat{A},\hat{x}]\phi_{(x)} = \left[\hat{x}\frac{d}{dx} + 2, \hat{x}\right]\phi_{(x)} = x\,\frac{d}{dx}(x\phi_{(x)}) + 2x\phi_{(x)} - x\left(x\frac{d\phi}{dx} + 2\phi\right)$$

$$= x\phi_{(x)} \Rightarrow \boxed{[\hat{A},\hat{x}] = \hat{x}}$$

$$\left[\hat{A},\frac{d}{dx}\right]\phi_{(x)} = \left(x\,d/dx + 2\right)\frac{d\phi}{dx} - \frac{d}{dx}\left(x\frac{d\phi}{dx} + 2\phi\right) = -\frac{d\phi}{dx} \Rightarrow \boxed{\left[\hat{A},\frac{d}{dx}\right] = -\frac{d}{dx}}$$

$$\left[\hat{A},\frac{d^2}{dx^2}\right]\phi_{(x)} = \left(x\frac{d}{dx} + 2\right)\frac{d^2\phi}{dx^2} - \frac{d^2}{dx^2}\left(x\frac{d\phi}{dx} + 2\phi\right)$$

$$= x\frac{d^3\phi}{dx^3} + 2\frac{d^2\phi}{dx^2} - d/dx\left(\frac{d\phi}{dx} + x\frac{d^2\phi}{dx^2} + 2\frac{d\phi}{dx}\right)$$

$$= x\frac{d^3\phi}{dx^3} + 2\frac{d^2\phi}{dx^2} - 3\frac{d^2\phi}{dx^2} - \frac{d^2\phi}{dx^2} - x\frac{d^3\phi}{dx^3} = -2\frac{d^2\phi}{dx^2} \Rightarrow \boxed{\left[A,\frac{d^2}{dx^2}\right] = -2\frac{d^2}{dx^2}}$$

$$[\hat{x},[\hat{A},\hat{x}]] = [\hat{x},\hat{x}] = 0$$

$$[d/dx,[\hat{A},d/dx]] = \left[d/dx, -\frac{d}{dx}\right] = 0$$

c

$$\hat{A}\,\omega_{(x,y)} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)e^{-i(x+y)} = -ie^{-i(x+y)} - ie^{-i(x+y)} = -2i\omega_{(x,y)} \quad ✓$$

بله، ویژه تابع است.

$$\hat{B}\,\omega_{(x,y)} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)e^{-i(x+y)} = 2\omega(x,y) \quad ✓$$

بله

$$[\hat{A},\hat{B}]\,\omega(x,y) = \hat{A}(2\omega) - \hat{B}(-2i\omega) = -4i\omega + 4i\omega = 0$$

در واقع، بصورت کلی می توان نشان داد که $[\hat{A},\hat{B}]=0$ چرا که ترتیب اثر مشتقات پاره ای مهم نیست. بنابراین

برای هر سه $[\hat{A},B]\psi = 0$ ، $[A,B]\varphi = 0$ ، $[A,B]\omega = 0$

$$\hat{A}\varphi = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)e^{-2(x^2+y^2)} = -4xe^{-2(x^2+y^2)} - 4ye^{-2(x^2+y^2)} \quad \times$$

$$\hat{B}\varphi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)e^{-2(x^2+y^2)} = \frac{\partial}{\partial x}\left(-4xe^{-2(x^2+y^2)}\right) + \frac{\partial}{\partial y}\left(-4ye^{-2(x^2+y^2)}\right) \quad \times$$

$$= -4e^{-2(x^2+y^2)} - 4x(-4x)e^{-2(x^2+y^2)} - 4e^{-2(x^2+y^2)} - 4y(-4y)e^{-2(x^2+y^2)} = 8e^{-2(x^2+y^2)}\left(-1 + 2(x^2+y^2)\right)$$

$$\hat{A}\,\psi(x,y) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \sin 2x \cos 5x = 2\cos(2x)\cos(5x) - 5\sin(2x)\sin(5x) \quad \times$$

$$\hat{B}\,\psi(x,y) = \frac{\partial^2}{\partial x^2} \sin 2x \cos 5x = -4\sin 2x \cos 5x - 10\cos 2x \sin 5x - 10\cos 2x \sin 5x$$

$$-25 \sin 2x \cos 5x \quad \times$$

2

با توجه به حل حالات مقید در چاه پتانسیل در کتاب، داریم:

$$\frac{2mE}{\hbar^2} = -\alpha^2 \quad , \quad q^2 = \frac{2m}{\hbar^2}(V_0 - |E|)$$

$$u(x) = A\cos qx + B\sin qx \qquad -a<x<a \quad , \quad \begin{cases} u(x) = C_1 e^{\alpha x} & x<-a \\ u(x) = C_2 e^{-\alpha x} & x>-a \end{cases}$$

$$\alpha/q = \frac{\sqrt{\frac{2m|E|}{\hbar^2}}}{\sqrt{\frac{2m}{\hbar^2}(V_0-|E|)}} = \tan qa \Rightarrow \tan(qa) = \sqrt{\frac{|E|}{V_0-|E|}}$$

با جایگذاری:

$$\hbar = 6.58\times10^{-16}\,ev \quad , \quad m_e = \frac{0.511\,Mev}{c^2} \quad , \quad a_0 \approx 0.529\times10^{-10}\,m \quad , \quad E = -13.6\,ev$$

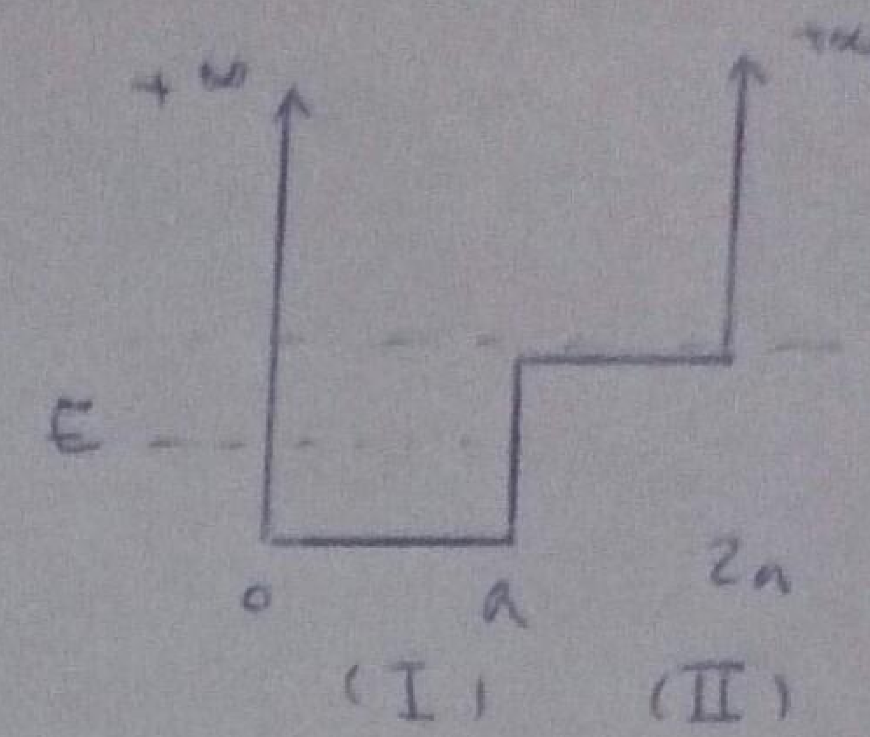
$$q^2 = \frac{2\times0.511\times10^{6}\,ev}{(6.58\times10^{-16}\,ev\,s)^2\times(3\times10^{8})^2}\left(V_0 - 13.6\,ev\right) = 2.62\times10^{19}\left(\frac{1}{ev}\right)(V_0 - 13.6)$$

با حل عددی رابطه فوق و جایگذاری مقادیر بدست آمده برای V_0 خواهیم داشت:

$$\boxed{V_0 \approx 23.67\,ev}$$

ناحیه (I) : $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} + \underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi = 0$

$\rightarrow \boxed{\psi = A\sin kx + B\cos kx}$



ناحیه (II) : $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$

$\Rightarrow \frac{d^2\psi}{dx^2} - \underbrace{\frac{(V_0 - E)2m}{\hbar^2}}_{K^2}\psi = 0 \Rightarrow \boxed{\psi = Ce^{-Kx} + De^{Kx}}$

$\psi_{(0)} = 0 \Rightarrow \boxed{B = 0}$; $\psi_{(2a)} = 0 \Rightarrow Ce^{-2Ka} + De^{2Ka} = 0$

$\Rightarrow C + De^{4Ka} = 0 \Rightarrow \boxed{C = -De^{4Ka}}$

$\psi^{I}_{(a)} = \psi^{II}_{(a)} \Rightarrow A\sin ka = -De^{4Ka-Ka} + De^{Ka}$

$\Rightarrow A\sin ka = D(e^{Ka} - e^{3Ka})$

$\psi'^{I}_{(a)} = \psi'^{II}_{(a)} \Rightarrow Ak\cos ka = -(De^{4Ka})(-Ke^{-Ka}) + DKe^{Ka}$

$\Rightarrow Ak\cos ka = DK(e^{3Ka} + e^{Ka})$

$\Rightarrow \frac{\tan ka}{k} = \frac{e^{Ka} - e^{3Ka}}{K(e^{Ka} + e^{3Ka})} = \frac{e^{-Ka} - e^{Ka}}{K(e^{-Ka} + e^{Ka})} = \frac{-1}{K}\tanh(Ka)$

$\Rightarrow \tan\eta = -\frac{k}{K}\tanh\xi \Rightarrow \boxed{\tan\eta = -\frac{\eta}{\xi}\tanh\xi}$

$Z_0^2 = \frac{2ma^2V_0}{\hbar^2} = \frac{2ma^2(V_0 - E)}{\hbar^2} + \frac{2mEa^2}{\hbar^2} = \eta^2 + \xi^2 \Rightarrow \xi = \pm\sqrt{Z_0^2 - \eta^2}$

$\Rightarrow \frac{\tan\eta}{\eta} = \mp\frac{\tanh(\pm\sqrt{Z_0^2 - \eta^2})}{\sqrt{Z_0^2 - \eta^2}} \Rightarrow \boxed{\frac{\tan\eta}{\eta} = \frac{\tanh(\sqrt{Z_0^2 - \eta^2})}{\sqrt{Z_0^2 - \eta^2}}}$

$K^2 = \frac{2m}{\hbar^2}(V_0 - E) \Rightarrow E = V_0 - \frac{\hbar^2K^2}{2m}$

با رسم نمودار فوق مشاهده می‌کنیم که در یک نقطه تقاطع رخ می‌دهد.

[a] این تابع موج جمع دو ویژه تابع انرژی است و بنابراین خودش ویژه تابع انرژی نیست.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(\sin\frac{2\pi x}{a}\right) = E_2 \sin\frac{2\pi x}{a} \Rightarrow E_2 = \frac{\hbar^2}{2ma^2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(\sin\frac{3\pi x}{a}\right) = E_3 \sin\frac{3\pi x}{a} \Rightarrow E_3 = \frac{9\hbar^2}{8ma^2}$$

$$\Rightarrow \Psi(x,t) = e^{-i\frac{\pi\hbar t}{ma^2}}\sqrt{\frac{2}{3a}}\sin\left(\frac{2\pi x}{a}\right) + e^{-i\frac{9\pi\hbar t}{4ma^2}}\sqrt{\frac{4}{3a}}\sin\left(\frac{3\pi x}{a}\right)$$

[b] & [c]

$$P\left(E=\frac{9\hbar^2}{8ma^2}\right) = \left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}$$

$$P\left(E=\frac{\hbar^2}{2ma^2}\right) = \left(\sqrt{\frac{1}{3}}\right)^2 = \frac{1}{3}$$

$$P\left(E=\frac{\hbar^2}{8ma^2}\right) = 0$$

[c]

$$\langle x\rangle = \int_0^a \left(e^{\frac{i\pi\hbar t}{ma^2}}\sqrt{\frac{2}{3a}}\sin\left(\frac{2\pi x}{a}\right) + e^{\frac{i9\pi\hbar t}{4ma^2}}\sqrt{\frac{4}{3a}}\sin\left(\frac{3\pi x}{a}\right)\right) x \left(e^{-\frac{i\pi\hbar t}{ma^2}}\sqrt{\frac{2}{3a}}\sin\left(\frac{2\pi x}{a}\right)\right.$$

$$\left.+ e^{-i\frac{9\pi\hbar t}{4ma^2}}\sqrt{\frac{4}{3a}}\sin\left(\frac{3\pi x}{a}\right)\right) dx = \int_0^a \left[\left(\frac{2}{3a}\right) x \sin^2\left(\frac{2\pi x}{a}\right) + \left(\frac{4}{3a}\right) x \sin^2\left(\frac{3\pi x}{a}\right)\right.$$

$$\left. + \sqrt{\frac{8}{9a^2}}\, x\left(\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right)\right)\left(e^{+i\frac{5\pi\hbar t}{4ma^2}} + e^{-i\frac{5\pi\hbar t}{4ma^2}}\right)\right] dx$$

با استفاده از

$$\left[\sin^2\alpha = \frac{1-\cos 2\alpha}{2}\right] = \frac{2}{3a}\left[\frac{x^2}{4}\Big|_0^a - \underbrace{\int_0^a \frac{x}{2}\cos\left(\frac{4\pi x}{a}\right)dx}_{\to 0}\right] + \frac{4}{3a}\left[\frac{x^2}{4}\Big|_0^a - \underbrace{\int_0^a \frac{x}{2}\cos\left(\frac{6\pi x}{a}\right)dx}_{\to 0}\right]$$

$$+2\sqrt{\frac{8}{9a^2}}\cos\left(\frac{5\pi\hbar t}{4ma^2}\right)\underbrace{\int_0^a x \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right)dx}_{\to 0}$$

$$= \frac{a}{6} + \frac{a}{3} - \frac{2}{3a}\left(\frac{-a + a\cos\left(\frac{4\pi a}{a}\right) + 4\pi\sin\left(\frac{4\pi a}{a}\right)}{32\pi^2}\right) - \frac{4}{3}\left(\frac{-a + a\cos\left(\frac{6\pi a}{a}\right) + 6\pi\sin\left(\frac{6\pi a}{a}\right)}{72\pi^2}\right)$$

$$= 3a/6 = a/2$$

و به همین صورت برای p هم مقدار $\langle p\rangle$ را محاسبه می‌کنیم